Adaptive wavelet methods for space-time variational formulations of evolutionary PDEs

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Motivation
Consider heat eq., implicit time discretization, AFEM for solving the seq. of elliptic problems. Issues:

- How to distribute ‘grid points’ optimally over space and time?
- Often the class of possible space-time grids doesn’t include ones that are suited for singularities that are local in space and time.
- Inherently sequential.
- When whole time evolution is needed, as with problems of optimal control, huge memory requirements.

Aim at adaptive solver for the problem as a whole, which is quasi-optimal within a class of trial spaces that contains one that gives best possible rate allowed by the order.

Furthermore, using that the space-time cylinder is a product domain, we apply (adaptive) tensor product approximation (cf. sparse grids). So we solve time-evolution at a complexity of an optimal solver for the stationary problem.
Topics

- Optimal adaptive wavelet method for solving well-posed (non-) linear operator eqs.

- New approximate residual evaluation scheme.

- To avoid $C^1$ wavelets: FOSLS formulations.

- Applications: elliptic; stationary NSE; parabolic; instat. NSE.

- Some numerical results.
Well-posed op. eqs.

Let $\mathcal{X}$, $\mathcal{Y}$ be sep. Hilbert spaces (over $\mathbb{R}$). Let $B \in \mathcal{L}(\mathcal{X}, \mathcal{Y}')$. Given $f \in \mathcal{Y}'$, we seek $u \in \mathcal{X}$ s.t.

$$Bu = f$$

Ex.:

• $(Bw)(v) := \int_\Omega \nabla w \cdot \nabla v \, dx$, $\mathcal{X} = \mathcal{Y} := H^1_0(\Omega)$ (Poisson problem),

• $(B(\bar{w}, p))(\bar{v}, q) := \int_\Omega \nabla \bar{w} : \nabla \bar{v} - \int_\Omega p \text{div} \bar{v} - \int_\Omega q \text{div} \bar{w} \, dx$, $\mathcal{X} = \mathcal{Y} := H^1_0(\Omega)^n \times L^2(\Omega)/\mathbb{R}$ (stat. Stokes problem),

• $(Bw)(v) := \frac{1}{4\pi} \int_{\partial\Omega} \int_{\partial\Omega} \frac{(w(y) - w(x))(v(y) - v(x))}{|x-y|^3} \, dx \, dy$, $\mathcal{X} = \mathcal{Y} := H^{1/2}(\partial\Omega)/\mathbb{R}$ (hypersingular boundary integral equation).

• $(Bw)(v) := \int_0^T \int_\Omega -w \frac{\partial v}{\partial t} + \nabla w \cdot \nabla v \, dx \, dt$, (heat eq.) $\mathcal{X} := L^2(I; H^1_0(\Omega))$, $\mathcal{Y} := L^2(I; H^1_0(\Omega)) \cap H^1_0,\{T\}(I, H^{-1}(\Omega))$. 
Reformulation as well-posed bi-infinite MV eq

Let \( \Psi^\mathcal{X} = \{ \psi^\mathcal{X}_\lambda : \lambda \in \vee \mathcal{X} \} \), \( \Psi^\mathcal{Y} = \{ \psi^\mathcal{Y}_\lambda : \lambda \in \vee \mathcal{Y} \} \) Riesz bases for \( \mathcal{X}, \mathcal{Y} \) (we have wavelet bases in mind). That is, the synthesis operator,

\[
\mathcal{F}_\mathcal{X} : c \mapsto c^\top \Psi^\mathcal{X} := \sum_{\lambda \in \vee \mathcal{X}} c_\lambda \psi^\mathcal{X}_\lambda \in \text{Lis}(\ell_2(\vee \mathcal{X}), \mathcal{X}),
\]

and so its adjoint, the analysis operator,

\[
\mathcal{F}'_\mathcal{X} : g \mapsto g(\Psi^\mathcal{X}) := [g(\psi^\mathcal{X}_\lambda)]_{\lambda \in \vee \mathcal{X}} \in \text{Lis}(\mathcal{X}', \ell_2(\vee \mathcal{X})).
\]

(analogously for \( \mathcal{F}_\mathcal{Y} \)).

\[
Bu = f \iff \underbrace{\mathcal{F}'_\mathcal{Y}BF_x F^{-1}_x}_B \underbrace{u}_u = \underbrace{\mathcal{F}'_\mathcal{Y}f}_f,
\]

where

\[
B = (B \Psi^\mathcal{X})(\Psi^\mathcal{Y}) \in \text{Lis}(\ell_2(\vee \mathcal{X}), \ell_2(\vee \mathcal{Y}))
\]

(infinite “stiffness” matrix),

\[
f = f(\Psi^\mathcal{Y}) \in \ell_2(\vee \mathcal{Y}) \quad \text{(infinite “load” vector).}
\]
Least squares problems

Let $B \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ with $\|B \cdot\|_{\mathcal{Y}} \approx \|\cdot\|_{\mathcal{X}}$. Then

$$\arg\min_{u \in \mathcal{X}} \frac{1}{2} \|Bu - f\|_{\mathcal{Y}}^2 \iff \arg\min_{u \in \ell_2(\vee \mathcal{X})} \frac{1}{2} \|Bu - f\|_{\ell_2(\vee \mathcal{Y})}^2 \iff B^\top(Bu - f) = 0,$$

where $B^\top B \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$.

We will apply these normal equations also when $B \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and thus $Bu = f$ is well-posed ($B \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$), but not $B = B^\top > 0$, because not $\mathcal{X} = \mathcal{Y}$, $\Psi^\mathcal{X} = \Psi^\mathcal{Y}$ and $B = B' > 0$. 
Adaptive Wavelet-Galerkin scheme ($Bu = f$)

([CDD01]) Let $B = B^\top > 0$. Otherwise apply to normal equations.

Goal: To generate sequence of approx. to $u$ that, whenever for some $s > 0$, $\|u\|_{A^s} := \sup_N N^s \|u - u_N\| < \infty$, converges with this rate $s$, at linear cost. (Here $u_N$ is a best approx. to $u$ with $\# \text{supp } u_N \leq N$.)

Notations: $\Lambda \subseteq \forall$, $I_\Lambda : \ell_2(\Lambda) \rightarrow \ell_2(\forall)$, $R_\Lambda = I_\Lambda^\top : \ell_2(\forall) \rightarrow \ell_2(\Lambda)$,

$$B_\Lambda := R_\Lambda B I_\Lambda, \quad u_\Lambda := B_\Lambda^{-1} R_\Lambda f, \quad \|\cdot\| := \langle B \cdot, \cdot \rangle^{\frac{1}{2}}$$

(we will identify $u_\Lambda$ with $I_\Lambda u_\Lambda$).

awgm:  
- Solve $B_{\Lambda_i} u_{\Lambda_i} = R_{\Lambda_i} f$;
- $\Lambda_{i+1} \supseteq \Lambda_i$ by bulk chasing on $f - Bu_{\Lambda_i}$;
- repeat with $i := i + 1$

(Familiar solve-estimate-mark-refine loop known from AFEM, with role a posteriori estimator played by the residual vector. Convergence and optimality not restricted to elliptic problems)
**Prop 1 ([CDD01])**. Let \( \theta \in (0, 1] \), \( \Lambda \subset \Xi \subset \mathcal{V} \), s.t.

\[
\|R_{\Xi}(f - Bu_{\Lambda})\| \geq \theta \|f - Bu_{\Lambda}\|.
\]

Then \( \|u - u_{\Xi}\| \leq \left[1 - \kappa(B)^{-1}\theta^2\right]^{1/2} \|u - u_{\Lambda}\| \).

**Prop 2 ([GHS07])**. If \( \theta < \kappa(B)^{-1/2} \) and \( \Xi \) is the smallest set satisfying bulk chasing criterium, then \( \#(\Xi \setminus \Lambda) \leq N \) for smallest \( N \) s.t.

\[
\|u - u_{N}\| \leq \left[1 - \theta^2 \kappa(B)\right]^{1/2} \|u - u_{\Lambda}\|.
\]

**Corol 1. awgm** realizes optimal rate \( s \) (\( N \lesssim \|u - u_{\Lambda}\|^{-1/s} \) & lin. conv.),\( 1 \) but, in this form, it is not implementable. \( 1 \)

**Thm 1. awgm** with **approx.** eval. of residual \( f - Bu_{\Lambda} \) and **approx.** solution of \( B_{\Lambda}u_{\Lambda} = R_{\Lambda} f \) within suff. small, but **fixed rel. tolerance** \( \delta \), also converges with optimal rate \( s \), \( 1 \) and, if such approx. eval. of \( f - Bw \) for \( w \in \ell_2(\Lambda) \) takes

\[
\mathcal{O}(\|u - w\|^{-1/s} + \#\Lambda) \text{ operations}
\]

(cost condition), then scheme has **optimal comput. compl.**
Nonlinear operator equations

([XZ03, Ste14]) Theorem 1 generalizes to equations

\[ F(u) = 0 \]

for \( F : \mathcal{X} \supset \text{dom}(F) \rightarrow \mathcal{Y}' \), written as

\[ \mathbf{F}(\mathbf{u}) = 0, \]

where \( \mathbf{F} := \mathcal{F}_{\mathcal{Y}}' \mathcal{F} \mathcal{F}_{\mathcal{X}} \), assuming that \( \mathcal{X} = \mathcal{Y} \), \( DF(u) \in \mathcal{L}is(\mathcal{X}, \mathcal{X}') \) and \( DF(u) = DF(u)' > 0 \),

or to

\[ \text{argmin}_{u \in \text{dom}(F)} \frac{1}{2} \| F(u) \|_{\mathcal{Y}'}^2, \]

written as

\[ DF(\mathbf{u})^\top \mathbf{F}(\mathbf{u}) = 0, \]

only assuming that \( \| DF(u)(\cdot) \|_{\mathcal{Y}'} \approx \| \cdot \|_{\mathcal{X}} \).
Verification of cost condition

Valid approx. eval. of $f - Bw$ from [CDD01] exploits near-sparsity of $f$, $B$, and $w$ ($u \in A^s \leadsto w \in A^s$). Depends non-linearly on $w$.

A quantitatively better scheme is obtained by not splitting the residual:

**Ex 1.** Poisson 1D. $w := w^\top \Psi$, to approx.

$$f - Bw = \left[ \int_0^1 f \psi_\lambda \right]_{\lambda \in \nabla} - \left[ \int_0^1 w' \psi_\lambda' \, dx \right]_{\lambda \in \nabla}$$

within $\delta \| u - w \|_{H^1(0,1)}$. Assuming $\Psi \subset H^2(0,1)$,

$$f - Bw = \left[ \int_0^1 (f + w'') \psi_\lambda \, dx \right]_{\lambda \in \nabla},$$

and $f + w''$ is piecewise polynomial w.r.t some mesh\(^1\), with $\| \cdot \|_{H^{-1}(0,1)}$-norm $\approx \| u - w \|_{H^1(0,1)}$. By dropping all $\psi_\lambda$ whose levels exceed level of local mesh by fixed increment, resulting linear approx. res. eval. scheme meets acc. req.

By putting tree constraint on wavelet index sets, multi- to locally single scale transforms in lin. complex, and so approx res. eval in $O(\# \text{supp } w)$ ops., thus meets cost condition.

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\(^1\)modulo data oscillation
• Scheme applies whenever operator applies to any wavelet in \textit{mild sense}, and also applies to \textit{semi-linear} PDEs. \textit{One} vanishing moment suffices.

• Instead of applying $C^1$-wavelets, we advocate to write second order PDE as (well-posed) \textbf{first order system least squares}. \textit{Always} possible.
FOSLS for semi-linear 2nd order PDEs

Let $F : \mathcal{X} \supset \text{dom}(F) \to \mathcal{Y}'$.

For some sep. H. space $\mathcal{P}$, let $F = F_0 + F_1 F_2$ where $F_2 \in \mathcal{L}(\mathcal{X}, \mathcal{P})$, $F_1 \in \mathcal{L}(\mathcal{P}, \mathcal{Y}')$. Then

$$F(u) = 0 \iff \vec{H}(u, \theta) := (F_0(u) + F_1 \theta, \theta - F_2 u) = \vec{0}.$$  

**Thm 2** ([RS16]).

$$\|DF(u)v\|_{\mathcal{Y}'} \simeq \|v\|_{\mathcal{X}} \implies \|D\vec{H}(u, \theta)(v, \eta)\|_{\mathcal{Y}' \times \mathcal{P}} \simeq \|(v, \eta)\|_{\mathcal{X} \times \mathcal{P}}.$$  

So $F(u) = 0$ can be found as

$$\text{argmin}_{(u, \theta) \in \text{dom}(F) \times \mathcal{P}} \frac{1}{2} \|\vec{H}(u, \theta)\|_{\mathcal{Y}' \times \mathcal{P}}^2,$$

i.e. as solution of $D\vec{H}(u, \theta) \top \vec{H}(u, \theta) = 0$.  

Application: Semi-linear elliptic eq.

\[
\begin{cases}
  -\Delta u + N(u) = f & \text{on } \Omega \subset \mathbb{R}^n \\
  u = 0 & \text{at } \partial \Omega.
\end{cases}
\]

\(-\Delta u\) may read as \(\nabla \cdot A\nabla u + b \cdot \nabla u + cu\); other (inhom.) b.c. can be used.

Let standard var. form. with \(X = Y = H^1_0(\Omega)\) be well-posed (i.e. solution \(u\) exists, linearized operator at \(u\) in \(\mathcal{L}(X,Y')\)).

Take \(P = L_2(\Omega)^n\). \(F_2 u := \nabla u, (F_1 \vec{\theta})(v) := \int_\Omega \vec{\theta} \cdot \nabla v \, dx\).

Well-posed FOSLS:

\[
\underset{(u,\vec{\theta}) \in X \times P}{\arg\min} \frac{1}{2} \left( \|v \mapsto \int_\Omega \vec{\theta} \cdot \nabla v \, dx + (N(u) - f)v \, dx\|_{Y'}^2 + \|\vec{\theta} - \nabla u\|_P^2 \right).
\]
Equip $\mathcal{X}, \mathcal{Y}, \mathcal{P}$, with wavelet Riesz bases $\Psi^\mathcal{X}, \Psi^\mathcal{X}, \Psi^\mathcal{P}$. To solve

$$\vec{0} = \left\langle \begin{bmatrix} \nabla \Psi^\mathcal{X} \\ -\Psi^\mathcal{P} \end{bmatrix}, \nabla u - \vec{\theta} \right\rangle_{L_2(\Omega)^n} +$$

$$\left[ \left\langle DN(u)\Psi^\mathcal{X}, \Psi^\mathcal{Y} \right\rangle_{L_2(\Omega)} \right] \left\langle \Psi^\mathcal{P}, \nabla \Psi^\mathcal{Y} \right\rangle_{L_2(\Omega)^n} \left\{ \left\langle \Psi^\mathcal{Y}, N(u) - f \right\rangle_{L_2(\Omega)} + \left\langle \nabla \Psi^\mathcal{Y}, \vec{\theta} \right\rangle_{L_2(\Omega)^n} \right\}$$

which rhs under the **additional condition** $\Psi^\mathcal{P} \subset H(\text{div}; \Omega)$, and $u, \vec{\theta}$ from the span of the wavelets, reads as

$$\left\langle \begin{bmatrix} \nabla \Psi^\mathcal{X} \\ -\Psi^\mathcal{P} \end{bmatrix}, \nabla u - \vec{\theta} \right\rangle_{L_2(\Omega)^n} + \left[ \left\langle DN(u)\Psi^\mathcal{X}, \Psi^\mathcal{Y} \right\rangle_{L_2(\Omega)} \right] \left\langle \Psi^\mathcal{P}, \nabla \Psi^\mathcal{Y} \right\rangle_{L_2(\Omega)^n} \left\{ \left\langle \Psi^\mathcal{Y}, N(u) - f - \text{div} \vec{\theta} \right\rangle_{L_2(\Omega)} \right\}.$$
Numerical experiment

(Nikolaos Rekatsinas (KdVI, Amsterdam))

\[
\begin{aligned}
-\Delta u + u^3 &= f \quad \text{on } \Omega \subset \mathbb{R}^2 \\
  u &= 0 \quad \text{at } \partial \Omega.
\end{aligned}
\]

L-shaped domain. FOSLS. Finite element wavelets. Continuous piecewise linears for \( \Psi^Y \) and \( \Psi^P \), continuous piecewise quadratics for \( \Psi^X \).

Figure 1: Approximate solution of \(-\Delta u + u^3 = 1\) on L-shaped, \(u = 0\) on bdr., as lin. combi of 202 wavelets.
Figure 2: Norm of residual vector, vs. number of wavelets, and optimal slope $\frac{3-1}{2} = 1$
Figure 3: Centers of supports wavelets (# = 5447) that were selected
Application: Stationary Navier-Stokes

For \( n \in \{2, 3\} \),

\[
\begin{cases}
-\Delta \vec{u} + \nu^{-3/2}(\vec{u} \cdot \nabla)\vec{u} + \nabla p = \vec{f} & \text{on } \Omega \\
\text{div } \vec{u} = 0 & \text{on } \Omega \\
\vec{u} = \vec{0} & \text{on } \partial \Omega.
\end{cases}
\]

Standard var. form. well-posed with \( X = Y = H^1_0(\Omega)^n \times L^2(\Omega)/\mathbb{R} \).

On \( H^1_0(\Omega)^n \times H^1_0(\Omega)^n \), \( \int_\Omega \nabla \vec{u} : \nabla \vec{v} - \text{div } \vec{u} \text{ div } \vec{v} - \text{curl } \vec{u} \cdot \text{curl } \vec{v} \, dx = 0 \).

Take \( \mathcal{P} = L^2(\Omega)^{2n-3} \) (recall \( F = F_0 + F_1 F_2 \))

\[
F_2(\vec{u}, p) = \text{curl } \vec{u}, \quad (F_1 \vec{\omega})(\vec{v}, q) = \int_\Omega \vec{\omega} \cdot \text{curl } \vec{v} \, dx.
\]

Well-posed FOSLS

\[
\text{argmin } (\vec{u}, p, \vec{\omega}) \in X \times \mathcal{P} \left\{ \frac{1}{2} \left( \| \vec{v} \mapsto \int_\Omega \vec{\omega} \cdot \text{curl } \vec{v} - p \text{ div } \vec{v} + \frac{(\vec{u} \cdot \nabla)\vec{u} \cdot \vec{v}}{\nu^{3/2}} - \vec{f} \cdot \vec{v} \, dx \|_{H^{-1}(\Omega)^n}^2 
\right.
\right.
\left. + \| \text{div } \vec{u} \|_{L^2(\Omega)}^2 + \| \vec{\omega} - \text{curl } \vec{u} \|_{L^2(\Omega)^{2n-3}}^2 \right) \right\}.
\]

(gives effortless proof of [CMM95, Thm. 2.1])
Equip $* \in \{ H^1_0(\Omega)^n, L_2(\Omega) / IR, L_2(\Omega)^{2n-3} \}$ with wavelet Riesz bases $\Psi^*$, apply \texttt{awgm} to

\[ \vec{0} = \begin{bmatrix} \langle \text{div } \Psi^{(H^1_0)^n}, \text{div } \vec{u} \rangle_{L_2(\Omega)} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \langle \text{curl } \Psi^{(H^1_0)^n}, \text{curl } \vec{u} - \vec{\omega} \rangle_{L_2(\Omega)^{2n-3}} \\ 0 \\ \langle \Psi^{L_2^{2n-3}}, \vec{\omega} - \text{curl } \vec{u} \rangle_{L_2(\Omega)^{2n-3}} \end{bmatrix} \]

\[ + \begin{bmatrix} \langle \frac{(\vec{u} \cdot \nabla) \Psi^{(H^1_0)^n} + (\Psi^{(H^1_0)^n} \cdot \nabla) \vec{u}}{\nu^{3/2}}, \Psi^{(H^1_0)^n} \rangle_{L_2(\Omega)^n} \\ -\langle \Psi^{L_2 / IR}, \text{div } \Psi^{(H^1_0)^n} \rangle_{L_2(\Omega)} \\ \langle \Psi^{L_2^{2n-3}}, \text{curl } \Psi^{(H^1_0)^n} \rangle_{L_2(\Omega)^{2n-3}} \end{bmatrix} \left\{ \langle \Psi^{(H^1_0)^n}, \frac{(\vec{u} \cdot \nabla) \vec{u}}{\nu^{3/2}} - \vec{f} \rangle_{L_2(\Omega)^n} + \langle \text{curl } \Psi^{(H^1_0)^n}, \vec{\omega} \rangle_{L_2(\Omega)^{2n-3}} - \langle \text{div } \Psi^{(H^1_0)^n}, p \rangle_{L_2(\Omega)} \right\} \]

which rhs under the conditions

\[ \Psi^{L_2 / IR} \subset H^1(\Omega), \ \Psi^{L_2^{2n-3}} \subset H(\text{curl}; \Omega), \]

and $\vec{u}, \vec{\omega}, p$ from the span of the wavelets, reads as
\[ \begin{bmatrix} \langle \text{div } \Psi(H_0^1)^n, \text{div } \vec{u} \rangle_{L_2(\Omega)} & \langle \text{curl } \Psi(H_0^1)^n, \text{curl } \vec{u} - \vec{\omega} \rangle_{L_2(\Omega)^{2n-3}} \\ \vec{0} & \vec{0} \end{bmatrix} + \begin{bmatrix} \langle \Psi L_2^{2n-3}, \vec{\omega} - \nabla \vec{u} \rangle_{L_2(\Omega)^{2n-3}} \\ \langle \Psi L_2^{2n-3} / \nu^{3/2}, \text{div } \Psi(H_0^1)^n \rangle_{L_2(\Omega)} - \langle \Psi L_2 / \nu^{3/2}, \text{div } \Psi(H_0^1)^n \rangle_{L_2(\Omega)} \end{bmatrix} + \langle \Psi(H_0^1)^n, \frac{(\vec{u} \cdot \nabla) \vec{u}}{\nu^{3/2}} - \vec{f} + \text{curl}' \vec{\omega} + \nabla p \rangle_{L_2(\Omega)^n}. \]
Application: Parabolic problems

With \( \Omega \subset \mathbb{R}^n \), \( I := (0, T) \),

\[
\begin{cases}
\frac{\partial u}{\partial t} - \nabla \cdot A \nabla u + Nu = g & \text{on } I \times \Omega, \\
u = 0 & \text{on } I \times \partial \Omega, \\
 u(0, \cdot) = h & \text{on } \Omega,
\end{cases}
\]

where \( \xi^\top A(\cdot)\xi \simeq \|\xi\|^2 \), \( N \) bounded first order PDO.

With

\[
\mathcal{X} := L_2(I; H^1_0(\Omega)) \cap H^1(I; H^{-1}(\Omega)), \quad \mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2 := L_2(I; H^1_0(\Omega)) \times L_2(\Omega),
\]

\[
(Bu)(v) := \int_I \int_\Omega \left( \frac{\partial u}{\partial t} + Nu \right) v_1 + A \nabla u \cdot \nabla v_1 \, dx \, dt + \int_\Omega u(0, \cdot) v_2 \, dx
\]
satisfies \( B \in \mathcal{L}_{is}(\mathcal{X}, \mathcal{Y}'). \)

With \( \mathcal{P} := L_2(I; L_2(\Omega)^n) \),

\[
F_2 u = A \nabla u, \quad (F_1 \vec{p})(v) = \int_I \int_\Omega \vec{p} \cdot \nabla v_1 \, dx.
\]
well-posed FOSLS (seems new)

$$\text{argmin}_{(u,p) \in \mathcal{X} \times \mathcal{P}} \frac{1}{2} \left( \left\| v_1 \mapsto \int_I \int_{\Omega} \left( \frac{\partial u}{\partial t} + N - g \right) v_1 + \vec{p} \cdot \nabla v_1 \, dx \, dt \right\|_{\mathcal{Y}_1'}^2 + \left\| u(0, \cdot) - h \right\|_{L_2(\Omega)}^2 + \left\| \vec{p} - A \nabla u \right\|_{\mathcal{P}}^2 \right).$$

Equip $* \in \{\mathcal{X}, \mathcal{Y}_1, \mathcal{P}\}$ with wavelet Riesz bases $\Psi^*$, apply \textbf{awgm} to

$$\tilde{0} = \left[ \langle \frac{\partial}{\partial t} + N \Psi^\mathcal{X}, \Psi^\mathcal{Y}_1 \rangle_{L_2(I \times \Omega)} \right] \left\{ \langle \Psi^\mathcal{Y}_1, \frac{\partial u}{\partial t} + N u - g \rangle_{L_2(I \times \Omega)} + \langle \nabla \Psi^\mathcal{Y}_1, \vec{p} \rangle_{L_2((I \times \Omega)^n)} \right\}
+ \left[ \langle \Psi^\mathcal{X}(0, \cdot), u(0, \cdot) - h \rangle_{L_2(\Omega)} \right] + \left[ -\langle A \nabla \Psi^\mathcal{X}, \vec{p} - A \nabla u \rangle_{L_2(I \times \Omega)^n} \right]
+ \left[ \langle \Psi^\mathcal{P}, \vec{p} - A \nabla u \rangle_{L_2(I \times \Omega)^n} \right],$$

which rhs under the additional condition $\Psi^\mathcal{P} \subset L_2(I; H(\text{div}; \Omega))$, and
$u, \vec{p}$ from the span of the wavelets, reads as

\[
\begin{align*}
\int \left[ \left( \frac{\partial}{\partial t} + N \right) \Psi_X, \Psi_{Y_1} \right]_{L^2(I \times \Omega)} \\
\int \left( \Psi_P, \nabla \Psi_{Y_1} \right)_{L^2(I \times \Omega)^n}
\right] \\
+ \int \left[ \left( \Psi_X(0, \cdot), u(0, \cdot) - h \right)_{L^2(\Omega)} \right] \\
+ \int \left[ -\left( A \nabla \Psi_X, \vec{p} - A \nabla u \right)_{L^2(I \times \Omega)^n} \right]
\end{align*}
\]
Tensor product bases

Bases needed for
\[ \mathcal{X} = L_2(I;H^1_0(\Omega)) \cap H^1(I;H^{-1}(\Omega)), \quad \mathcal{Y} = L_2(I;H^1_0(\Omega)) \times L_2(\Omega), \quad \mathcal{P} = L_2(I;L_2(\Omega)^n). \]

Let, prop. sc., \( \Sigma \) (wavelet) Riesz basis for \( H^1_0(\Omega) \) and \( H^{-1}(\Omega) \), and, prop. sc., \( \Theta \) (wavelet) Riesz basis for \( L_2(I) \) and \( H^1(I) \). Then, prop. sc., \( \Theta \otimes \Sigma \) is a Riesz basis for \( \mathcal{X} \) (tensor product or anisotropic wavelets).

Similar construction of the bases for \( \mathcal{Y}_1 \) and \( \mathcal{P} \).

**Advantage**: In any case for sufficiently smooth solutions, convergence rates possible as for the stationary problem.

Index set \( \bigvee \mathcal{X} = \bigvee \Theta \times \bigvee \Sigma \). We restrict to subsets that satisfy a double-tree constraint.

For elliptic problems, anis. reg. results ([CDN12]) combined with approximation results ([DS10]), show that on 2 and 3-dim. polytopes, for suff. smooth forcing best (piecewise) multi-tree tensor approx. yields optimal rates as for 1D problems.

We conjecture that with sufficiently smooth forcing, best double-tree tensor approximation yields rates as for the stationary problem, i.e. that the singularities induced by the bottom corners of space-time cylinder are sufficiently smooth.
Residual evaluation with tensor product bases

Returning to the residual evaluation

\[
\begin{bmatrix}
\langle (\frac{\partial}{\partial t} + N)\psi^x, \psi^y_1 \rangle_{L^2(I \times \Omega)} \\
\langle \psi^p, \nabla \psi^y_1 \rangle_{L^2(I \times \Omega)^n}
\end{bmatrix} \langle \psi^y_1, \frac{\partial u}{\partial t} + Nu - g - \text{div} \vec{p} \rangle_{L^2(I \times \Omega)}
\]

\[+ \begin{bmatrix}
\langle \psi^x(0, \cdot), u(0, \cdot) - h \rangle_{L^2(\Omega)} \\
0
\end{bmatrix} + \begin{bmatrix}
-\langle A \nabla \psi^x, \vec{p} - A \nabla u \rangle_{L^2(I \times \Omega)^n} \\
\langle \psi^p, \vec{p} - A \nabla u \rangle_{L^2(I \times \Omega)^n}
\end{bmatrix}
\]

For \( u \) and \( \vec{p} \) from the span of tensor product wavelets with indices in double-trees \( \Lambda^x \) and \( \Lambda^p \), the restriction of this residual vector to \( \bar{\Lambda}^x \times \bar{\Lambda}^p \), for double trees \( \bar{\Lambda}^x \) and \( \bar{\Lambda}^p \) with \( \#(\bar{\Lambda}^x \cup \bar{\Lambda}^p) \lesssim \#(\Lambda^x \cup \Lambda^p) \), achieves a relative tolerance \( \delta \).

How to do this evaluation in linear complexity? Multi-to-locally single scale, followed by ‘stiffness’ in single-scale, followed by transpose of multi-to-locally single scale doesn’t work.
Let $\Lambda$ be a ‘sparse grid’. To apply $R_\Lambda(B \otimes A)I_\Lambda$.

[BZ96]: Write $A = L(\rightarrow) + U(\leftarrow)$, and

$$\updownarrow \circ (\rightarrow + \leftarrow) = \updownarrow \circ \rightarrow + \updownarrow \circ \leftarrow$$

$$\rightarrow \circ \uparrow + \updownarrow \circ \leftarrow$$

Application in linear complexity, when $A$ and $B$ are sparse in single scale coordinates.

Generalization to multi-trees in [KS14].
Numerical results heat eqn

(Nabi Chegini (Univ. of Tafresh, Iran)). Heat eqn. in one spatial dimension. No FOSLS. Special quartic wavelets that yield truly sparse stiffness matrix.

Figure 4: Right-hand side $g = 1$ and initial condition $u_0 = 0$. $\| Bu_\varepsilon - f \| / \| f \|$ vs. $N = \#\text{supp } u_\varepsilon$ for the awgm (solid), full-grid (dashed) and sparse-grid method (dashed-dotted). The dotted line is a multiple of $N^{-5}(\log N)^{5 \frac{1}{2}}$. 
Figure 5: Heat eqn. in $n = 1$ spatial dimension, right-hand side $g = 1$ and initial condition $u_0 = 1$. $\|Bu_\varepsilon - f\|/\|f\|$ vs. $N = \#\text{supp } u_\varepsilon$ for the awgm (solid). The dotted line is a multiple of $N^{-5}(\log N)^{5\frac{1}{2}}$. 

Figure 6: Heat eqn. in $n = 1$ spatial dimension and right-hand side $g = 1$. Centers of the supports of the wavelets selected by the awgm. Left $u_0 = 0$ and $\#u_\varepsilon = 13420$. Right $u_0 = 1$ and $\#u_\varepsilon = 13917$. A zoom in near $t = 0$ is given at the bottom row.
Can be reduced, for $h = 0$, to parabolic for velocities, but then arising spaces will be spaces of divergence-free functions. We enforce incompressibility constraint via Lagrange multiplier. Saddle point form.

Space-time variational form: With

\[
\begin{align*}
    c(u, v) &:= \int_I \int_{\Omega} \frac{\partial u}{\partial t} \cdot v + \nu \nabla u : \nabla v \, dxdt, \\
    d(p, v) &:= -\int_I \int_{\Omega} p \text{div} \, v \, dxdt, \\
    f(v, q) &:= \int_I \int_{\Omega} g \cdot v + hq \, dxdt,
\end{align*}
\]

find $(u, p)$ in some suitable space, such that

\[
(S(u, p))(v, q) := c(u, v) + d(p, v) + d(q, u) = f(v, q)
\]

for all $(v, q)$ from another suitable space.
For $\delta \in \{0, T\}$,
\[
\tilde{H}_0^s(\{\delta\})(I) := [L_2(I), H^1_0(\{\delta\})(I)]_s,
\]
\[
H^s(\Omega) := [L_2(\Omega), H^2(\Omega) \cap H^1_0(\Omega)]^{\frac{s}{2}},
\]
\[
\tilde{H}^s(\Omega) := [(H^1(\Omega)/\mathbb{R})', H^1(\Omega)/\mathbb{R})]^{\frac{s+1}{2}},
\]
\[
\mathcal{U}^s_\delta := L_2(I; H^{2s}(\Omega)^n) \cap \tilde{H}^s_0(\{\delta\})(I; L_2(\Omega)^n),
\]
\[
\mathcal{P}^s_\delta := (L_2(I; \tilde{H}^{2s-1}(\Omega)'), \tilde{H}^{1-s}_0(\{\delta\})(I; \tilde{H}^1(\Omega)'))'.
\]

**Thm 3 ([SS15]).** For $\Omega \subset \mathbb{R}^n$ a bounded Lipschitz domain, and $s \in (\frac{1}{4}, \frac{3}{4})$, it holds that
\[
S \in \text{Lis}(\mathcal{U}^s_0 \times \mathcal{P}^s_T, (\mathcal{U}^{1-s}_T \times \mathcal{P}^{1-s}_0)').
\]

- For $\partial \Omega \in C^2$, result also valid for $s \in [0, 1]$. $s \in \{0, 1\}$ avoids fractional Sob. sp., but $\mathcal{U}^1_\delta$ involves $H^2(\Omega)$.

- For $s \in (\frac{1}{4}, \frac{3}{4})$, all arising spaces can be ‘conveniently’ equipped with wavelet Riesz bases.

- Generalizes to NSE for $n = 2$; for $n = 3$ we need ‘$s' > \frac{3}{4}$ which requires smooth $\partial \Omega$ or convex domains, and $C^1$-wavelets.

- Well-posed FOSLS.
Main ingredients of proof of Thm 3 about the boundedness of $S^{-1}$:

- $[\mathcal{U}_\delta^0, \mathcal{U}_\delta^1]_s \simeq \mathcal{U}_\delta^s$, $[\mathcal{P}_\delta^0, \mathcal{P}_\delta^1]_s \simeq \mathcal{P}_\delta^s$ ($s \in [0, 1]$).

- Right-inverse $\text{div}^+$ from [Bog79] satisfies $\text{div}^+ \in \mathcal{L}((\bar{H}^{-1}(\Omega), L^2(\Omega))^n)$ and, for $s \in [0, \frac{3}{4})$, $\text{div}^+ \in \mathcal{L}((\bar{H}^{2s-1}(\Omega), H^{2s}(\Omega))^n)$, so $I \otimes \text{div}^+ \in \mathcal{L}((\mathcal{P}_\delta^{1-s})', \mathcal{U}_\delta^s)$, so $I \otimes \text{div} \in \mathcal{L}(\mathcal{U}_\delta^s, (\mathcal{P}_\delta^{1-s})')$ is surjective, i.e.,

$$\inf_{0 \neq q \in \mathcal{P}_\delta^{1-s}} \sup_{0 \neq u \in \mathcal{U}_\delta^s} \frac{d(u, q)}{\|u\|\mathcal{U}_\delta^s\|q\|\mathcal{P}_\delta^{1-s}} > 0.$$

Remains to show that $(Cu)(\nu) := c(u, \nu)$ is boundedly invertible between $\{u \in \mathcal{U}_0^s : d(\mathcal{P}_0^{1-s}, u) = 0\}$ and $(\{u \in \mathcal{U}_T^{1-s} : d(\mathcal{P}_T^s, u) = 0\})'$.

- Using $\text{div}^+$ again, first space (second similar)

$\simeq L^2(I; H^{2s}(\text{div} 0; \Omega)) \cap \overset{s}{\overset{\text{H}}{\Omega}}, \{0\} (I; H^0(\text{div} 0; \Omega))$

$\simeq L^2(I; [H^0(\text{div} 0; \Omega), D(A)]_s) \cap \overset{s}{\overset{\text{H}}{\Omega}}, \{0\} (I; H^0(\text{div} 0; \Omega))$

using elliptic regularity of stat. Stokes op. $A$ on div.-free functions.

- Proof completed by maximal regularity results (e.g. [PS01])

$C \in \text{Lis}(L^2(I; D(A)) \cap H^1_{0,\{\alpha\}} (I; H^0(\text{div} 0; \Omega)), L^2(I; H^0(\text{div} 0; \Omega)))$

$C \in \text{Lis}(L^2(I; H^0(\text{div} 0; \Omega)), (L^2(I; D(A)) \cap H^1_{0,\{\beta\}} (I; H^0(\text{div} 0; \Omega)))'$, and interpolation.
Conclusions

• Adaptive wavelet method solves general well-posed operator equations with the best possible rate in linear complexity.

• Besides optimal ‘preconditioning’, main advantage is that ‘efficient and reliable’ a posteriori estimator, being the residual of operator equation in wavelet coordinates, is not restricted to elliptic problems.

• Tensor product wavelets for time-dependent problems give complexity reduction.

• Well-posedness of space-time variational formulations of evolution problems is not only of interest for wavelet methods.

Thanks for your patience/attention!
References


