Branching Diffusions Representation for Nonlinear PDEs

Nizar Touzi

Ecole Polytechnique, Paris

RICAM, Linz
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Objective

Design numerical approximation for the equation :

$$\partial_t v + \mu \cdot Dv + \frac{1}{2}\sigma^2 : D^2 v + F(t, x, v, Dv, D^2 v) = 0, \quad v(T, .) = g$$

• Finite differences, finite elements : very efficient in $1-2$ dim, curse of dimensionality, path dependency increases dimension

• **Probabilistic representation** $\Longrightarrow$ Monte Carlo methods + extension to the path-dependent case...

• Extension to stochastic PDEs and general initial value problems

Intuition from the linear case

The heat equation :

$$\partial_t v + \frac{1}{2}\Delta v = 0, \quad v(T,.) = g$$

has the following probabilistic representations :

$$v(0,x) \quad = \quad \mathbb{E}\big[g(B_T)|B_0 = x\big]$$

where B a Brownian motion

$\Longrightarrow$ opens the door to Monte Carlo approximation

dimension-free rate of convergence

$\Longrightarrow$ allows for path-dependency

A first Monte Carlo scheme for fully nonlinear PDEs

Consider the backward numerical scheme

$$
\begin{aligned}
Y^n_{t_n} &= g\left(W_{t_n}\right) \;, \\
Y^n_{t_{i-1}} &= \mathbb{E}^n_{i-1}\left[Y^n_{t_i}\right] + f\left(W_{t_{i-1}}, Y^n_{t_{i-1}}, Z^n_{t_{i-1}}, \Gamma^n_{t_{i-1}}\right)\Delta t_i \;, \quad 1 \le i \le n \;, \\
Z^n_{t_{i-1}} &= \mathbb{E}^n_{i-1}\left[Y^n_{t_i}\,\frac{\Delta W_{t_i}}{\Delta t_i}\right] \\
\Gamma^n_{t_{i-1}} &= \mathbb{E}^n_{i-1}\left[Y^n_{t_i}\,\frac{|\Delta W_{t_i}|^2 - \Delta t_i}{|\Delta t_i|^2}\right]
\end{aligned}
$$

Then $Y^n_0 \longrightarrow v(0,x)$, where v is the unique solution of

$$\partial_t v + f(x, v, Dv, D^2 v) = 0, \quad v_T = g.$$

Error estimate... Curse of dimensionality is back

Our main objective

An alternative representation of the heat equation

$$v(t,x) = e^{\beta T}\mathbb{E}\big[g(B_T)\mathbb{1}_{\{T<\tau\}}|B_0 = x\big], \quad B \text{ BM}, \quad \tau \sim \text{Expo}(\beta) \perp B$$

Extend this representation for nonlinear PDEs

$\Longrightarrow$ Branching diffusions...

- ⊕ Purely forward Monte Carlo
- ⊕ Suitable for path-dependency
- ⊕ Very easy to implement, complexity linear in d^2
- ⊖ Need to control from explosion of solution

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Outline

From linear representation (ii) to nonlinear

- Consider KPP equations

$$\text{(KPP)} \quad \partial_t v + \mu \cdot Dv + \tfrac{1}{2}\sigma^2 : D^2 v + \beta\Big(\textstyle\sum_{i=1}^n p_i v^i - v\Big) = 0$$

with $p_i > 0$ and $\sum_{k=1}^n p_i = 1$

- Branching diffusions representation :

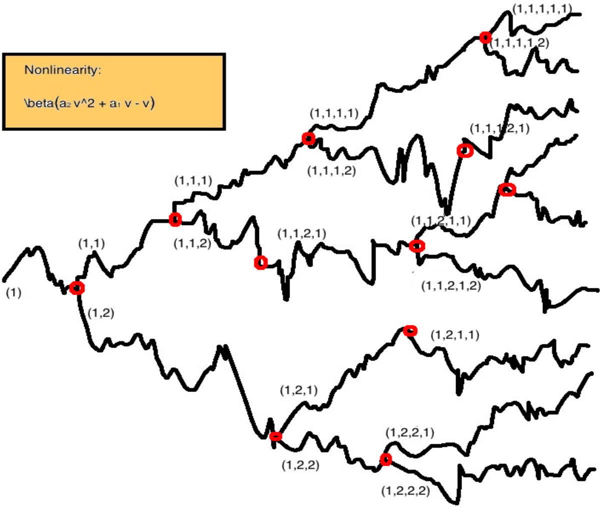
$$v(0,x) \;=\; \mathbb{E}\Big[\prod_{k\in\mathcal{K}_T} g(Z_T^k)\Big], \quad \text{where } \; Z^k : \; k - \text{th particle}$$

and

$$\mathcal{K}_t \;:=\; \big\{\text{All particles alive at time } t\big\}$$

[Skorokhod, Watanabe, McKean]

Branching diffusion ($n = 2$)



Generalized KPP equation

Let $a_i(t,x)$ be bounded functions, and consider the PDE

$$\partial_t v + \mu(t,x)\cdot Dv + \frac{1}{2}\sigma^2(t,x) : D^2 v + \sum_{i=0}^{n} p_i a_i(t,x) v^i = 0$$

$$v(T,.) = g$$

Introduce the branching diffusion :

- $(\tau_k)_k$ iid $\sim \rho$, and $T_k := T \wedge (\tau_1 + \ldots \tau_k)$: branching times
- $(I_k)_k$ iid Multinomial$(p_0,\ldots,p_n)$: number of decendents
- Particle k dies out at the branching event T_k, and I_k independent particles follow the diffusion with drift and diffusion (μ,σ)

The branching diffusion representation

Recall

- $\mathcal{K}_T :=$ {particles present at T}
- $\overline{\mathcal{K}}_T := \cup_{t \le T} \mathcal{K}_t$: all particles

Theorem (Henry-Labordère, Tan & NT SPA '14)

Let $\rho > 0$ density on $(0, \infty)$, and $\bar{F}_\rho(t) := \int_t^\infty \rho(s)ds$. Then

$$v(0,x) \quad = \quad \mathbb{E}_{0,x}\Big[\prod_{k \in \mathcal{K}_T} \frac{g(Z_T^k)}{\bar{F}_\rho(\Delta T)} \prod_{k \in \overline{\mathcal{K}}_T \setminus \mathcal{K}_T} \frac{a_{I_k}(T_k, Z_{T_k}^k)}{\rho(\Delta T_k)} \Big]$$

Moreover, this representation extends to the path-dependent case

- Numerical implications

- In the rest of he talk : extension to more general nonlinearities

Sketch of proof

By the Feynman-Kac formula

$$
\begin{aligned}
v(0,x) &= \mathbb{E}_{0,x}\left[\frac{\bar{F}_\rho(T)}{\bar{F}_\rho(T)} g(X_T) + \int_0^T \sum_{i=1}^n \frac{p_i a_i(t,X_t)}{\rho(t)} v(t,X_t)^i \rho(t) dt\right] \\
&= \mathbb{E}_{0,x}\left[\mathbb{1}_{\tau_1 > T} \frac{g(X_T)}{\bar{F}_\rho(T)} + \mathbb{1}_{\tau_1 \le T} \frac{a_{l_1}(\tau_1, X_{\tau_1})}{\rho(\tau_1)} v(\tau_1, X_{\tau_1})^{l_1}\right]
\end{aligned}
$$

Continue with the nonlinear term to get rid of the regression :

$$
\begin{aligned}
v(\tau_1, X_{\tau_1})^{l_1} &= \mathbb{E}_{\tau_1, X_{\tau_1}}\left[\phi(\tau_2, X_{\tau_2}, l_2)\right]^{l_1} \\
&= \underbrace{\mathbb{E}_{\tau_1, X_{\tau_1}}\left[\phi(\tau_2, X_{\tau_2}, l_2)\right] \times \ldots \times \mathbb{E}_{\tau_1, X_{\tau_1}}\left[\phi(\tau_2, X_{\tau_2}, l_2)\right]}_{l_1 \text{ times}} \\
&= \mathbb{E}_{\tau_1, X_{\tau_1}}\left[\phi(\tau_{1,1}, Z^{1,1}_{\tau_{1,1}}, l_{1,1}) \times \ldots \times \phi(\tau_{1,l}, Z_{\tau_{1,l}}, l_{1,l})\right]
\end{aligned}
$$

.... Tower property... Number of branching $\longrightarrow \infty$

Sketch of proof

By the Feynman-Kac formula

$$\begin{aligned} v(0,x) &= \mathbb{E}_{0,x}\Big[\frac{\bar F_\rho(T)}{\bar F_\rho(T)} g(X_T) + \int_0^T \sum_{i=1}^n \frac{p_i a_i(t,X_t)}{\rho(t)} v(t,X_t)^i \rho(t)dt\Big] \\ &= \mathbb{E}_{0,x}\Big[\mathbb{1}_{\tau_1>T}\frac{g(X_T)}{\bar F_\rho(T)} + \mathbb{1}_{\tau_1\le T}\frac{a_{I_1}(\tau_1,X_{\tau_1})}{\rho(\tau_1)} v(\tau_1,X_{\tau_1})^{I_1}\Big] \end{aligned}$$

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$$\begin{aligned} v(\tau_1,X_{\tau_1})^{I_1} &= \mathbb{E}_{\tau_1,X_{\tau_1}}\big[\phi(\tau_2,X_{\tau_2},I_2)\big]^{I_1} \\ &= \underbrace{\mathbb{E}_{\tau_1,X_{\tau_1}}\big[\phi(\tau_2,X_{\tau_2},I_2)\big]\times\ldots\times\mathbb{E}_{\tau_1,X_{\tau_1}}\big[\phi(\tau_2,X_{\tau_2},I_2)\big]}_{I_1 \text{ times}} \\ &= \mathbb{E}_{\tau_1,X_{\tau_1}}\big[\phi(\tau_{1,1},Z^{1,1}_{\tau_{1,1}},I_{1,1})\times\ldots\times\phi(\tau_{1,I},Z_{\tau_{1,I}},I_{1,I})\big] \end{aligned}$$

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By the Feynman-Kac formula

$$v(0,x) = \mathbb{E}_{0,x}\Big[\frac{\bar F_\rho(T)}{\bar F_\rho(T)} g(X_T) + \int_0^T \sum_{i=1}^n \frac{p_i a_i(t,X_t)}{\rho(t)} v(t,X_t)^i \rho(t)dt\Big]$$

$$= \mathbb{E}_{0,x}\Big[\mathbb{1}_{\tau_1>T}\frac{g(X_T)}{\bar F_\rho(T)} + \mathbb{1}_{\tau_1\le T}\frac{a_{I_1}(\tau_1,X_{\tau_1})}{\rho(\tau_1)} v(\tau_1,X_{\tau_1})^{I_1}\Big]$$

Continue with the nonlinear term to get rid of the regression :

$$v(\tau_1,X_{\tau_1})^{I_1} = \mathbb{E}_{\tau_1,X_{\tau_1}}\big[\phi(\tau_2,X_{\tau_2},I_2)\big]^{I_1}$$

$$= \underbrace{\mathbb{E}_{\tau_1,X_{\tau_1}}\big[\phi(\tau_2,X_{\tau_2},I_2)\big] \times \ldots \times \mathbb{E}_{\tau_1,X_{\tau_1}}\big[\phi(\tau_2,X_{\tau_2},I_2)\big]}_{I_1 \text{ times}}$$

$$= \mathbb{E}_{\tau_1,X_{\tau_1}}\big[\phi(\tau_{1,1},Z^{1,1}_{\tau_{1,1}},I_{1,1}) \times \ldots \times \phi(\tau_{1,I},Z_{\tau_{1,I}},I_{1,I})\big]$$

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A class of semilinear PDEs with polynomial nonlinearity

Consider the PDE (unit diffusion for simplicity)

$$\partial_t u + \frac{1}{2}\Delta u + f(t, x, u, Du) = 0, \quad u_T = g$$

with nonlinearity

$$f(t, x, y, z) \quad = \sum_{(\ell_i)_{0 \le i \le n} \in L} p_\ell c_\ell(t, x) y^{\ell_0} \prod_{i=1}^{n} \left(b_i(t, x) \cdot z\right)^{\ell_i}$$

- L finite subset of $\mathbb{N}^{n+1}$
- $p_\ell > 0$ with $\sum_{\ell \in L} p_\ell = 1$
- $b_i(t, x)$ bounded functions

Sketch of proof for the Burger's equation

By the Feynman-Kac formula

$$\begin{aligned} v(0,x) &= \mathbb{E}_{0,x}\Big[\frac{\bar F_\rho(T)}{\bar F_\rho(T)} g(X_T) + \int_0^T \frac{1}{\rho(t)} v(t,X_t)Dv(t,X_t)\rho(t)dt\Big] \\ &= \mathbb{E}_{0,x}\Big[\mathbb{1}_{\tau_1>T}\frac{g(X_T)}{\bar F_\rho(T)} + \mathbb{1}_{\tau_1\le T}\frac{1}{\rho(\tau_1)} v(\tau_1,X_{\tau_1})Dv(\tau_1,X_{\tau_1})\Big] \end{aligned}$$

Continue with the nonlinear term to get rid of the regression :

$$\begin{aligned} (vDv)(\tau_1,X_{\tau_1}) &= \mathbb{E}_{\tau_1,X_{\tau_1}}\big[\phi(\tau_2,X_{\tau_2})\big]\partial_x\mathbb{E}_{\tau_1,X_{\tau_1}}\big[\phi(\tau_2,X_{\tau_2})\big] \\ &= \mathbb{E}_{\tau_1,X_{\tau_1}}\big[\phi(\tau_2,X_{\tau_2})\big]\mathbb{E}_{\tau_1,X_{\tau_1}}\Big[\frac{W_{\tau_2}-W_{\tau_1}}{\tau_2-\tau_1}\phi(\tau_2,X_{\tau_2})\Big] \\ &= \mathbb{E}_{\tau_1,X_{\tau_1}}\Big[\phi(\tau_{1,1},Z^{1,1}_{\tau_{1,1}})\frac{W_{\tau_2}-W_{\tau_1}}{\tau_2-\tau_1}\phi(\tau_{1,l},Z_{\tau_{1,l}})\Big] \end{aligned}$$

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Continue with the nonlinear term to get rid of the regression :

$$(vDv)(\tau_1,X_{\tau_1}) = \mathbb{E}_{\tau_1,X_{\tau_1}}\big[\phi(\tau_2,X_{\tau_2})\big]\partial_x\mathbb{E}_{\tau_1,X_{\tau_1}}\big[\phi(\tau_2,X_{\tau_2})\big]$$
$$= \mathbb{E}_{\tau_1,X_{\tau_1}}\big[\phi(\tau_2,X_{\tau_2})\big]\mathbb{E}_{\tau_1,X_{\tau_1}}\Big[\frac{W_{\tau_2}-W_{\tau_1}}{\tau_2-\tau_1}\phi(\tau_2,X_{\tau_2})\Big]$$
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Continue with the nonlinear term to get rid of the regression :

$$(vDv)(\tau_1,X_{\tau_1}) = \mathbb{E}_{\tau_1,X_{\tau_1}}\big[\phi(\tau_2,X_{\tau_2})\big]\partial_x\mathbb{E}_{\tau_1,X_{\tau_1}}\big[\phi(\tau_2,X_{\tau_2})\big]$$
$$= \mathbb{E}_{\tau_1,X_{\tau_1}}\big[\phi(\tau_2,X_{\tau_2})\big]\mathbb{E}_{\tau_1,X_{\tau_1}}\Big[\frac{W_{\tau_2}-W_{\tau_1}}{\tau_2-\tau_1}\phi(\tau_2,X_{\tau_2})\Big]$$
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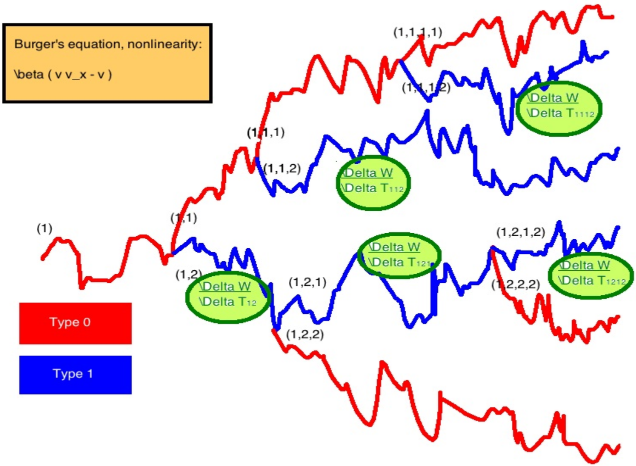
.... Tower property... Number of branching $\longrightarrow \infty$...

Branching diffusion for the Burger equation

Example : Burgers equation $d = 1$ and $f(t, x, u, u_x) = u\ u_x$



Automatic differentiation

For simplicity, $d = 1$, constant μ, $\sigma = 1$: $X_h = x + \mu h + W_h$

$$\partial_x \mathbb{E}[\phi(X_h)] = \mathbb{E}\left[\phi(X_h)\frac{W_h}{h}\right]$$

Direct Integration by parts

$$\partial_x \mathbb{E}\big[\phi(X_h)\big] = \mathbb{E}\big[\phi_x(X_h)\big] = \int \phi_x(x+y)\frac{e^{\frac{-1}{2h}(y-\mu h)^2}}{\sqrt{2\pi h}}dy$$

$$= \int \phi(x+y)\frac{(y-\mu h)}{h}\frac{e^{\frac{-1}{2h}(y-\mu h)^2}}{\sqrt{2\pi h}}dy$$

$$= \mathbb{E}\left[\phi(X_h)\frac{W_h}{h}\right]$$

Marked branching mechanism

- $(\tau_k)_k$ iid arrival times, $T_k := \tau_k \wedge T$
- $(I_k)_k$ iid Multinomial$\big(\ell, p_\ell, \ell \in L\big)$
- If $T_1 < T$: particle dies out at T_1
  - $I_1 = \ell = (\ell_0, \ldots, \ell_n)$ with probability p_ℓ
  - birth of new particles :

$$\ell_i \text{ particles of type } i,\ i = 0, \ldots, n$$

- $\mathcal{K}_t = \{k \text{ alive at time } t\}$, $\overline{\mathcal{K}}_t := \cup_{s \le t} \mathcal{K}_s$, for all $k \in \overline{\mathcal{K}}_T$:
  - $D(k)$ its type
  - $k-$ its parent particle

    $\Longrightarrow$ Particle k lives between T_{k-} and T_k

Using automatic differentiation

- Automatic differentiation :

$$\mathcal{W}_k := \mathbb{1}_{\{D(k)=0\}} + \mathbb{1}_{\{D(k)\neq 0\}} b_{D(k)}(T_{k-}, X^k_{T_{k-}}) \cdot \frac{\Delta W_{T_k}}{\Delta T_k}$$

The limiting random variable is :

$$\psi \;:=\; \prod_{k\in\mathcal{K}_T} \bar{F}_\rho(\Delta T_k)^{-1}\left[g\left(X^k_T\right) - \mathbb{1}_{\{D_k\neq 0\}} g\left(X^k_{T_{k-}}\right)\right]\mathcal{W}_k$$
$$\times \prod_{k'\in\overline{\mathcal{K}}_T\setminus\mathcal{K}_T} \left[\rho(\Delta T_{k'})\right]^{-1} b_{I_{k'}}(T_{k'}, X^{k'}_{T_{k'}})\mathcal{W}_{k'}$$

Sufficient condition for square integrability

For independent BM W, $\tau \sim \rho$, and $T_1 := \tau \wedge T$, define :

$$A_p := \max_{\ell} \frac{|g|_\infty^p \vee \|W_{T_1}\|^p \|b_\ell \cdot \frac{W_{T_1}}{T_1}\|^p}{\bar{F}_\rho(T)^{p-1}}$$

$$B_p := \max_{\ell} \|\Delta\tau\|^{p/2} \Big\| b_\ell \cdot \frac{W_{T_1}}{T_1} \Big\|^p \Big[|b_\ell|_\infty \sup_{t \le T} \frac{t^{-\frac{p}{2(p-1)}}}{\rho(t)} \Big]^{p-1}$$

Theorem (Henry-Labordère, Oudjane, Tan, NT, Warin '16)

Assume that g Lipschitz and, for some $p > 1$,

$$\int_{A_p}^{\infty} \Big[B_p \sum |b_\ell|_\infty |x|^{|\ell|} \Big]^{-1} dx \quad > \quad T$$

Then $v(0,x) = \mathbb{E}_{0,x}[\psi]$, and $\psi \in \mathbb{L}^2$

Numerical example

- Define

$$u(t,x) = \cos(x_1 + \cdots + x_d)\exp(\alpha(T-t))$$

is solution of semilinear PDE

$$\partial_t u + \frac{1}{2}\Delta u + c\, u(b_1 \cdot Du) + b_0 \;=\; 0.$$

- For numerical implementation, we choose

$$\alpha = 0.2, \quad c = 0.15, \quad b_1 = (1+\frac{1}{d}, 1+\frac{2}{d}, \cdots, 2).$$

A numerical example of dimension $d = 20$

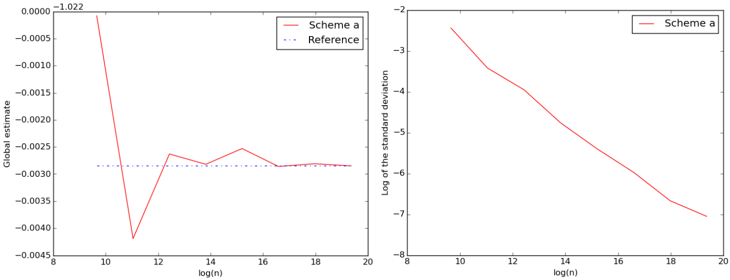


Figure – Estimation and standard deviation observed in dimension $d = 20$ depending on the log of the number of simulation used.

Comments on the Monte-Carlo method

- Choice of ρ, $(p_\ell)_{\ell \in L}$, the expression of nonlinearity ...
- possible to use importance sampling, particles method,...
- open to parallel computing

Monte Carlo approximation of nonlinear PDEs

Fully nonlinear PDEs... (e.g. HJB equations)

If $T_1 < T$: particle dies out, and is replaced with probability p_ℓ by

- i_ℓ particles of type 0
- j_ℓ particles of type 1 $\Longrightarrow$ first order differentiation weight
- h_ℓ particles of type 2 $\Longrightarrow$ second order differentiation weight

Automatic differentiation for particles k of type $D(k) = 2$:

$$\frac{\Delta W_T^2 - \Delta T}{(\Delta T)^2} \quad !!$$

Outline

Linear initial value problems with constant coefficients

Consider the general PDE :

$$\sum_{j=1}^{n} a_j \partial_t^j u - \sum_{j=1}^{m} b_j \cdot D^j u = f_n \text{ on } \mathbb{R}_+ \times \mathbb{R}^d,$$

$$\partial_t^j u(0,.) = f_j \text{ on } \mathbb{R}^d, \quad j = 0, \ldots, n-1,$$

where

$f_j : \mathbb{R}^d \longrightarrow \mathbb{R}$, $j < n$ and $f_n : \mathbb{R}_+ \times \mathbb{R}^d \longrightarrow \mathbb{R}$, bounded continuous

Examples :

- Wave equation : $\partial_t^2 u - \Delta u = f_2$
- KdV equation : $\partial_t u - b_3 \cdot D^3 u = f_1$

Duhamel's Representation

For all $t \geq 0$, and $x \in \mathbb{R}^d$, we have

$$\begin{aligned} u(t,x) &= \sum_{j=1}^{n} \int f_{j-1}(y) g_j(t, x-y) dy \\ &\quad + \int_0^t \int f_n(s,y) g_n(t-s, x-y) dy\, ds \end{aligned}$$

where $g_j(r,z) := (2\pi)^{-d} \int \left\{ e^{B(\xi) r} \right\}_{1,j} e^{-i\xi \cdot z} d\xi$, and

$$B(\xi) := \left(\begin{array}{c|ccc} 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \\ \hline b(\xi) & -a_1 & \cdots & -a_{n-1} \end{array} \right)$$

Probabilistic representation

Assume $G_j(t) := \int |g_j(t,y)| dy < \infty$ for all $t \geq 0$, $1 \leq j \leq n$

$$\text{Then,} \quad u(t,x) = \mathbb{E}\Big[\mathbb{1}_{\{\tau > t\}} m_I(t, Z_t^{1+I}) f_I(X_t^{1+I}) + \mathbb{1}_{\{\tau \leq t\}} m_n(\tau, Z_\tau^n) f_n(t-\tau, X_\tau^n) \Big]$$

$I, \tau, X_t^j := x - Z_t^j, 1 \leq j \leq n$ independent with

$$\textstyle\sum_{j=0}^{n-1} p_j = 1, \;\; p_j := \mathbb{P}[I = j]$$

$$\mathbb{P}[\tau \in dt] = \rho(t) \mathbb{1}_{\{t \geq 0\}} dt, \;\; \rho > 0$$

$$\mathbb{P}\big[Z_t^j \in dz\big] = \frac{|g_j(t,z)|}{G_j(t)} dz$$

$$m_n(t,y) := \frac{\mathrm{sg}\{g_n(t,y)\} G_n(t)}{\rho(t)}, \; m_j(t,y) := \frac{\mathrm{sg}\{g_{j+1}(t,y)\} G_{j+1}(t)}{p_j \bar{\rho}(t)}, \; j < n$$

Representation of semilinear initial value problems

Theorem (Henry-Labordère & NT)

Under some conditions, we have

$$\xi_{t,x} \;:=\; \prod_{k\in\mathcal{K}_t} m_{I_k}\big(\Delta T_k^t, Z_t^{1+I_k}\big) f_{I_k}\big(X_t^{1+I_k}\big)$$

$$\times \prod_{k\in\overline{\mathcal{K}}_t\setminus\mathcal{K}_t} m_n\big(\Delta T_k^t, Z_{T_k^t}^k\big) c_{J_k}\big(t - T_k^t, X_{T_k^t}^n\big) \;\in\; \mathbb{L}^1$$

and $u(t,x) := \mathbb{E}\big[\xi_{t,x}\big]$ *is the unique solution of the semilinear IVP*

$$\sum_{j=1}^n a_j \partial_t^j u - \sum_{j=1}^m b_k \cdot D^k u \;=\; \sum_{\ell=1}^L q_\ell\, c_\ell\, u^\ell \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^d$$

$$\partial_t^j u(0,.) = f_j \quad \text{on } \mathbb{R}^d, \quad j = 0,\ldots,n-1$$

Outline

Representation of semilinear initial value problems

Consider the SPDE driven by Brownian motion W :

$$\partial_t u+\frac{1}{2}\Delta u+\beta(\frac{1}{3}u^2-u)+\dot{W}=0,\ t<T,\ x\in\mathbb{R}^d,\ \ u(T,.)=g(.) \text{ on } \mathbb{R}^d$$

Notice that $u^0(t) := \int_t^T e^{-\beta(s-t)}dW_s$ is the solution of

$$\partial_t u^0+\frac{1}{2}\Delta u^0+-\beta u^0+\dot{W}=0,\ t<T,\ x\in\mathbb{R}^d,\ \ u^0(T,.)=g(.) \text{ on } \mathbb{R}^d$$

Then, $U(t,x) := u(t,x) - u^0(t)$ satisfies :

$$\begin{cases} \partial_t U + \frac{1}{2}\Delta U_t dt + \beta\Big(\frac{U_t^2 + 2u_t^0 U_t + (u_t^0)^2}{3} - U_t\Big) = 0,\ t<T,\ x\in\mathbb{R}^d \\ U(T,x) = g(x),\ x\in\mathbb{R}^d \end{cases} \quad (1)$$

Branching diffusion representation of SPDE

Theorem (Henry-Labordère, Matoussi & NT)

Under convenient conditions, the solution of the SPDE is $u := U + u^0$ *where*

$$U_t(x,\omega) = \mathbb{E}_{t,x}\Big[\prod_{k\in\mathcal{K}_T} g(W^k_T) \prod_{k\in\bar{\mathcal{K}}_T\setminus\mathcal{K}_T} \big(2^{I_k} u^0_{T_k}\big)^{2-I_k}\Big],$$

satisfies $\mathbb{E}\Big[\int_0^T \big(|u(t,.)|_{\mathbb{L}^\infty(\mathbb{R}^d)} + |Du(t,.)|_{\mathbb{L}^\infty(\mathbb{R}^d)}\big)\,dt\Big] < \infty.$

THANK YOU FOR YOUR ATTENTION