Solving Hamilton-Jacobi-Bellman equations by combining a max-plus linear approximation and a probabilistic numerical method

Marianne Akian

INRIA Saclay - Île-de-France and CMAP École polytechnique CNRS

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A finite horizon diffusion control problem involving "discrete" and "continuum" controls

The state $\xi_s \in \mathbb{R}^d$ satisfies the stochastic differential equation

$$d\xi_s = f^{\mu_s}(\xi_s, u_s)ds + \sigma^{\mu_s}(\xi_s, u_s)dW_s \ ,$$

where $(W_s)_{s\geq 0}$ is a d-dimensional Brownian motion,
$\mu := (\mu_s)_{0\leq s\leq T}$, and $u := (u_s)_{0\leq s\leq T}$ are admissible control processes,
$\mu_s \in \mathcal{M}$ a finite set and $u_s \in \mathcal{U} \subset \mathbb{R}^p$.

The problem consists in maximizing the finite horizon discounted payoff ($\delta^m \geq 0$):

$$J(t,x,\mu,u) := \mathbb{E}\left[\int_t^T e^{-\int_t^s \delta^{\mu_\tau}(\xi_\tau,u_\tau)d\tau}\ell^{\mu_s}(\xi_s,u_s)ds \right.$$
$$\left. + e^{-\int_t^T \delta^{\mu_\tau}(\xi_\tau,u_\tau)d\tau}\psi(\xi_T) \mid \xi_t = x\right] \ .$$

The Hamilton-Jacobi-Bellman (HJB) equation

Define the value function $v : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ as:

$$v(t, x) = \sup_{\mu, u} J(t, x, \mu, u) \ .$$

Under suitable assumptions, it is the unique (continuous) viscosity solution of the HJB equation

$$\begin{aligned} &-\tfrac{\partial v}{\partial t} - \mathcal{H}(x, v(t,x), Dv(t,x), D^2 v(t,x)) = 0, \quad x \in \mathbb{R}^d, \ t \in [0, T), \\ &v(T, x) = \psi(x), \quad x \in \mathbb{R}^d, \end{aligned}$$

satisfying also some growth condition at infinity (in space), where the Hamiltonian $\mathcal{H} : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ is given by:

$$\mathcal{H}(x, r, p, \Gamma) := \max_{m \in \mathcal{M}} \mathcal{H}^m(x, r, p, \Gamma) \ ,$$

with

$$\begin{aligned} \mathcal{H}^m(x, r, p, \Gamma) := \quad & \max_{u \in \mathcal{U}} \Big\{ \tfrac{1}{2} \operatorname{tr}\left(\sigma^m(x,u)\sigma^m(x,u)^\top \Gamma\right) \\ & + f^m(x,u) \cdot p - \delta^m(x,u) r + \ell^m(x,u) \Big\} \ . \end{aligned}$$

Standard grid based discretizations solving HJB equations suffer the curse of dimensionality malediction: for an error of ϵ, the computing time of finite difference or finite element methods is at least in the order of $(1/\epsilon)^{d/2}$.

Some possible curse of dimensionality-free methods:

- Idempotent methods introduced by McEneaney (2007) in the deterministic case, and by McEneaney, Kaise and Han (2011) in the stochastic case.
- Probabilistic numerical methods based on a backward stochastic differential equation interpretation of the HJB equation, simulations and regressions:
  - Quantization Bally, Pagès (2003) for stopping time problems.
  - Introduction of a new process without control: Bouchard, Touzi (2004) when σ does not depend on control; Cheridito, Soner, Touzi and Victoir (2007) and Fahim, Touzi and Warin (2011) in the fully-nonlinear case.
  - Control randomization: Kharroubi, Langrené, Pham (2013).
  - Fixed point iterations: Bender, Zhang (2008) for semilinear PDE (which are not HJB equations).

The idempotent method of McEneaney, Kaise and Han

Given m and u, denote by $\hat{\xi}^{m,u}$ the Euler discretization of the process ξ with time step h:

$$\hat{\xi}^{m,u}(t+h) = \hat{\xi}^{m,u}(t) + f^m(\hat{\xi}^{m,u}(t), u)h + \sigma^m(\hat{\xi}^{m,u}(t), u)(W_{t+h} - W_t) \ .$$

Define the dynamic programming operators:

$$T^m_{t,h}(\phi)(x) = \sup_{u \in \mathcal{U}} \left\{ h\ell^m(x,u) + e^{-h\delta^m(x,u)} \mathbb{E}\left[\phi(\hat{\xi}^{m,u}(t+h)) \mid \hat{\xi}^{m,u}(t) = x\right] \right\} \ ,$$

and

$$T_{t,h}(\phi)(x) = \max_{m \in \mathcal{M}} T^m_{t,h}(\phi)(x) \ .$$

The HJB equation can be discretized in time by:

$$v^h(t,x) = T_{t,h}(v^h(t+h,\cdot))(x), \quad t \in \mathcal{T}_h := \{0, h, 2h, \ldots, T-h\} \ .$$

Under appropriate assumptions, this scheme converges to the solution of HJB eq. when h goes to zero.

- In the deterministic case ($\sigma^m = 0$), $T^m_{t,h}$ and $T_{t,h}$ are max-plus linear:

$$v^h(t+h,x) = \max_{i=1,\dots,N}(\lambda_i + q_i^{t+h}(x))\ \forall x \Rightarrow$$
$$v^h(t,x) = \max_{i=1,\dots,N}(\lambda_i + q_i^{t}(x))\ \forall x \text{ with } q_i^t = T_{t,h}(q_i^{t+h})\ .$$

- We only need to compute the effect of the dynamic programming operator on the finite basis q_i^T, $i = 1, \dots, N$, for instance by computing their projection on a fixed basis (see Fleming and McEneaney (2000) and A.,Gaubert,Lakoua (2008)).
- However, the q_i^T are difficult to compute in general, or the size of the basis need to be exponential in d.
- If $T^m_{t,h}(q)$ is a quadratic form when q is a quadratic form, and if it easy to compute (for instance when the $\mathcal{H}^m$ correspond to linear quadratic problems), and if the q_i^T are quadratic forms, then the q_i^t are finite suppremum of quadratic forms easy to compute (see McEneaney (2006)). The number of quadratic forms for $v^h(0,\cdot)$ is exponential in the number of time step only.
- This idea was extended to the stochastic case by McEneaney, Kaise and Han (2011).

Theorem (McEneaney, Kaise and Han (2011))

Assume $\delta^m = 0$, σ^m *is constant,* f^m *is affine,* ℓ^m *is concave quadratic (with respect to* (x, u)*), and* ψ *is the supremum of a finite number of concave quadratic forms. Then, for all* $t \in \mathcal{T}_h$*, there exists a set* Z_t *and a map* $g_t : \mathbb{R}^d \times Z_t \to \mathbb{R}$ *such that for all* $z \in Z_t$*,* $g_t(\cdot, z)$ *is a concave quadratic form and*

$$v^h(t, x) = \sup_{z \in Z_t} g_t(x, z) \ .$$

Moreover, the sets Z_t *satisfy*

$$Z_t = \mathcal{M} \times \{\bar{z}_{t+h} : \mathcal{W} \to Z_{t+h} \mid \textit{Borel measurable}\} \ ,$$

where $\mathcal{W} = \mathbb{R}^d$ *is the space of values of the Brownian process.*

- Here a concave quadratic form is any map $\mathbb{R}^d \to \mathbb{R}$ of the form $x \mapsto q(x, z) := \frac{1}{2} x^\top Qx + b \cdot x + c$, with $z = (Q, b, c) \in \mathcal{Q}_d = \mathbb{S}_d^- \times \mathbb{R}^d \times \mathbb{R}$
- The proof uses the max-plus (infinite) distributivity property.

- In the deterministic case, the sets Z_t are finite, and their cardinality is exponential in time: $\#Z_t = M \times \#Z_{t+h} = \cdots = M^{N_t} \times \#Z_T$ with $M = \#\mathcal{M}$ and $N_t = (T-t)/h$.
- In the stochastic case, the sets Z_t are infinite as soon as $t < T$.
- If the Brownian process is discretized in space, then $\mathcal{W}$ can be replaced by the finite subset with fixed cardinality p, and the sets Z_t become finite.
- Nevertheless, their cardinality increases doubly exponentially in time: $\#Z_t = M \times (\#Z_{t+h})^p = \cdots = M^{\frac{p^{N_t}-1}{p-1}} \times (\#Z_T)^{p^{N_t}}$ where $p \geq 2$ ($p = 2$ for the Bernouilli discretization).
- Then, McEneaney, Kaise and Han proposed to apply a pruning method to reduce at each time step $t \in \mathcal{T}_h$ the cardinality of Z_t.
- In this talk, we shall replace pruning by random sampling.
- The idea is to use only quadratic forms that are optimal in the points of a sample of the process.

Consider the case with no continuous control u and no discount factor.

- Then $\hat{\xi}^m(t+h) = S^m_{t,h}(\hat{\xi}^m(t), W_{t+h} - W_t)$ with
$$S^m_{t,h}(x,w) = x + f^m(x)h + \sigma^m(x)w \ .$$
and
$$T^m_{t,h}(\phi)(x) = h\ell^m(x) + \mathbb{E}\big[\phi(S^m_{t,h}(x, W_{t+h} - W_t))\big] \ .$$

- Assume that $\phi(x) = \max_{z \in Z_{t+h}} q(x,z)$, $Z_{t+h} \subset \mathcal{Q}_d = \mathbb{S}^-_d \times \mathbb{R}^d \times \mathbb{R}$, and $q(x,z) := \frac{1}{2}x^\mathsf{T} Qx + b \cdot x + c$, $z = (Q,b,c) \in \mathcal{Q}_d$.
- Then, for each $x \in \mathbb{R}^d$, there exists $\bar{z}^m_x : \mathcal{W} \to Z_{t+h}$ measurable s.t.
$$\phi(S^m_{t,h}(x, W_{t+h} - W_t)) = q\big(S^m_{t,h}(x, W_{t+h} - W_t), \bar{z}^m_x(W_{t+h} - W_t)\big) \ .$$

- Moreover, under the previous assumptions on ℓ^m, f^m and σ^m, we have, for all $x' \in \mathbb{R}^d$,
$$\mathbb{E}\big[h\ell^m(x') + q\big(S^m_{t,h}(x', W_{t+h} - W_t), \bar{z}^m_x(W_{t+h} - W_t)\big)\big] = q(x', z^m_x)$$
for some $z^m_x \in \mathcal{Q}_d$, and so
$$T^m_{t,h}(\phi)(x) = q(x, z^m_x) = \sup_{x' \in \mathbb{R}^d} q(x, z^m_{x'}) \ .$$

The sampling algorithm

- Let $M = \#\mathcal{M}$ and choose $N = (N_{\rm in}, N_{\rm rg})$ giving size of samples.
- Choose $Z_T \subset \mathcal{Q}_d$ such that $|\psi(x) - \max_{z \in Z_T} q(x,z)| \leq \epsilon$. Define $v^{h,N}(T,x) = \max_{z \in Z_T} q(x,z)$, for $x \in \mathbb{R}^d$.
- Construct a sample of $((\hat{\xi}^m(0))_{m \in \mathcal{M}}, (W_{t+h} - W_t)_{t \in \mathcal{T}_h})$ of size $N_{\rm in}$ indexed by $\omega \in \Omega_{N_{\rm in}} := \{1, \ldots, N_{\rm in}\}$, and deduce $\hat{\xi}^m(t,\omega)$, $m \in \mathcal{M}$.
- For $t = T - h, T - 2h, \ldots, 0$ do:
  1. For each $\omega \in \Omega_{N_{\rm in}}$ and $m \in \mathcal{M}$, denote $x = \hat{\xi}^m(t,\omega)$, and construct a subsample of size $N_{\rm rg}$ of elements (ω_i, ω_i') of $(\Omega_{N_{\rm in}})^2$, $i \in \Omega_{N_{\rm rg}}$. Let $\bar{z}_x^m : \mathcal{W} \to Z_{t+h}$ (as above), be computed at the points $(W_{t+h} - W_t)(\omega_i')$ only.
     Consider
     $$\tilde{q}(x', w) = h\ell^m(x') + q\big(S_{t,h}^m(x', w), \bar{z}_x^m(w)\big) \ .$$
     Approximate z_x^m such that $q(x', z_x^m) = \mathbb{E}[\tilde{q}(x', W_{t+h} - W_t)]$ by doing a regression of $\tilde{q}(\hat{\xi}^m(t), W_{t+h} - W_t)$ using a (usual) basis of quadratic forms of $\hat{\xi}^m(t)$, and the sample $(\hat{\xi}^m(t,\omega_i), (W_{t+h} - W_t)(\omega_i'))$, $i \in \Omega_{N_{\rm rg}}$.
  2. Let Z_t be the set of the parameters $z_x^m \in \mathcal{Q}_d$ of all the quadratic forms obtained in Step 2. Define $v^{h,N}(t,x) = \max_{z \in Z_t} q(x,z)$.

The sampling algorithm

- Several subsamplings of elements (ω_i, ω'_i), $i \in \Omega_{N_{\rm rg}}$, of $(\Omega_{N_{\rm in}})^2$ have been tested:
  1. the initial sampling: $N_{\rm rg} = N_{\rm in}$ and $\omega_i = \omega'_i = i$.
  2. $N_{\rm rg} = N_x \times N_w$, choose, once for all $\omega \in \Omega_{N_{\rm in}}$ and $m \in \mathcal{M}$, a random sampling of size N_x of the elements of $\Omega_{N_{\rm in}}$ and an independent random sampling of size N_w and make the product: so ω_i and ω'_i are independent.
  3. Do as in Method 2, but choose different samplings for each $\omega \in \Omega_{N_{\rm in}}$ and $m \in \mathcal{M}$, independently.
  4. Do as in Method 2, but replace the second random sampling of $\Omega_{N_{\rm in}}$ by $\Omega_{N_{\rm in}}$ itself, so $N_w = N_{\rm in}$.
  5. Do as in Method 2, but replace both random samplings of $\Omega_{N_{\rm in}}$ by $\Omega_{N_{\rm in}}$ itself, so $N_{\rm rg} = N_{\rm in}^2$.
- For each time step $t \in \mathcal{T}_h$, we have $\#Z_t \leq M \times N_{\rm in}$, and the number of computations is in $O((M \times N_{\rm in})^2 \times N_{\rm rg})$ and in $O((M \times N_{\rm in})^2 \times N_w)$ for subsamplings 2,3,4.

We consider a simple problem of pricing and hedging an option with uncertain volatility and two underlying processes, tested in Kharroubi, Langrené, Pham (2013).

- $d = 2$, $\mathcal{M} = \{\rho_{\min}, \rho_{\max}\}$ with $-1 \leq \rho_{min} \leq \rho_{max} \leq 1$.
- The dynamics of the processes are given, for all $m \in \mathcal{M}$, by $f^m = 0$ and

$$\sigma^m(\xi) = \begin{bmatrix} \sigma_1 \xi_1 & 0 \\ \sigma_2 m \xi_2 & \sigma_2 \sqrt{1 - m^2} \xi_2 \end{bmatrix}, \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \ ,$$

  with $\sigma_1, \sigma_2 > 0$, that is

$$d\xi_i = \sigma_i \xi_i dB_i, \quad \langle dB_1, dB_2 \rangle = \mu_s ds \ .$$

- The final reward is given by

$$\psi(\xi) = (\xi_1 - \xi_2 - K_1)^+ - (\xi_1 - \xi_2 - K_2)^+, \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \ ,$$

  with $x^+ = \max(x, 0)$, $K_1 < K_2$.

- We restrict the state space to the set of $\xi \in \mathbb{R}_+^2$ s.t. $\xi_1 - \xi_2 \in [-100, 100]$.
- We take $\sigma_1 = 0.4,\ \sigma_2 = 0.3,\ K_1 = -5,\ K_2 = 5,\ T = 0.25$, and fix the time discretization step to $h = 0.01$.
- $\mathcal{M} = \{\rho\}$ with $\rho = -0.8$ or $\rho = 0.8$, or $\mathcal{M} = \{-0.8, 0.8\}$.
- When $\mathcal{M} = \{\rho\}$, the value function is known, so we can compute the error.
- We tested in that case each subsampling method: Method 1 (initial sampling) gives very bad results even at time $T - h$. Method 5 need too much space and time even for $N_{\text{in}} = 1000$. Method 2 seems to be the best one.

ρ	N_{in}	N_{rg}	N_x	N_w	N_{m}	e_∞	e_1
-0.8	1000	10000	10	1000	2	0.521	0.173
0.8	1000	10000	10	1000	2	0.157	0.074
-0.8	1000	1000	10	100	2	0.75	0.41
0.8	1000	1000	10	100	2	0.36	0.11
-0.8	1000	1000	10	100	3	3.48	1.92
0.8	1000	1000	10	100	3	3.05	0.81
-0.8	100	1000	10	100	2	1.95	0.46
0.8	100	1000	10	100	2	1.81	0.33
-0.8	100	10000	10	1000	2	2.09	0.53
0.8	100	10000	10	1000	2	1.79	0.36
-0.8	100	1000	10	100	4	2.15	0.55
0.8	100	1000	10	100	4	1.80	0.39

Table : Sup-norm and normalized ℓ^1 norm of the error on the value function at time $t = 0$, and states $\xi_2 = 50$ and $\xi_1 \in [20, 80]$, denoted e_∞ and e_1 resp., when $\mathcal{M} = \rho$, and the subsampling number N_{m} is used.

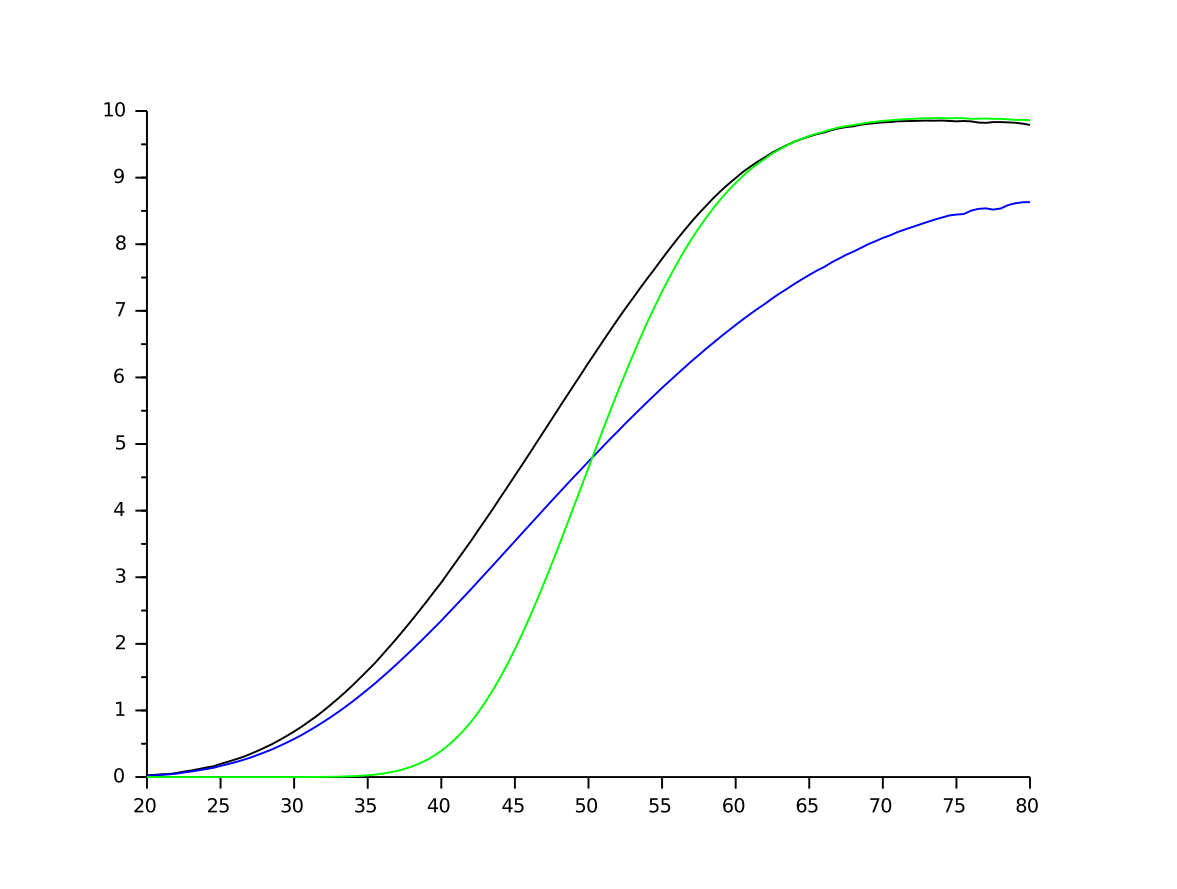

Figure : Value function obtained at $t = 0$, and $\xi_2 = 50$ as a function of $\xi_1 \in [20, 80]$. Here $N_{\text{in}} = 1000$, $N_{\text{rg}} = N_x \times N_w$, $N_x = 10$, $N_w = 1000$ and the subsampling method 2 is used. In blue, $\mathcal{M} = \{-0.8\}$, in green $\mathcal{M} = \{0.8\}$, and in black $\mathcal{M} = \{-0.8, 0.8\}$.

In the general case:

- Discretize the continuous control u and apply the previous method would be too expensive since one need to simulate a process for each m and u.
- So we need to consider simulations associated to a process independent of the control u at least.
- The probabilistic method of Fahim, Touzi and Warin (2011) uses the simulation of one process only.
- We shall use the simulation of one process of each discrete control m.

The algorithm of Fahim, Touzi and Warin (2011)

For a fixed $m \in \mathcal{M}$, consider the equation:

$$-\frac{\partial v}{\partial t} - \mathcal{H}^m(x, v(t,x), Dv(t,x), D^2 v(t,x)) = 0, \quad x \in \mathbb{R}^d,\ t \in [0, T) \ .$$

Decompose the Hamiltonian $\mathcal{H}^m$ as $\mathcal{H}^m = \mathcal{L}^m + \mathcal{G}^m$ with

$$\mathcal{L}^m(x, r, p, \Gamma) := \frac{1}{2}\,\mathrm{tr}\,(a^m(x)\Gamma) + \underline{f}^m(x) \cdot p \ ,$$

$a^m(x) = \underline{\sigma}^m(x)\underline{\sigma}^m(x)^\mathsf{T}$ and $\mathcal{G}^m$ such that $\partial_\Gamma \mathcal{G}^m$ is positive semidefinite (that is $\mathcal{G}^m$ is elliptic).

The idea of the algorithm: if X_t^m is the diffusion with generator $\mathcal{L}^m$, and v^m the viscosity solution of the above equation, then $Y_t = v^m(X_t^m)$ is a backward process with drift $-\mathcal{G}^m(X_t^m, Y_t, Z_t, \Gamma_t)$.

Denote by $\hat{X}^m$ the Euler discretization of X_t^m:

$$\hat{X}^m(t+h) = \hat{X}^m(t) + \underline{f}^m(\hat{X}^m(t))h + \underline{\sigma}^m(\hat{X}^m(t))(W_{t+h} - W_t) \ .$$

Then, v^m is approximated (Fahim, Touzi and Warin) by $v^{m,h}$ satisfying:

$$v^{m,h}(t,x) = T_{t,h}^m(v^{m,h}(t+h,\cdot))(x), \quad t \in \mathcal{T}_h \ ,$$

and

$$T_{t,h}^m(\phi)(x) = \mathcal{D}_{m,t,h}^0(\phi)(x) + h\mathcal{G}^m(x, \mathcal{D}_{m,t,h}^0(\phi)(x), \mathcal{D}_{m,t,h}^1(\phi)(x), \mathcal{D}_{m,t,h}^2(\phi)(x))$$

with $\mathcal{D}_{m,t,h}^i(\phi)$ being the approximation of the ith derivative of ϕ given by

$$\mathcal{D}_{m,t,h}^i(\phi)(x) = \mathbb{E}(\phi(\hat{X}^m(t+h))\mathcal{P}_{m,t,x,h}^i(W_{t+h} - W_t) \mid \hat{X}^m(t) = x) \ ,$$

where, for all m, t, x, h, i, $\mathcal{P}_{m,t,x,h}^i$ is a polynomial of degree i with values in an appropriate finite dimensional space. For instance, when $\underline{\sigma}^m$ is the identity:

$$\mathcal{P}_{m,t,x,h}^0 = 1, \quad \mathcal{P}_{m,t,x,h}^1(w) = \frac{w}{h}, \quad \mathcal{P}_{m,t,x,h}^2(w) = \frac{ww^T - hI}{h^2} \ .$$

- The time approximation of (Fahim, Touzi and Warin) works when $\mathrm{tr}(a^m(x)^{-1}\partial_\Gamma \mathcal{G}^m) \leq 1$ and $\mathcal{G}^m$ is Lipschitz continuous. Indeed, this implies that $T^m_{t,h}$ is a monotone operator over the set of Lipschitz continuous functions $\mathbb{R}^d \to \mathbb{R}$:

$$\phi \leq \psi \implies T^m_{t,h}(\phi) \leq T^m_{t,h}(\psi) \ .$$

- $T^m_{t,h}$ is further approximated by using a regression scheme.
- Under some technical assumptions, the above algorithm converges.
- The above assumptions do not allow in general to handle the case of the Hamiltonian $\mathcal{H}$ directly, since it may be nonsmooth and with noncomparable diffusion coefficients.
- Note also that theoretically, the sample size to obtain the convergence of the estimator is at least in the order of $1/h^{d/2}$. Also the dimension of the linear regression space should be in this order. The curse of dimensionality persists, but in practice a much smaller sample size is sufficient.

Combining idempotent and probabilistic algorithms

- When the above assumptions are satisfied for the $\mathcal{H}^m$, one consider the following mixed scheme:

$$v^h(t,x) = T_{t,h}(v^h(t+h,\cdot))(x), \quad t \in \mathcal{T}_h \ ,$$

with

$$T_{t,h}(\phi)(x) = \max_{m \in \mathcal{M}} T^m_{t,h}(\phi)(x) \ ,$$

and $T^m_{t,h}$ as above, for each $m \in \mathcal{M}$.

- The mixed scheme consists also in the approximation of the above $v^h(t,\cdot)$ as a finite supremum of quadratic forms, using a sampling algorithm similar to the previous one, where we apply a regression over the set of quadratic forms to compute $\mathcal{D}^i_{m,t,h}(\phi)$, when ϕ is the supremum of quadratic forms.
- To do this, we need a result comparable to the one of McEneaney, Kaise and Han (2011) with the new operator $T_{t,h}$.
- We know that $T_{t,h}$ is monotone.

The following result generalizes the max-plus distributivity. It says that any monotone continuous map distributes over max.

Theorem

Let $\mathcal{W} = \mathbb{R}^d$ and $\mathcal{D}$ be the set of measurable functions $\mathcal{W} \to \mathbb{R}$ with at most exponential growth rate, and and let G be a monotone additively α-subhomogeneous operator from $\mathcal{D} \to \mathbb{R}$, for some constant $\alpha > 0$. Let $(Z, \mathfrak{A})$ be a measurable space, and let $\mathcal{W}$ be endowed with its Borel σ-algebra. Let $\phi : \mathcal{W} \times Z \to \mathbb{R}$ be a measurable map such that for all $z \in Z$, $\phi(\cdot, z)$ is continuous and belongs to $\mathcal{D}$. Let $v \in \mathcal{D}$ be such that $v(W) = \sup_{z \in Z} \phi(W, z)$. Assume that v is continuous. Then,

$$G(v) = \sup_{\bar{z} \in \overline{Z}} G(\bar{\phi}^{\bar{z}})$$

where $\bar{\phi}^{\bar{z}} : \mathcal{W} \to \mathbb{R},\ W \mapsto \phi(W, \bar{z}(W))$, and

$$\overline{Z} = \{\bar{z} : \mathcal{W} \to Z, \text{ measurable and such that } \bar{\phi}^{\bar{z}} \in \mathcal{D}\}.$$

Then, if the operators $T^m_{t,h}$ preserve random quadratic forms, the conclusion of the theorem of McEneaney, Kaise and Han (2011) holds for the mixed scheme.

And a sampling algorithm can be applied to compute the sets Z_t such that $v^h(t,x) = \max_{z \in Z_t} q(x,z)$.

Moreover, the previous result and method can also be applied to HJB-Isaacs equations of zero sum games, as soon as the above property holds.

- However, the assumptions necessary for the scheme of (Fahim, Touzi and Warin) to work may be difficult to obtain for the Hamiltonians $\mathcal{H}^m$ too: linear quadratic problems have unbounded coefficients, and in financial applications, the diffusions coefficients are highly unnoncomparable.
- Recently, we managed to get rid of these assumptions by using approximations of the first and second derivative of the function v_h by using different polynomials in $(W_{t+h} - W_t)^+$ and $(W_{t+h} - W_t)^-$ and $W_{t+h} - W_t$.
- We hope to use this new scheme to obtain more appealing nunmerical results in the future.

Conclusion

- We proposed an algorithm to solve HJB equations, combining ideas included in the idempotent algorithm of McEneaney, Kaise and Han (2011) and in the probabilistic numerical scheme of Fahim, Touzi and Warin (2011).
- The advantages with respect to the pure probabilistic scheme are that the regression estimation is over a linear space of small dimension, and that one may bypass in some cases the restrictive condition of application of the method.
- The advantages with respect to the pure idempotent scheme is that one may avoid the pruning step: the number of quadratic forms generated by the algorithm is linear with respect to the sampling size times the number of discrete controls.
- The theoretical results suggest that it can also be applied to Isaacs equations of zero-sum games.
- We find only recently an improvement of the probabilistic scheme of Fahim, Touzi and Warin (2011), which will allows us to do numerical tests in high dimension and with fully nonlinear Hamiltonians.