Numerical approximations for nonlinear stochastic partial differential equations

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Some examples of SDEs

- **Heston model** Consider $\alpha, \gamma \in \mathbb{R}$, $\beta, \delta, X_0^{(1)}, X_0^{(2)} > 0$, $\rho \in [-1, 1]$ and

$$\frac{\partial}{\partial t} X_t^{(1)} = \alpha X_t^{(1)} + \sqrt{X_t^{(2)}}\, X_t^{(1)} \frac{\partial}{\partial t} W_t^{(1)}$$

$$\frac{\partial}{\partial t} X_t^{(2)} = \delta - \gamma X_t^{(2)} + \beta \sqrt{X_t^{(2)}} \left(\rho \frac{\partial}{\partial t} W_t^{(1)} + \sqrt{1-\rho^2} \frac{\partial}{\partial t} W_t^{(2)}\right)$$

  for $t \in [0, T]$, where $(W_t)_{t \in [0,T]} = ((W_t^{(1)}, W_t^{(2)}))_{t \in [0,T]}$ is a two-dim. BM.

- **Nonlinear stochastic heat equation** (**P**arabolic **A**nderson **m**odel)

$$\frac{\partial}{\partial t} X_t(x) = \frac{\partial^2}{\partial x^2} X_t(x) + b(X_t(x)) \frac{\partial}{\partial t} W_t(x)$$

  with $X_t(0) = X_t(1) = 0$ for $x \in (0, 1)$, $t \in [0, T]$, where $(W_t)_{t \in [0,T]}$ is a cylindrical $\mathrm{Id}_{L^2((0,1);\mathbb{R})}$-Wiener process and $b\colon \mathbb{R} \to \mathbb{R}$ is regular.

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- **Stochastic Burgers equation**

$$\frac{\partial}{\partial t} X_t(x) = \frac{\partial^2}{\partial x^2} X_t(x) - X_t(x) \cdot \frac{\partial}{\partial x} X_t(x) + \frac{\partial}{\partial t} W_t(x)$$

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Some difficulties in the numerical approximations of SDEs

Theorem (Hairer, Hutzenthaler & J 2015 AOP)

Let $T \in (0, \infty)$, $d \in \{4, 5, \dots\}$, $\xi \in \mathbb{R}^d$. *Then there exist globally bounded* $\mu, \sigma \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ *such that for every probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, *every Brownian motion* $W\colon [0, T] \times \Omega \to \mathbb{R}$, *every solution* $X\colon [0, T] \times \Omega \to \mathbb{R}^d$ *of*

$$dX_t = \mu(X_t) + \sigma(X_t)\, dW_t, \qquad t \in [0, T], \qquad X_0 = \xi,$$

and every $Y^N\colon \{0, 1, \dots, N\} \times \Omega \to \mathbb{R}^4$, $N \in \mathbb{N}$, *with* $\forall\, N \in \mathbb{N}, n \in \{0, 1, \dots, N-1\}\colon\ Y_0^N = X_0$ *and*

$$Y_{n+1}^N = Y_n^N + \mu(Y_n^N)\tfrac{T}{N} + \sigma(Y_n^N)\big(W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}}\big)$$

(Euler-Maruyama approximations) we have $\forall\, \alpha \in [0, \infty)$:

$$\lim_{N\to\infty} \left(N^\alpha \left\| \mathbb{E}[X_T] - \mathbb{E}[Y_N^N] \right\|\right) = \begin{cases} 0 & : \alpha = 0 \\ \infty & : \alpha > 0 \end{cases}.$$

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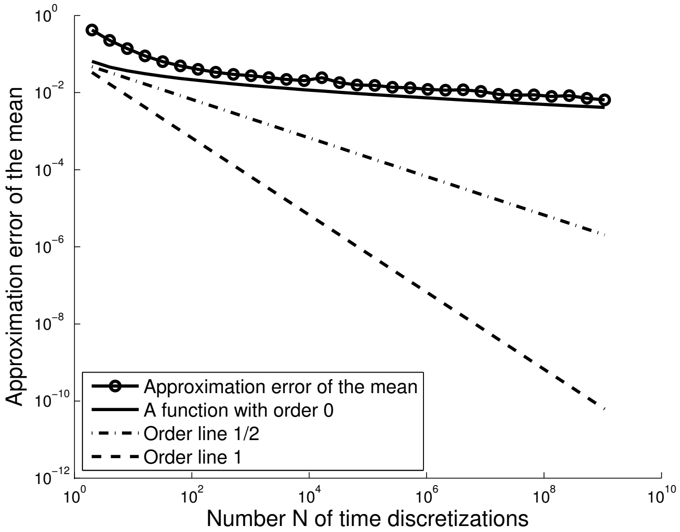
Plot of $\left\|\mathbb{E}\left[X_T\right] - \mathbb{E}\left[Y_N^N\right]\right\|$ for $T = 2$ and $N \in \{2^1, 2^2, \ldots, 2^{30}\}$.



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Let $T \in (0, \infty)$, $d \in \{2, 3, 4, \dots\}$, $\xi \in \mathbb{R}^d$, $(a_N)_{N \in \mathbb{N}} \subseteq \mathbb{R}$ satisfy $\lim_{N \to \infty} a_N = 0$. Then there exist globally bounded $\mu, \sigma \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$ such that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every Brownian motion $W \colon [0, T] \times \Omega \to \mathbb{R}$, every solution $X \colon [0, T] \times \Omega \to \mathbb{R}^d$ of

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dW_t, \qquad t \in [0, T], \qquad X_0 = \xi,$$

and every $N \in \mathbb{N}$ we have

$$\inf_{s_1, \dots, s_N \in [0, T]} \inf_{\substack{u \colon \mathbb{R}^N \to \mathbb{R}^d \\ \text{measurable}}} \mathbb{E}\Big[\big\| X_T - u\big(W_{s_1}, \dots, W_{s_N}\big) \big\|\Big] \geq a_N.$$

- **Dimension $d \geq 4$:** J, Müller-Gronbach & Yaroslavtseva 2016 CMS
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Theorem (Hefter & J 2016)

Let $T, \delta, \beta \in (0, \infty)$, $\gamma, \xi \in [0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W \colon [0, T] \times \Omega \to \mathbb{R}$ be a Brownian motion, let $X \colon [0, T] \times \Omega \to \mathbb{R}$ be a solution of

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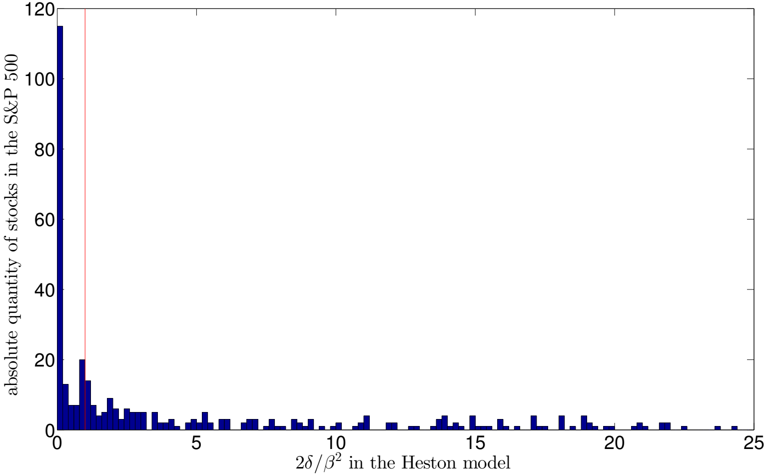
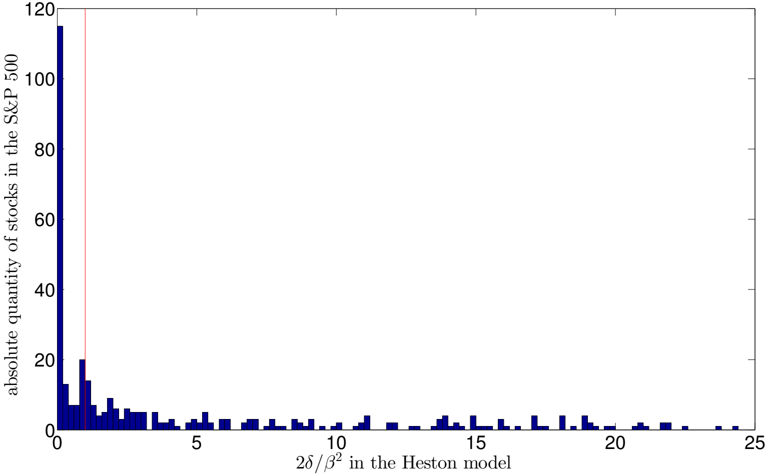


The **S&P 500** (the Standard & Poor's 500) is a stock market index.

In Hutzenthaler, J & Noll 2016 we calibrate 498 stocks from the S&P 500 within the **Heston model**: 359 stocks satisfy $\frac{2\delta}{\beta^2} \leq 25$, 162 stocks ($\approx 32\%$) satisfy $\frac{2\delta}{\beta^2} < 1$. More than 100 stocks ($= 20\%$) satisfy $\frac{2\delta}{\beta^2} \leq \frac{1}{10}$.



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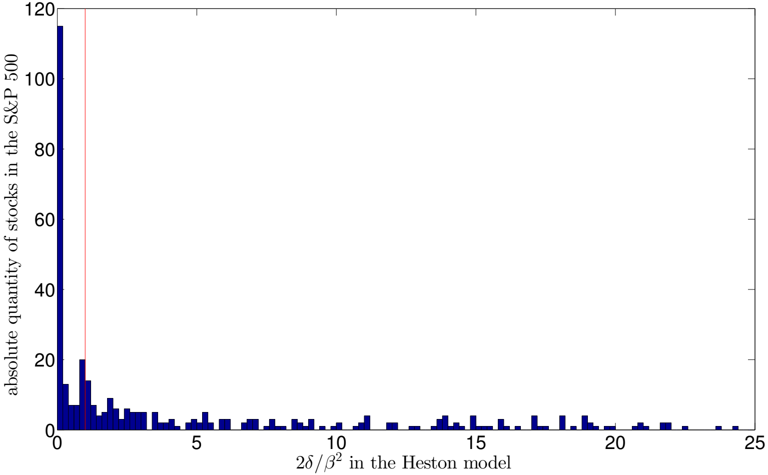

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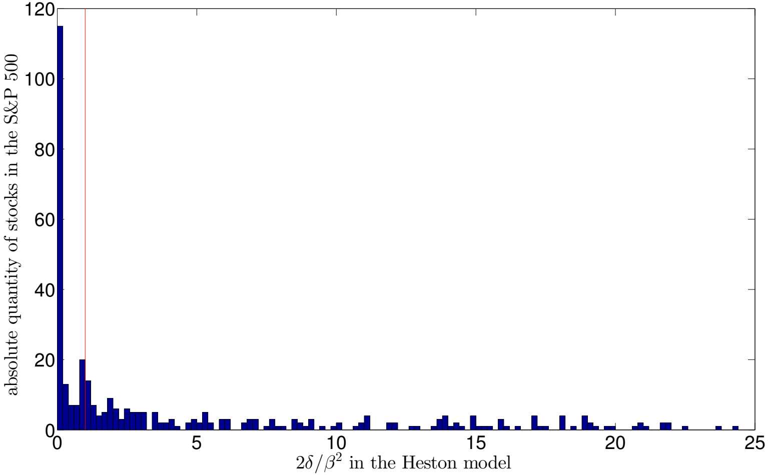

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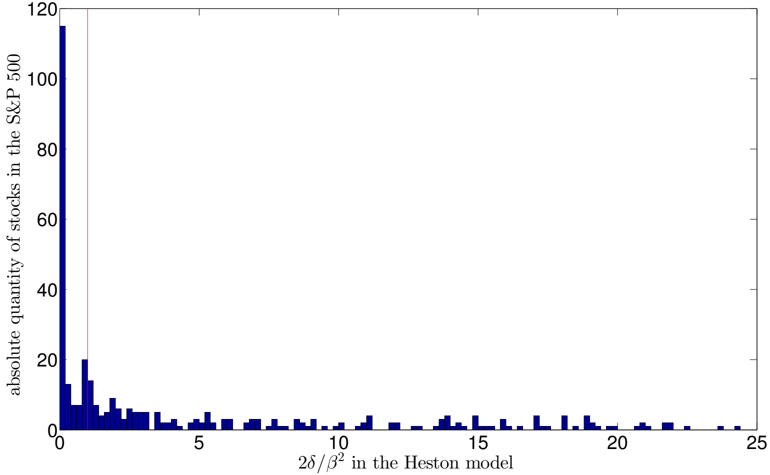

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Convergence results

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Let $T > 0$, $p \geq 2$, let $f, b\colon \mathbb{R} \to \mathbb{R}$ and $\varphi\colon L^p((0,1);\mathbb{R}) \to \mathbb{R}$ be $\mathcal{C}^4$ with Lipschitz continuous and bounded derivatives, let $\xi\colon (0,1) \to \mathbb{R}$ be measurable and bounded, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(W_t)_{t\in[0,T]}$ be an $\mathrm{Id}_{L^2((0,1);\mathbb{R})}$-cylindrical Wiener process, let $X\colon [0,T] \times \Omega \to L^p((0,1);\mathbb{R})$ be a continuous mild solution of

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Theorem (Hyperbolic Anderson model; Jacobe de Naurois, J & Welti 2015)

Let $H = L^2((0,1);\mathbb{R})$, $\varphi \in C_b^2(H,\mathbb{R})$, $\xi = (\xi_0, \xi_1) \in H_0^1((0,1);\mathbb{R}) \times H$, $f \in C_b^3((0,1) \times \mathbb{R}, \mathbb{R})$, *for every* $n \in \mathbb{N}$ *let* $e_n(\cdot) = \sqrt{2}\sin(n\pi(\cdot)) \in H$ *and for every* $N \in \mathbb{N} \cup \{\infty\}$ *let* $P_N(\cdot) = \sum_{n=1}^{N} \langle e_n, \cdot \rangle_H \, e_n \in L(H)$ *and let* $X^N \colon [0,T] \times \Omega \to P_N(H)$ *be a mild solution of*

$$\ddot{X}_t(x) = \tfrac{\partial^2}{\partial x^2} X_t(x) + P_N \, f(x, X_t(x)) + P_N \, X_t(x) \, \dot{W}_t(x)$$

with $X_0(x) = \xi_0(x)$ *and* $\dot{X}_0(x) = \xi_1(x)$ *for* $t \in [0,T]$, $x \in (0,1)$. *Then* $\forall \, \varepsilon > 0 \colon \exists \, C \geq 0 \colon \forall \, N \in \mathbb{N} \colon$

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- Rate can essentially not be improved.
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Theorem (Stochastic Burgers equation – Debussche, J, & Welti 2016)

Let $H = L^2((0,1);\mathbb{R})$, $\xi \in H$, $\varphi \in C_b^3(H,\mathbb{R})$, $\kappa \in \mathbb{R}$, *for every* $n \in \mathbb{N}$ *let* $e_n(\cdot) = \sqrt{2}\sin(n\pi(\cdot)) \in H$, *and for every* $N \in \mathbb{N} \cup \{\infty\}$ *let* $P_N(\cdot) = \sum_{n=1}^{N} \langle e_n, \cdot \rangle_H \, e_n \in L(H)$ *and let* $X^N \colon [0,T] \times \Omega \to P_N(H)$ *be a continuous mild solution of*

$$\dot{X}_t^N(x) = \tfrac{\partial^2}{\partial x^2} X_t^N(x) + P_N \left(\kappa \cdot X_t^N(x) \cdot \tfrac{\partial}{\partial x} X_t^N(x) \right) + P_N \, \dot{W}_t(x)$$

with $X_0^N(x) = (P_N \xi)(x)$ *for* $t \in [0,T]$, $x \in (0,1)$. *Then* $\exists\, C \geq 0 \colon \forall\, N \in \mathbb{N}\colon$

$$\left| \mathbb{E}\big[\varphi(X_T^\infty)\big] - \mathbb{E}\big[\varphi(X_T^N)\big] \right| \leq C \cdot N^{-1}.$$

- In the case $\kappa = 0$ it holds $\exists\, \varphi \in C_b^3(H,\mathbb{R}), c, C \in (0,\infty) \colon \forall\, N \in \mathbb{N}\colon$

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Let $H = L^2((0,1);\mathbb{R})$, $\xi \in H$, $\varphi \in \mathcal{C}^3_b(H,\mathbb{R})$, $\kappa \in \mathbb{R}$, *for every* $n \in \mathbb{N}$ *let* $e_n(\cdot) = \sqrt{2}\sin(n\pi(\cdot)) \in H$, *and for every* $N \in \mathbb{N} \cup \{\infty\}$ *let* $P_N(\cdot) = \sum_{n=1}^{N} \langle e_n, \cdot \rangle_H \, e_n \in L(H)$ *and let* $X^N \colon [0,T] \times \Omega \to P_N(H)$ *be a continuous mild solution of*

$$\dot{X}^N_t(x) = \tfrac{\partial^2}{\partial x^2} X^N_t(x) + P_N\left(\kappa \cdot X^N_t(x) \cdot \tfrac{\partial}{\partial x} X^N_t(x)\right) + P_N\,\dot{W}_t(x)$$

with $X^N_0(x) = (P_N\xi)(x)$ *for* $t \in [0,T]$, $x \in (0,1)$. *Then* $\exists\, C \geq 0 \colon \forall\, N \in \mathbb{N} \colon$

$$\left|\mathbb{E}\big[\varphi(X^\infty_T)\big] - \mathbb{E}\big[\varphi(X^N_T)\big]\right| \leq C \cdot N^{-1}.$$

- In the case $\kappa = 0$ it holds $\exists\, \varphi \in \mathcal{C}^3_b(H,\mathbb{R}), c, C \in (0,\infty) \colon \forall\, N \in \mathbb{N} \colon$

$$c \cdot N^{-1} \leq \left|\mathbb{E}\big[\varphi(X^\infty_T)\big] - \mathbb{E}\big[\varphi(X^N_T)\big]\right| \leq C \cdot N^{-1}.$$

Cahn-Hilliard Cook equation driven by a standard Wiener process:

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for $x \in (0,1)$ with Neumann and no-flux boundary conditions and regular initial value.

Kovacs, Larsson, Mesforush 2011, in particular, implies

$\forall\, \alpha \in (0,2), \varepsilon > 0 \colon \exists\, C_{\alpha,\varepsilon} \geq 0 \colon \forall\, h > 0 \colon \exists\, \Omega_{\varepsilon,h} \subseteq \Omega \colon$

$$\mathbb{P}(\Omega_{\varepsilon,h}) \geq 1 - \varepsilon \qquad \text{and} \qquad 1_{\Omega_{\varepsilon,h}} \sup_{t \in [0,T]} \left\| X_t - Y_t^h \right\| \leq C_{\alpha,\varepsilon}\, h^{\alpha}$$

(**semi strong convergence rate**) and hence

$$\lim_{h \searrow 0} \mathbb{E}\left[\sup_{t \in [0,T]} \left\| X_t - Y_t^h \right\|^2 \right] = 0.$$

Hutzenthaler & J 2014: $\forall\, \alpha \in (0,2), p > 0 \colon \exists\, C_{\alpha,p} \geq 0 \colon \forall\, N \in \mathbb{N} \colon$

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Theorem (Gerencsér, J, & Salimova 2016)

Let $T \in (0,\infty)$, $d \in \{2,3,4,\ldots\}$, $\xi \in \mathbb{R}^d$, $(a_N)_{N\in\mathbb{N}} \subseteq \mathbb{R}$ *satisfy* $\lim_{N\to\infty} a_N = 0$. *Then there exist globally bounded* $\mu, \sigma \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$ *such that for every probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, *every Brownian motion* $W\colon [0,T]\times\Omega \to \mathbb{R}$, *every solution* $X\colon [0,T]\times\Omega\to\mathbb{R}^d$ *of*

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and every $N\in\mathbb{N}$ *we have*

$$\inf_{s_1,\ldots,s_N\in[0,T]} \;\inf_{\substack{u\colon \mathbb{R}^N\to\mathbb{R}^d \\ \text{measurable}}} \mathbb{E}\Big[\big\|X_T - u\big(W_{s_1},\ldots,W_{s_N}\big)\big\|\Big] \geq a_N.$$

Remark: $\forall\, \alpha \in (0,\frac{1}{2}), \varepsilon > 0\colon \exists\, C_{\alpha,\varepsilon} \geq 0\colon \forall\, N \in \mathbb{N}\colon \exists\, \Omega_{\varepsilon,N} \subseteq \Omega\colon$

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Remark: $\forall\, \alpha \in (0, \tfrac{1}{2}), \varepsilon > 0 \colon \exists\, C_{\alpha,\varepsilon} \geq 0 \colon \forall\, N \in \mathbb{N} \colon \exists\, \Omega_{\varepsilon,N} \subseteq \Omega \colon$

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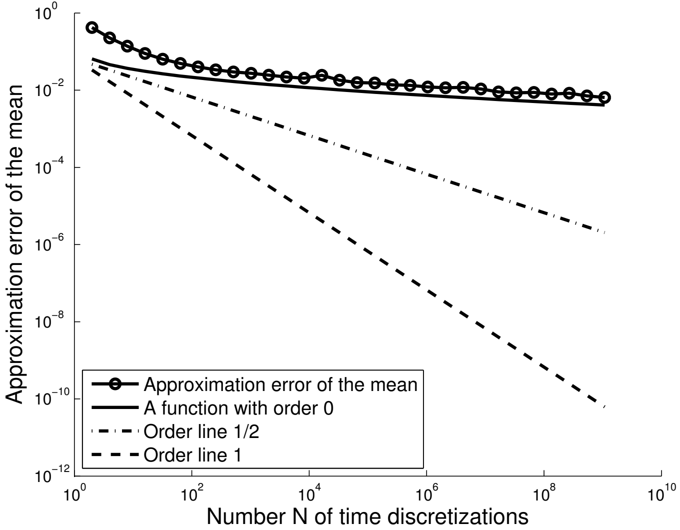
Plot of $\left\| \mathbb{E}\left[X_T\right] - \mathbb{E}\left[Y_N^N\right] \right\|$ for $T = 2$ and $N \in \{2^1, 2^2, \ldots, 2^{30}\}$.



Methods of proof

Corollary (Mild Itô formula; Da Prato, J & Röckner 2012 Trans. AMS (to appear))

Let V be a separable Hilbert space and let $\varphi \in C^2(H, V)$, $t_0, t \in [0, T]$ with $t_0 \leq t$. Then it holds $\mathbb{P}$-a.s. that

$$\varphi(X_t) = \varphi(e^{A(t-t_0)} X_{t_0}) + \int_{t_0}^{t} \varphi'(e^{A(t-s)} X_s)\, e^{A(t-s)} F(X_s)\, ds$$
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Mild Itô formula shows

$$\mathbb{E}\big[\varphi(X_t)\big] = \mathbb{E}\big[\varphi(e^{At} X_0)\big]$$
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Corollary (Mild Itô formula; Da Prato, J & Röckner 2012 Trans. AMS (to appear))

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$$+ \tfrac{1}{2} \sum_{u \in \mathbb{U}} \int_0^t \mathbb{E}\big[\varphi''(e^{A(t-s)} P_N(X_s)) \big(e^{A(t-s)} P_N B(X_s) u, e^{A(t-s)} P_N B(X_s) u \big)\big]\, ds.$$

Corollary (Mild Itô formula; Da Prato, J & Röckner 2012 Trans. AMS (to appear))

Let V be a separable Hilbert space and let $\varphi \in \mathcal{C}^2(H, V)$, $t_0, t \in [0, T]$ with $t_0 \le t$. Then it holds $\mathbb{P}$-a.s. that

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Theorem (Mild Itô formula; Da Prato, J & Röckner 2012 Trans. AMS (to appear))

Let X be a mild Itô process with evolution family S, mild drift Y, and mild diffusion Z. Then for all $\varphi \in \mathcal{C}^2(H, V)$, $t_0, t \in [0, T]$ with $t_0 < t$ it holds $\mathbb{P}$-a.s. that

$$\int_{t_0}^{t} \|\varphi'(S_{s,t}X_s)S_{s,t}Y_s\|_V + \|\varphi'(S_{s,t}X_s)S_{s,t}Z_s\|^2_{HS(U,V)}\, ds < \infty,$$

$$\int_{t_0}^{t} \|\varphi''(S_{s,t}X_s)\|_{L^{(2)}(H,V)}\, \|S_{s,t}Z_s\|^2_{HS(U,H)}\, ds < \infty$$

and

$$\varphi(X_t) = \varphi(S_{t_0,t}X_{t_0}) + \int_{t_0}^{t} \varphi'(S_{s,t}X_s)\, S_{s,t}Y_s\, ds + \int_{t_0}^{t} \varphi'(S_{s,t}X_s)\, S_{s,t}Z_s\, dW_s$$
$$+ \frac{1}{2} \sum_{u \in \mathbb{U}} \int_{t_0}^{t} \varphi''(S_{s,t}X_s)\, (S_{s,t}Z_s u, S_{s,t}Z_s u)\, ds.$$

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