Filtered schemes for Hamilton-Jacobi equations

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joint work with Adam Oberman

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Numerical methods for Hamilton-Jacobi equations
in optimal control and related fields
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McGill

Outline

1. Hamilton-Jacobi equations
2. Filtered Schemes
3. Numerical schemes
   - Schemes for the Eikonal equation
   - Schemes for Hamilton-Jacobi equations
4. Explicit formulas
   - Eikonal equation
   - HJ equations
5. Results
6. Conclusions

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Hamilton-Jacobi equations

Consider the Hamilton-Jacobi (HJ) equation

$$\begin{cases} H(x, \nabla u) = f(x), & x \in \Omega \setminus \Gamma, \\ u(x) = g(x), & x \in \Gamma \subset \Omega, \end{cases}$$

where

- ∇u is the gradient of the function u,
- Ω is a bounded open set,
- Γ is a subset of Ω,
- H is a nonlinear Lipschitz continuous function.

Eikonal equation

In particular, we will focus on the the Eikonal equation

$$\begin{cases} |\nabla u(x)| = f(x), & \text{for x outside } \Gamma, \\ u(x) = g(x), & \text{for x on } \Gamma, \end{cases}$$

where $f > 0$ and Γ is closed, bounded set.

Fact: Solutions are usually piecewise smooth and globally Lipschitz.

Related work

- Semi-Lagrangian schemes (Falconi, Ferretti '02 and Cristiani, Falconi '07)
- Central schemes (Lin, Tadmor '00)
- ENO and WENO (Osher, Shu '91)
- Compact upwind second order scheme (Benamou, Luo, Zhao '03)
- Abgrall blended scheme (Abgrall '09)

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Motivation

In the one-dimensional case we are interested in solving $|u_x| = f(x)$. The *monotone* (upwind) scheme is given by

$$\begin{aligned} \left|u_x^h\right|^M &= \max\left\{-\frac{u(x+h)-u(x)}{h}, \frac{u(x)-u(x-h)}{h}, 0\right\} \\ &= \max\left\{-D^+u(x), D^-u(x), 0\right\} \end{aligned}$$

This scheme is also *consistent* and *stable* and so, by the Barles and Souganidis theory, it converges to the unique viscosity solution.

Fact: Monotone schemes are provably convergent but only first order accurate.

We could consider the *second order accurate* centered scheme

$$\left|u_x^h\right|^A = \frac{|u(x+h) - u(x-h)|}{2h}.$$

However this scheme is not *monotone* nor *stable*.

Fact: In general, higher order finite difference schemes for HJ equations are neither monotone nor stable.

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Up to a scaling in h, the difference of the schemes is the centered finite difference approximation for $|u_{xx}|$:

$$\left|\left|u_x^h\right|^A - \left|u_x^h\right|^M\right| = \frac{h}{2}\frac{|u(x+h) - 2u(x) + u(x-h)|}{h^2} = \frac{h}{2}u_{xx}^h(x).$$

We can then use this as a (local) criteria to decide whether or not to use the accurate scheme.

Filtered Schemes

The filtered scheme, $|\nabla u^h|^F$, then uses the following simple formula:

$$\left|u_x^h\right|^F = \begin{cases} \left|u_x^h\right|^A, & \text{if } \left|\left|u_x^h\right|^A - \left|u_x^h\right|^M\right| \leq \sqrt{h} \\ \left|u_x^h\right|^M, & \text{otherwise.} \end{cases}$$

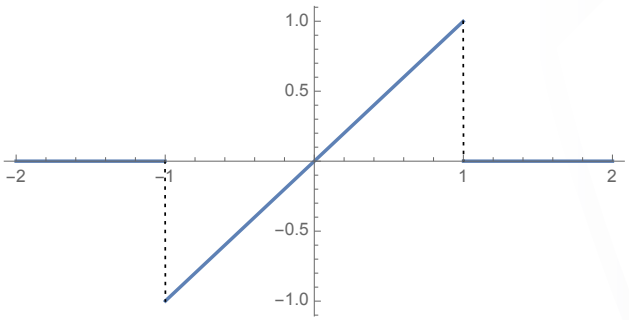
The filtered scheme, which is consistent provided both underlying schemes are consistent, is usually not monotone. However it is almost monotone, since

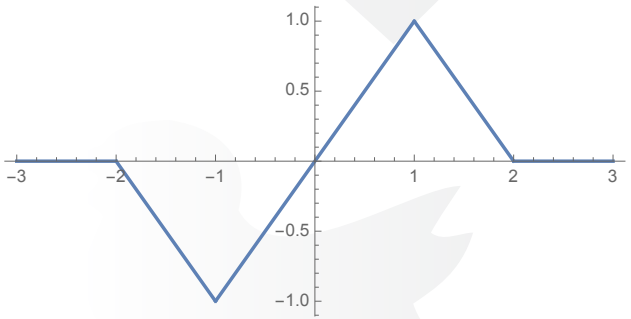
$$\left|u_x^h\right|^F = \left|u_x^h\right|^M + \mathcal{O}(\sqrt{h}).$$

Filtered Schemes

Definition

S is a filter function if it is a bounded function that is equal to the identity in a neighborhood of the origin and zero outside.





Definition

The filtered scheme is given by $F^h[u] = F^h_M[u] + \epsilon(h) S\left(\frac{F^h_A[u] - F_M[u]}{\epsilon(h)}\right)$.

Theorem (Froese, Oberman 2013)

Let F_M^h be a consistent, stable, monotone scheme. Assume S is a continuous filter function and $\epsilon(h) \to 0$ as $h \to 0$. For each $h > 0$, let u_h be a solution of $F^h[u] = 0$, where the filtered scheme is given by

$$F^h[u] = F_M^h[u] + \epsilon(h) S\left(\frac{F_A^h[u] - F_M[u]}{\epsilon(h)}\right).$$

Then u_h converges to the unique viscosity solution of the underlying PDE.

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Why use filtered schemes?

- Simplicity.
- Provably convergent.
- Achieve higher accuracy where the solution is smooth.
- In practice, it avoids the use of a wide stencil in second order equations (e.g. Monge-Ampère equation).

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Schemes for the Eikonal equation

- Monotone scheme

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- High-order upwind scheme

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- ENO schemes

$$|u_x^h|^{E,n} = \max\left\{-\frac{d}{dx}E^{n,\frac{1}{2}}[u](x), \frac{d}{dx}E^{n,-\frac{1}{2}}[u](x), 0\right\}$$

Schemes for Hamilton-Jacobi equations

- Monotone scheme (Lax-Friedrichs)

$$H^h_{LF}[u](x) = H^h_{LF}(x, D^+u(x), D^-u(x)) = H\left(x, \frac{D^+u(x) + D^-u(x)}{2}\right) - \frac{1}{2}\sigma_x(D^+u(x) - D^-u(x))$$

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$$H^h_{U,n}[u](x) = H^h_{LF}\left(x, \frac{d}{dx}P^{+,n}[u](x), \frac{d}{dx}P^{-,n}[u](x)\right)$$

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- ENO schemes

$$H^h_{E,n}[u](x) = H^h_{LF}\left(x, \frac{d}{dx}E^{n,\frac{1}{2}}[u](x), \frac{d}{dx}E^{n,-\frac{1}{2}}[u](x)\right)$$

Two-dimensional schemes

- Eikonal equation

$$\left|\nabla u^h\right| = N\left(\left|u_x^h\right|, \left|u_y^h\right|\right) \quad \text{where } N(x,y) = \sqrt{x^2+y^2}$$

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- Hamilton-Jacobi equations

$$\begin{aligned} H_{LF}^h[u](x) &= H_{LF}^h(x, p^+, p^-, q^+, q^-) \\ &= H\left(x, \frac{p^+ + p^-}{2}, \frac{q^+ + q^-}{2}\right) - \sigma_x \frac{p^+ - p^-}{2} - \sigma_y \frac{q^+ - q^-}{2} \end{aligned}$$

where $p^\pm = D_x^\pm u(x)$ and $q^\pm = D_y^\pm u(x)$.

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Explicit formulas for Eikonal equation

Solving $\left|u_x^h\right|^M = f$ for the reference variable, $u(x)$, leads to

$$\begin{aligned}\left|u_x^h\right|^M = f &\iff \max\left\{-\frac{u(x+h)-u(x)}{h}, \frac{u(x)-u(x-h)}{h}, 0\right\} = f(x)\\ &\iff \max\{u(x)-u(x+h), u(x)-u(x-h), 0\} = hf(x)\\ &\iff u(x) = \min\{u(x+h), u(x-h)\} + hf(x)\end{aligned}$$

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Similarly, for the second order upwind scheme we have

$$\begin{aligned}\left|u_x^h\right|^{U,2} = f &\iff \max\{3u(x) - 4u(x\pm h) + u(x\pm 2h), 0\} = 2hf(x)\\ &\iff u(x) = \frac{1}{3}\min\{4u(x\pm h) - u(x\pm 2h)\} + \frac{2}{3}hf(x)\end{aligned}$$

Two-dimensional case

Solving

$$\left|\nabla u^h\right| = f \iff \left|u_x^h\right|^2 + \left|u_y^h\right|^2 = f(x,y)^2$$

for the reference variable $u(x,y)$ requires solving nonlinear equation

$$\max\{z-a,0\}^2 + \max\{z-b,0\}^2 = c^2$$

for the unknown z where a, b and $c > 0$ are constants. The unique solution is given by

$$z = \begin{cases} \min\{a,b\} + c, & |a-b| \geq c, \\ \frac{a+b+\sqrt{2c^2-(a-b)^2}}{2}, & |a-b| < c. \end{cases}$$

Explicit formulas for the HJ equations

Recall that

$$D^+u(x) = \frac{u(x+h)-u(x)}{h}, \quad D^-u(x) = \frac{u(x)-u(x-h)}{h}.$$

Solving $H^h_{LF}[u] = f$ for the reference variable, $u(x)$, leads to

$$H\left(x, \frac{D^+u(x)+D^-u(x)}{2}\right) - \frac{1}{2}\sigma_x\left(D^+u(x)-D^-u(x)\right) = f(x)$$

$$\iff H\left(x, \frac{u(x+h)-u(x-h)}{2h}\right) - \frac{1}{2}\sigma_x\frac{u(x+h)-2u(x)+u(x-h)}{h} = f(x)$$

$$\iff u(x) = \frac{1}{\sigma_x}\left[f(x) - H\left(x, \frac{u(x+h)-u(x-h)}{2h}\right) + \sigma_x\frac{u(x+h)+u(x-h)}{2h}\right].$$

Explicit formulas for the HJ equations

Consider now the second order upwind scheme, $H^h_{U,2}[u] = f$, rewritten as

$$H\left(x, \frac{\frac{d}{dx}P^{+,2}[u](x) + \frac{d}{dx}P^{-,2}[u](x)}{2}\right) - \frac{1}{2}\sigma_x\left(\frac{d}{dx}P^{+,2}[u](x) - \frac{d}{dx}P^{-,2}[u](x)\right) = f(x).$$

where

$$\frac{d}{dx}P^{+,2}[u](x) = \frac{-3u(x) + 4u(x+h) - u(x+2h)}{2h},$$
$$\frac{d}{dx}P^{-,2}[u](x) = \frac{3u(x) - 4u(x-h) + u(x-2h)}{2h}.$$

Thus, solving for the reference variable, $u(x)$, leads to

$$u(x) = \frac{2}{3\sigma_x}\left[f(x) - H\left(x, \frac{-u(x+2h) + 4u(x+h) - 4u(x-h) + u(x-2h)}{4h}\right)\right.$$
$$\left. + \sigma_x \frac{-u(x+2h) + 4u(x+h) + 4u(x-h) - u(x-2h)}{4h}\right].$$

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One-dimensional examples

The 1st example is given by

$$|u_x| = 1 + \cos(x), \quad u(x) = 3 - |x + \sin(x)|$$

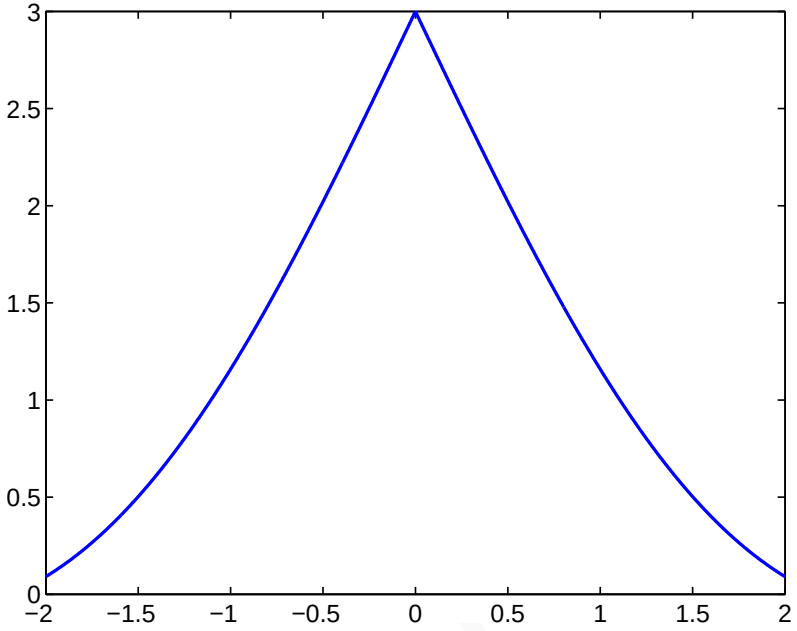


Figure: Profile of the solution.

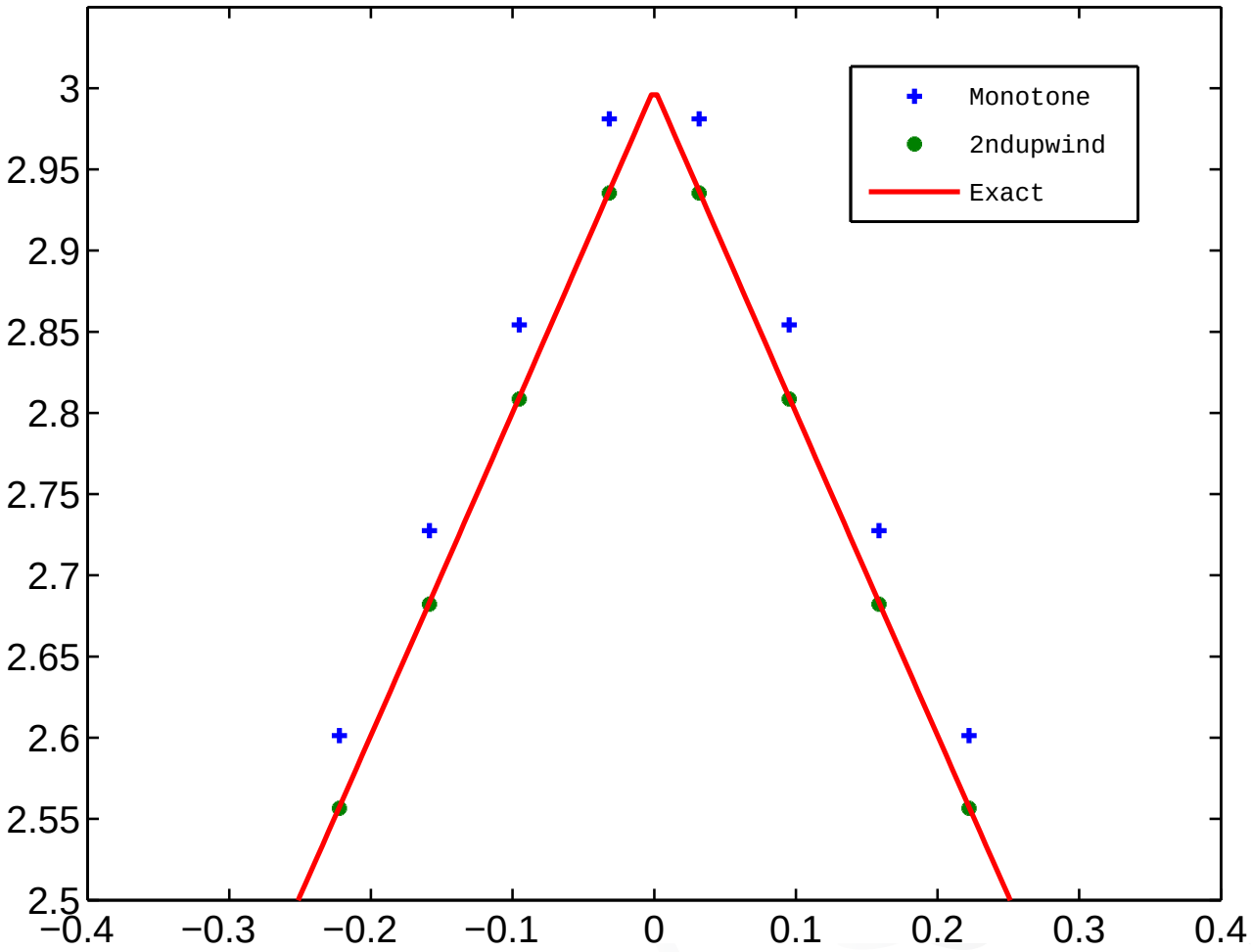


Figure: Exact solution and solutions obtained with the monotone scheme and the 2nd order upwind filtered scheme with 50 grid points.

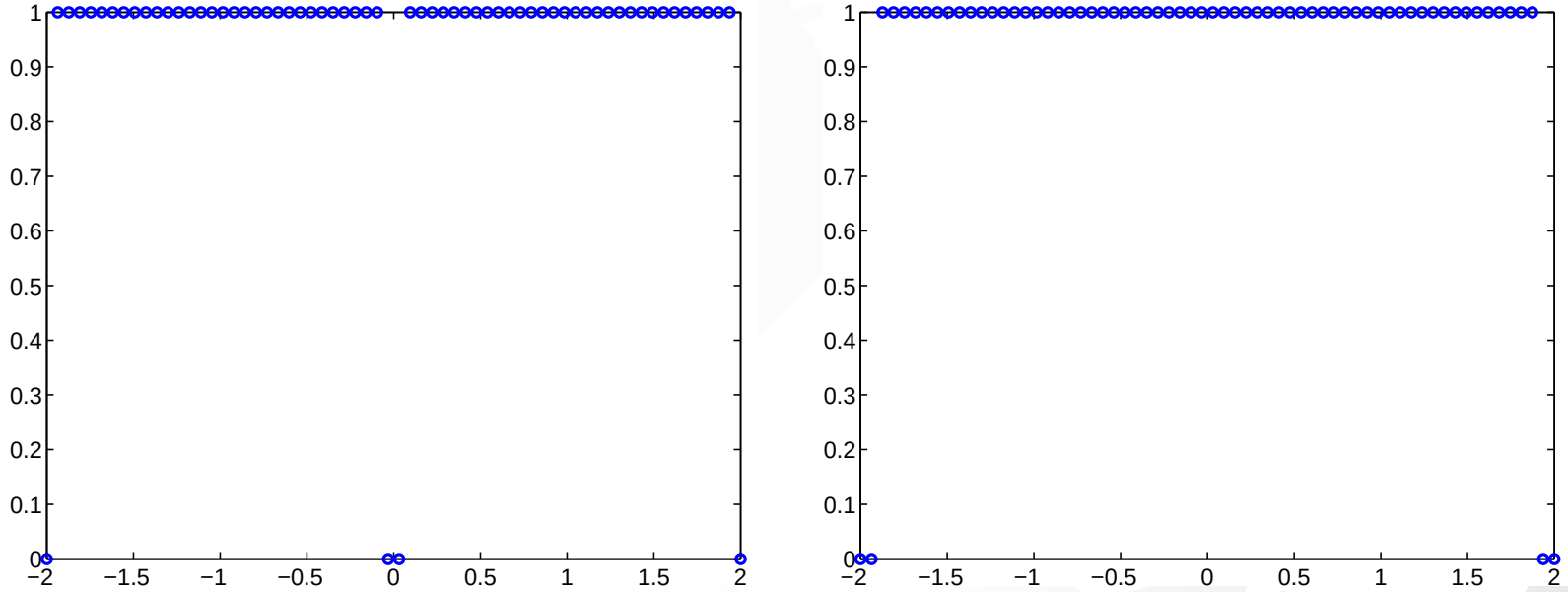


Figure: Active scheme for the solutions of the 1st example: filtered scheme with centred scheme (left) and filtered scheme with 2nd order upwind scheme (right).

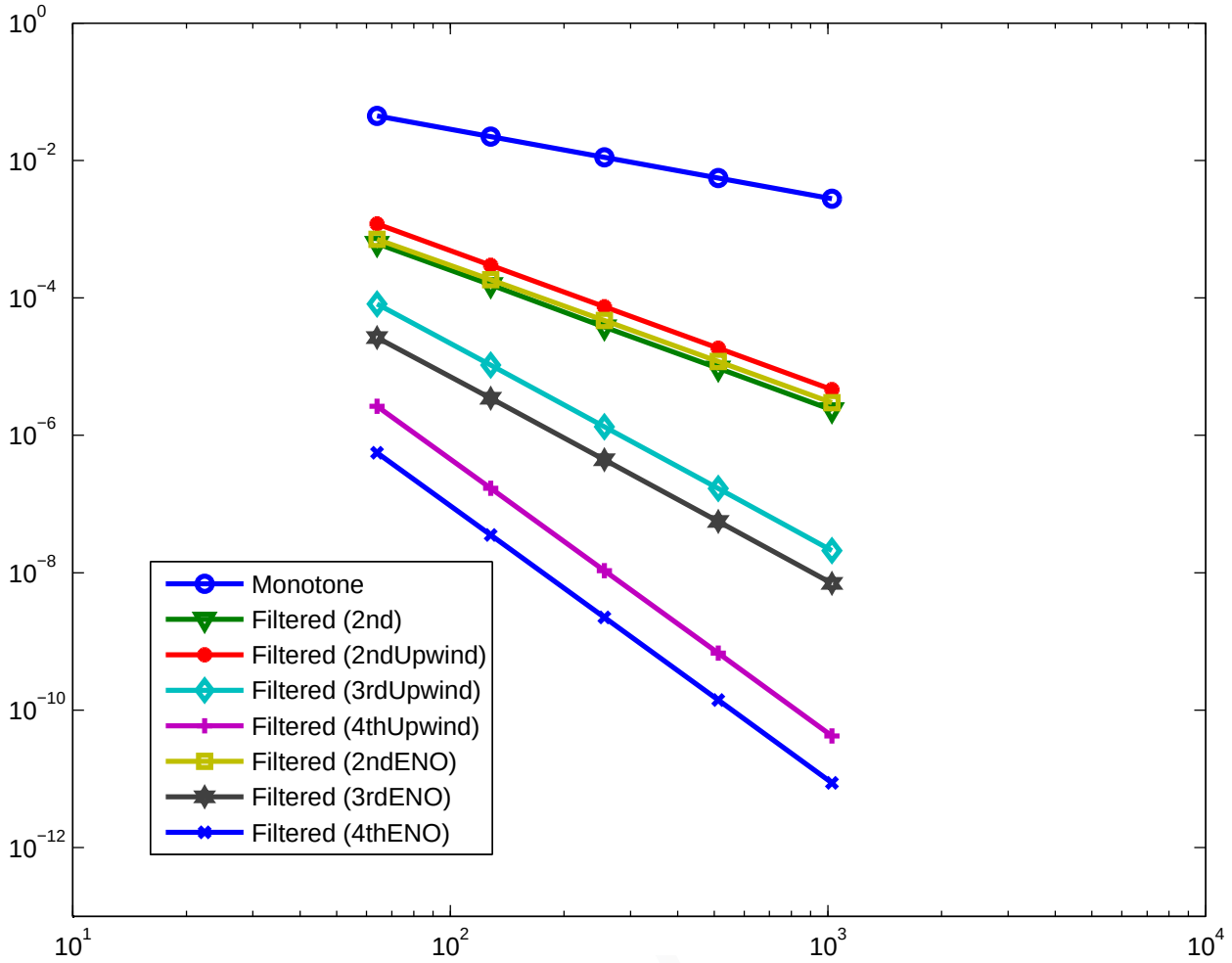
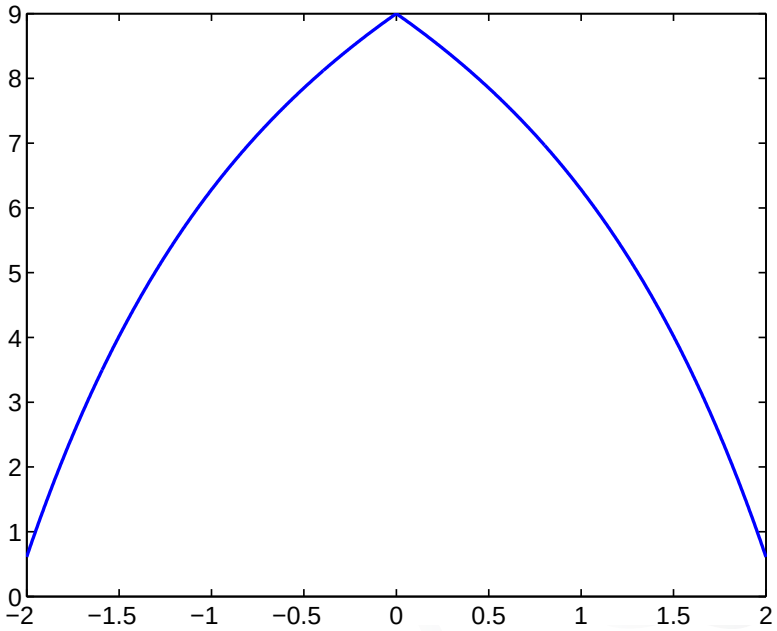


Figure: Loglog plot of the errors for the 1st Example.

The 2nd example is given by

$$|u_x| = 1 + e^{|x|}, \quad u(x) = 10 - |x| - e^{|x|}$$



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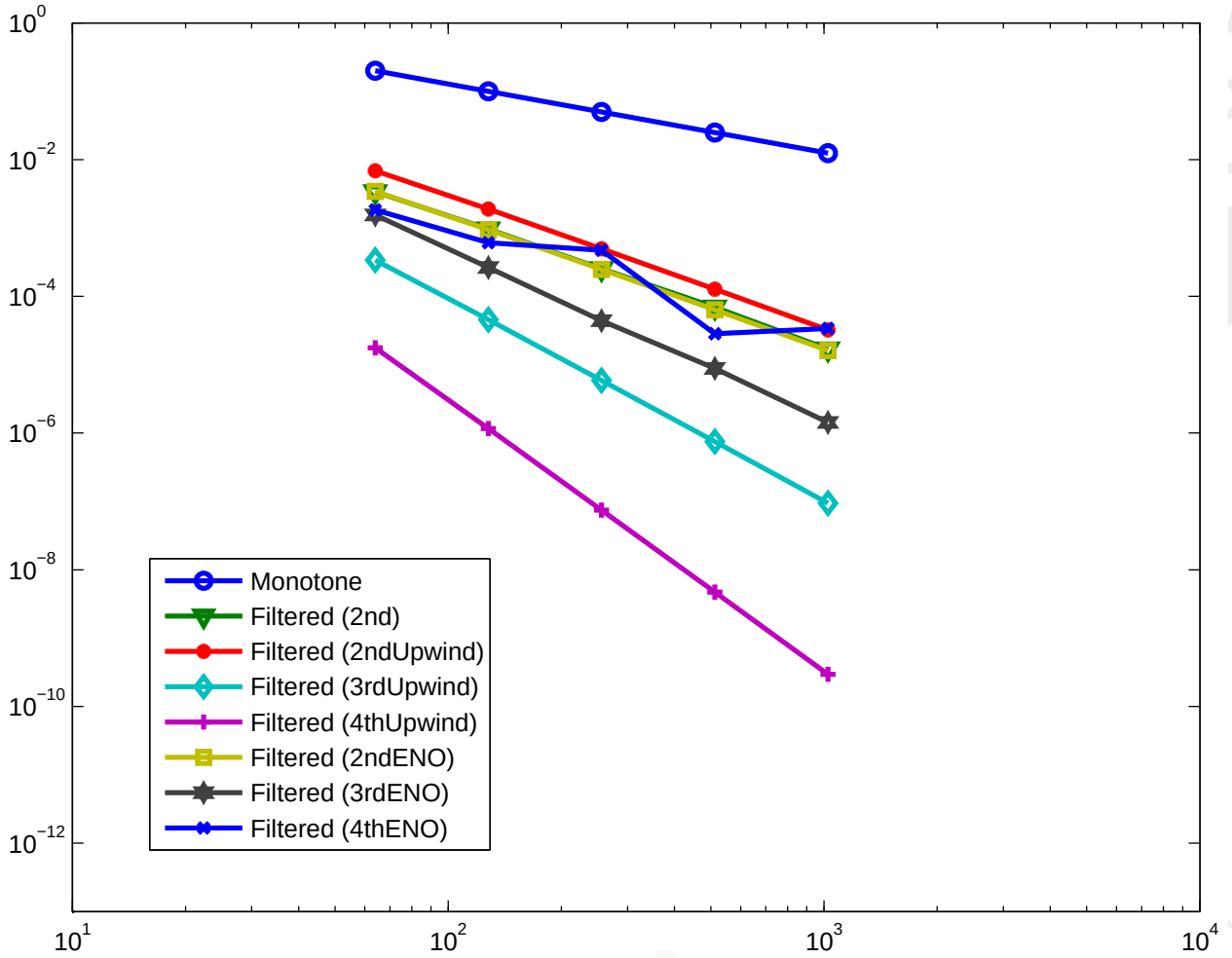
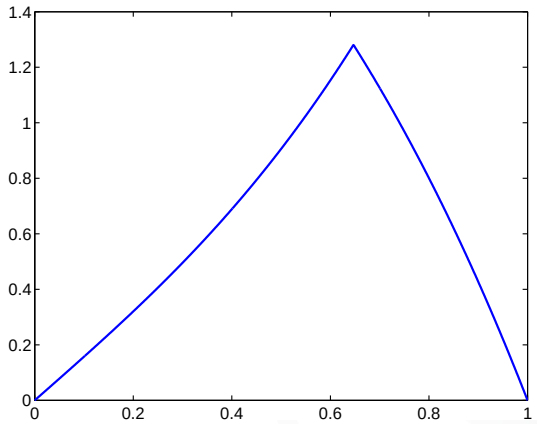


Figure: Loglog plot of the errors for the 2nd Example.

The 3rd example is given by

$$|u_x| = 3x^2 + a, \quad u(x) = \begin{cases} x^3 + ax, & x \in [0, x_0], \\ 1 + a - ax - x^3, & x \in [x_0, 1], \end{cases}$$

with $a = \frac{1-2x_0^3}{2x_0-1}$, $x_0 = \frac{\sqrt[3]{2}+2}{4\sqrt[3]{2}}$.



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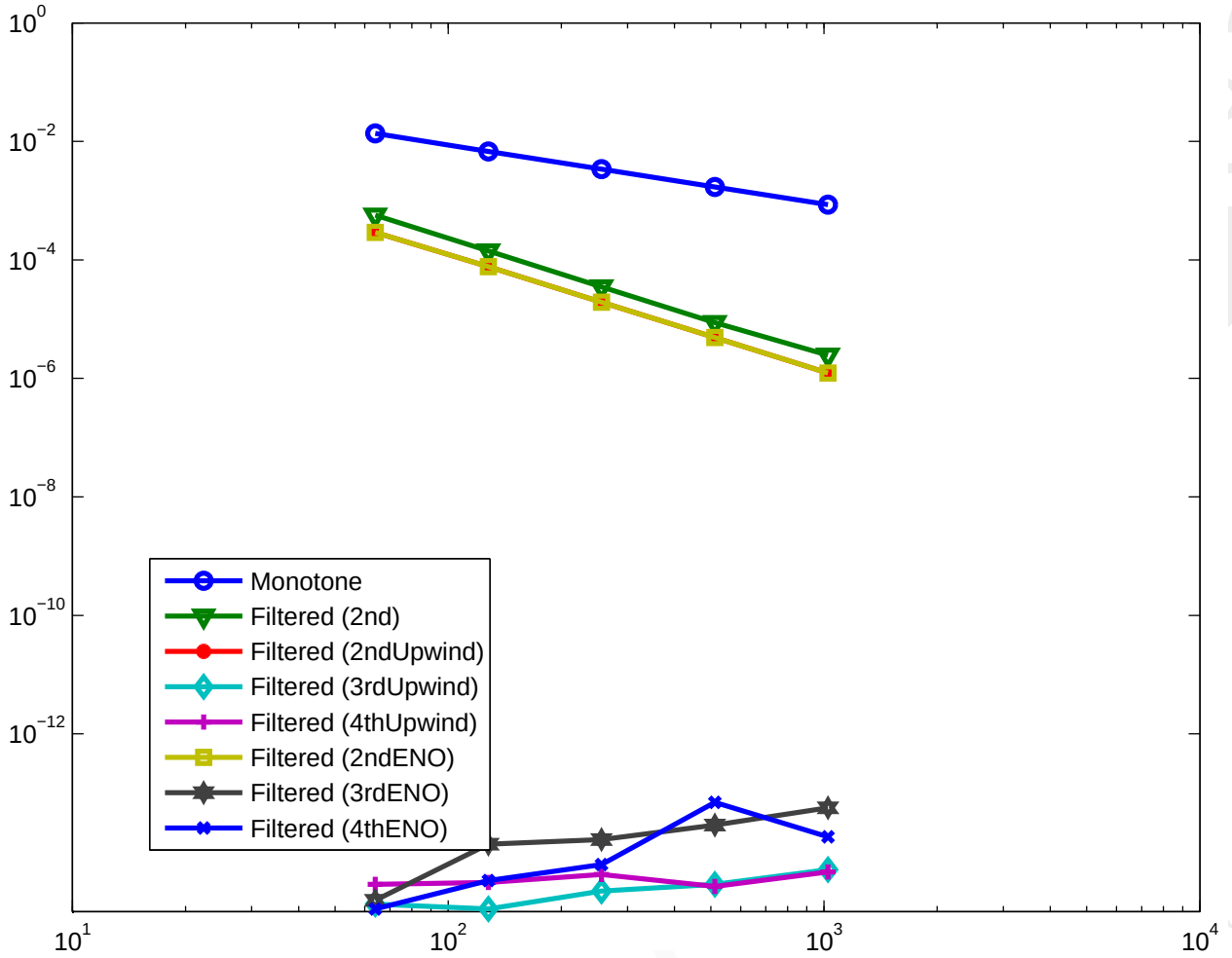
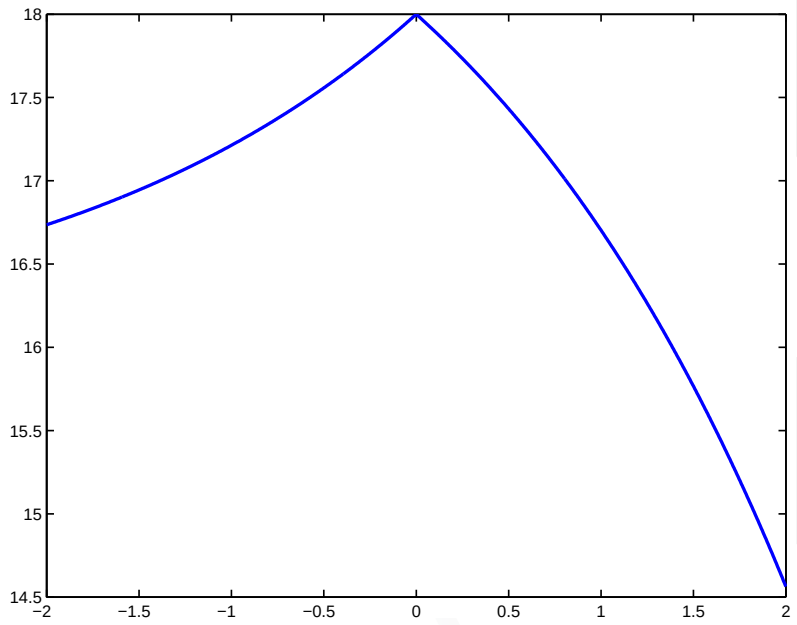


Figure: Loglog plot of the errors for the 3rd Example.

The 4th example is given by

$$|u_x|^2 = e^x, \quad u(x) = \begin{cases} 2e^{\frac{x}{2}} + 16, x & x \in [-2, 0], \\ -2e^{\frac{x}{2}} + 20, & x \in [0, 2]. \end{cases}$$



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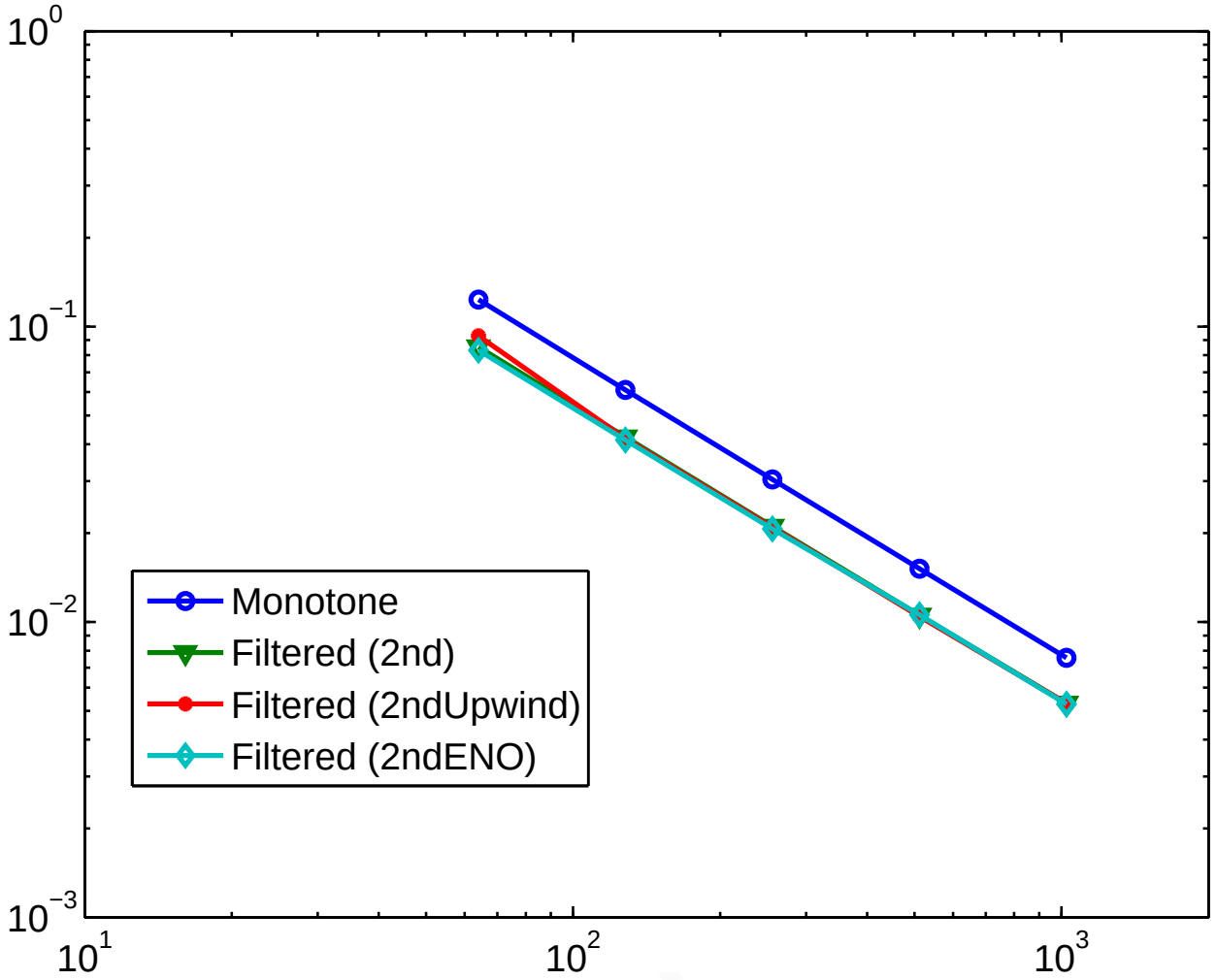
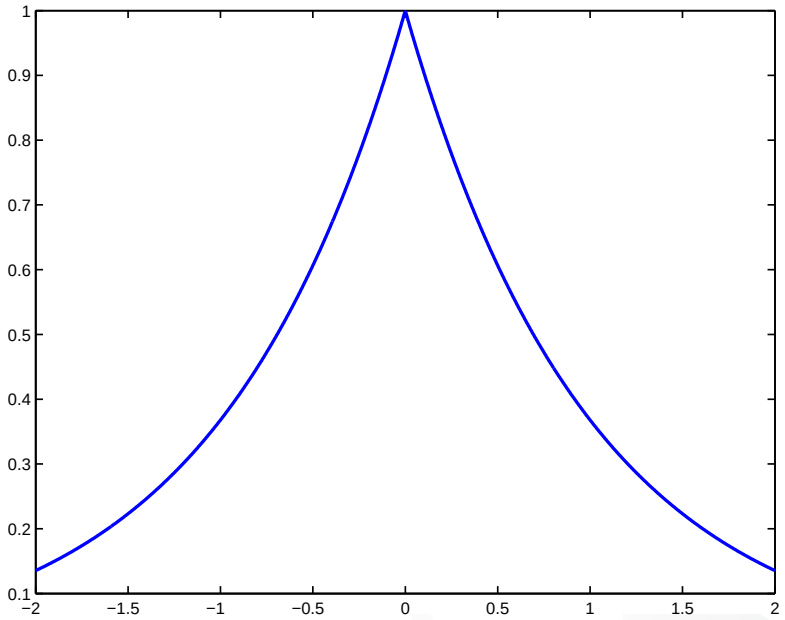


Figure: Loglog plot of the errors for the 4th Example.

For the last example we take

$$cos^2(u_x) + |u_x| = \cos^2\left(e^{-|x|}\right) + e^{-|x|}, \quad u(x) = e^{-|x|}$$



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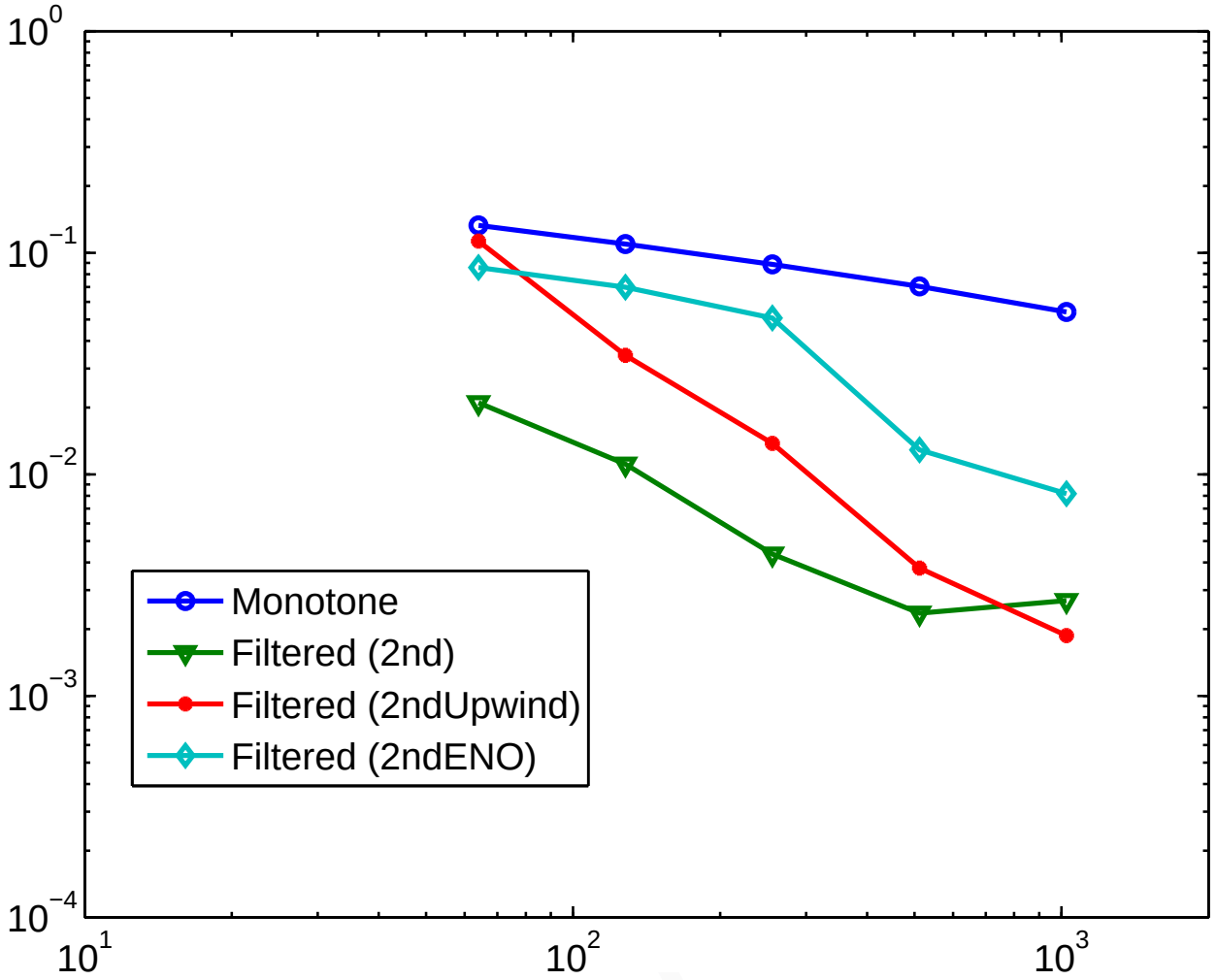

Figure: Loglog plot of the errors for the 5th Example.

Two-dimensional examples

We consider three distinct examples all of them solution to the Eikonal equation

$$\begin{cases} |\nabla u| = 1 & \text{for } x \text{ outside } \Gamma, \\ u = 0 & \text{for } x \text{ on } \Gamma \end{cases}$$

with the computational domain being $[-2,2]^2$ taking for Γ:

1. $\Gamma = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\right\}$,
2. $\Gamma = \left\{\left(\frac{1}{2}, \frac{1}{2}\right), \left(-\frac{1}{2}, -\frac{1}{2}\right)\right\}$,
3. $\Gamma = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, x \geq 0\right\}$.

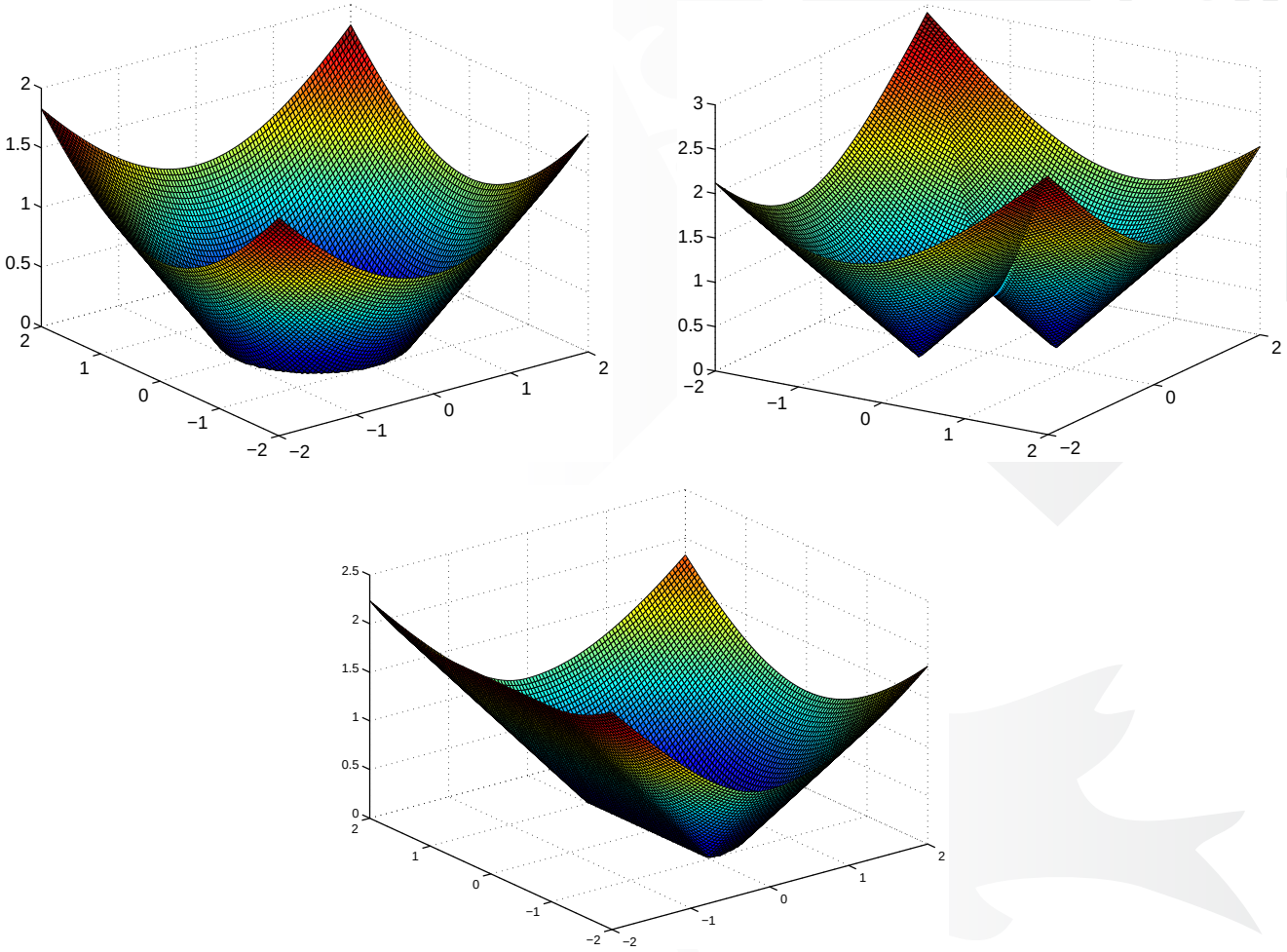

Figure: Profile of the solutions of the three examples considered in 2D.

Distance to a circle

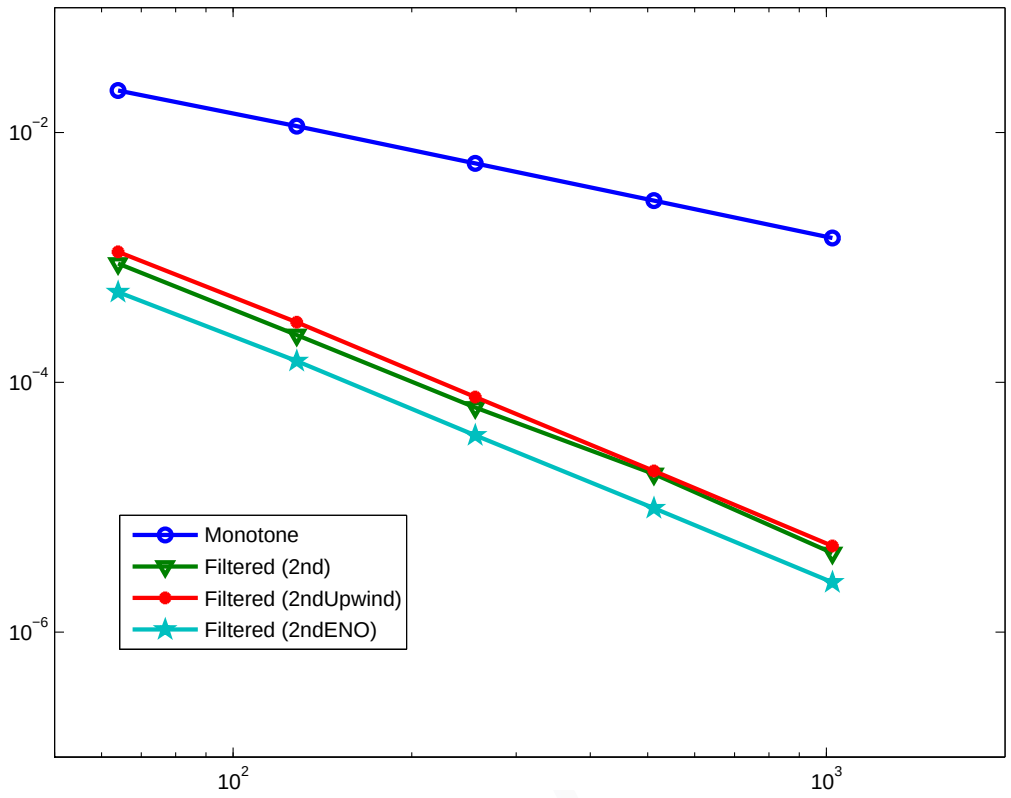

Figure: Log-log plot of the errors for the first example in the L^∞ norm.

Distance to two points

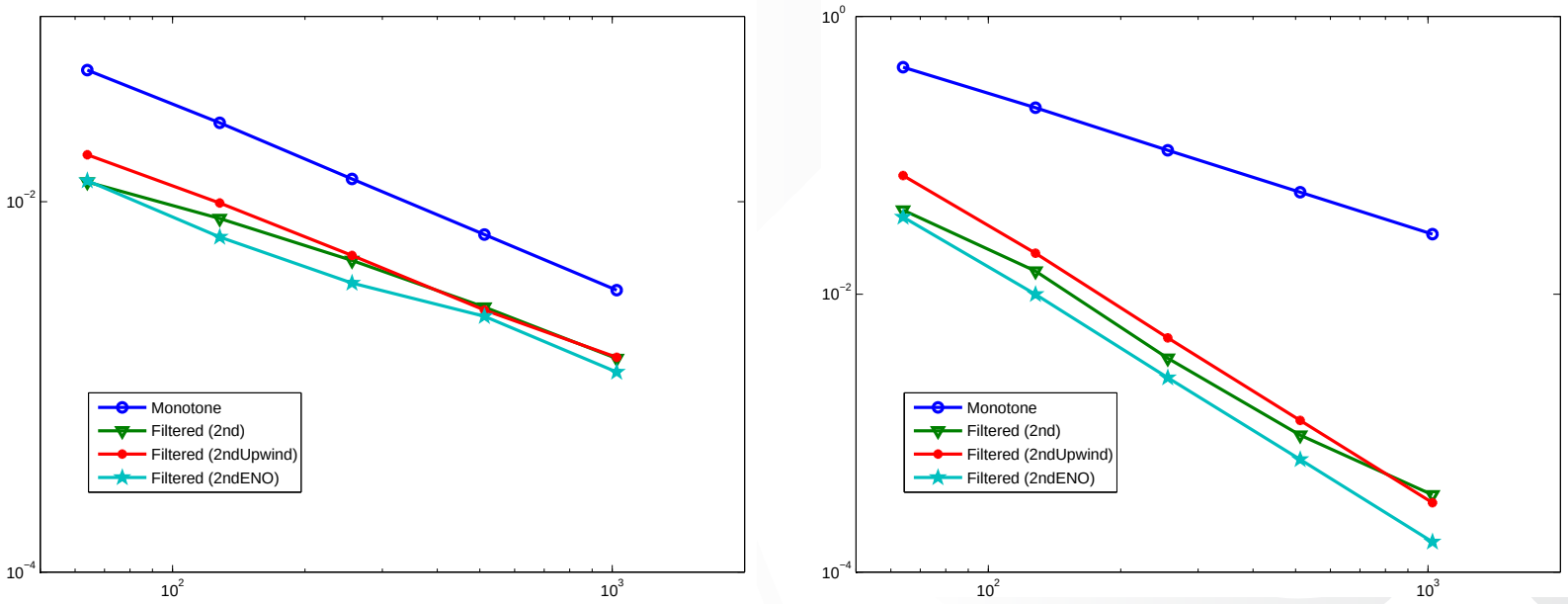

Figure: Log-log plot of the errors for the second example in the L^∞ norm (left) and L^1 norm (right).

Distance to a semi-circle

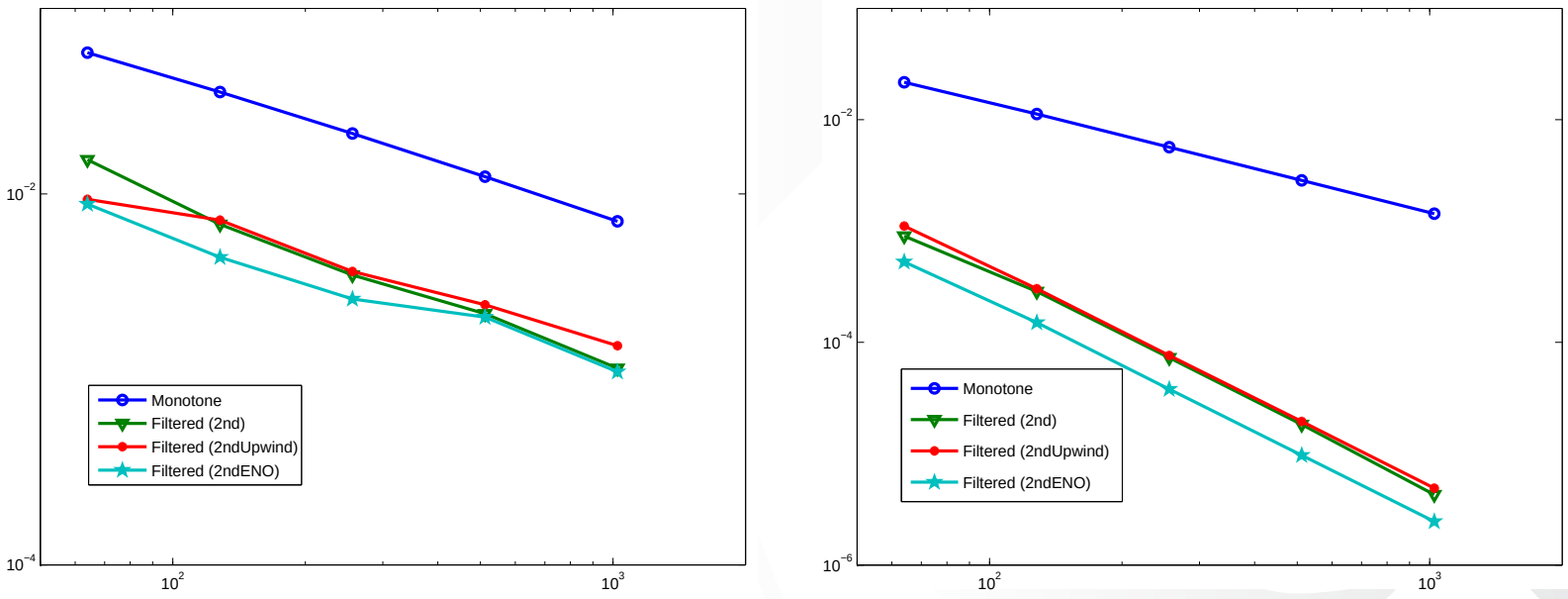

Figure: Log-log plot of the errors for the third example in the L^∞ norm in $[-2,2]$ (left) and L^∞ norm in $\left\{(x,y)\in\mathbb{R}^2 : x^2+y^2\geq 1, x>0.1\right\}$ (right).

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  - In general, solutions are piecewise smooth and we only recover second order convergence in the smooth regions.

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2. Filtered Schemes
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- in practice, it avoids the use of a wide stencil in second order equations (e.g. Monge-Ampère equation).

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