Mapped Tent Pitching Method for Hyperbolic Conservation Laws

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Outline

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Problem description

Let be $\Omega \subset \mathbb{R}^N$. Find $u : \Omega \times (0, T] \to \mathbb{R}^n$ such that

$$\partial_t u(x, t) + \text{div}_x f(x, t, u(x, t)) = 0 \quad \forall (x, t) \in \Omega \times (0, T],$$

$$u(x, 0) = u_0(x) \quad \forall x \in \Omega,$$

with the given flux function

$$f : \Omega \times (0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n \times N},$$

$$(x, t, u(x, t)) \longmapsto f(x, t, u(x, t)).$$
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\]

Hyperbolicity

We call the system hyperbolic in the t-direction, if the matrix

\[
D_u(f \nu)
\]

has real eigenvalues (characteristic speeds) \( \lambda_1, \ldots, \lambda_n \) for all directions \( \nu \in S^{N-1} \).
Mapped Tent Pitching (MTP) method

- Construct space-time mesh using a tent pitching algorithm
- Map conservation law on each tent to a space-time cylinder
- Spatially discretize using a discontinuous Galerkin method
- Apply high order time stepping on the cylinder
Mapped Tent Pitching (MTP) method

- **construct space-time mesh using a tent pitching algorithm**
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Tent pitching algorithm in 1D

\[ \tau_i \left( x \right) := \left( 1 - \hat{t} \right) \left( x \right) + \hat{t} \tau_i \left( x \right) \]

\[ \hat{u} = \hat{u} - \nabla \phi \]

\[ \nabla \phi = 0 \Rightarrow \hat{U} = \hat{u} \]
Tent pitching algorithm in 1D

\[ x \propto \bar{c}, \bar{c} \ldots \text{maximal characteristic speed} \]
Tent pitching algorithm in 1D

- Advancing front \( \tau \)

\[ \propto \frac{1}{\bar{c}}, \bar{c} \ldots \text{maximal characteristic speed} \]

- Local CFL-condition \(|\nabla \tau| < \frac{1}{\bar{c}}|\]

\[ \hat{U} = \hat{u} - f(\hat{u}) \]

\[ \nabla \hat{u} = 0 \Rightarrow \hat{U} = \hat{u} \]
Tent pitching algorithm in 1D

Advancing front \( \tau \)

\[
\tau (x, \hat{t}) := (1 - \hat{t}) \tau_{i-1}(x) + \hat{t} \tau_i(x)
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\[ \text{Advancing front } \tau \]

\[ \phi(x, \hat{t}) := (1 - \hat{t}) \tau_{i-1}(x) + \hat{t} \tau_i(x) \]

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Advancing front \( \tau \)
Tent pitching algorithm in 1D

Advancing front $\tau$

$\tau_i^t(x) := (1 - \hat{t}) \tau_{i-1}(x) + \hat{t} \tau_i(x)$

$\hat{U} = \hat{u} - f(\hat{u}) \nabla \phi \Rightarrow \hat{U} = \hat{u}$

Diagram showing steps of the advancing front.
Tent pitching algorithm in 1D

Advancing front $\tau$

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\phi(x, \hat{t}) := (1 - \hat{t}) \tau_{i-1}(x) + \hat{t} \tau_i(x)
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\hat{u} = \nabla \phi \quad \nabla \phi = 0 \quad \Rightarrow \quad \hat{U} = \hat{u} - f(\hat{u}) \nabla \phi
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Tent pitching algorithm in 1D

Advancing front $\tau$

\[ \phi(x, \hat{t}) := (1 - \hat{t}) \tau_{i-1}(x) + \hat{t} \tau_i(x) \]

\[ \hat{U} = \hat{u} - \nabla \phi \]

\[ \nabla \phi = 0 \]
Tent pitching algorithm in 2D

Gray tents: Level 0 tents, can be solved in parallel
Tent pitching algorithm in 2D

Gray tents: Level 1 tents, can be solved in parallel
Tent pitching algorithm in 2D

Gray tents: Level 2 tents, can be solved in parallel
Gray tents: Level 3 tents, can be solved in parallel


Space-time cylinder $\hat{K}_i := \Omega_{v(i)} \times (0, 1)$ over the vertex patch $\Omega_{v(i)}$. 
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**Duffy-like transformation**

$\Phi : \hat{K}_i \rightarrow K_i,$

$(x, \hat{t}) \mapsto (x, \varphi(x, \hat{t})),$
Space-time cylinder $\hat{K}_i := \Omega_{v(i)} \times (0, 1)$ over the vertex patch $\Omega_{v(i)}$

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Mapped conservation laws

\[ F(u) := (f, u) \in \mathbb{R}^{n \times (N+1)} \]

\[ \partial_t u + \text{div}_x f = 0 \iff \text{div}_{(x, t)} F(u) = 0 \quad (1) \]
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\[
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\]  \hspace{1cm} (1)

Piola transformation

\[
\hat{F}_l = \det(D\Phi) [D\Phi]^{-1} F_l \circ \Phi \quad \forall l \in \{1, \ldots, n\}
\]
### Mapped conservation laws

\[
F(u) := (f, u) \in \mathbb{R}^{n \times (N+1)}
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\partial_t u + \text{div}_x f = 0 \quad \Leftrightarrow \quad \text{div}_{(x,t)} F(u) = 0
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(1)

### Piola transformation

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\hat{F}_l = \det(\hat{\mathbf{D}}\Phi)[\hat{\mathbf{D}}\Phi]^{-1} F_l \circ \Phi \quad \forall l \in \{1, \ldots, n\}
\]

\[
\frac{1}{\det(\hat{\mathbf{D}}\Phi)} \text{div}_{(x, \hat{t})} \hat{F}(u \circ \Phi) = 0
\]

\[
=:\hat{u}
\]  

(2)
Mapped conservation laws

\[ F(u) := (f, u) \in \mathbb{R}^{n \times (N+1)} \]
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conservation law on the space-time cylinder \( \hat{K}_i \)
Problem description

Find $\hat{u} : \hat{K}_i \to \mathbb{R}^n$ such that

$$\partial_t (\hat{u} - f(\hat{u}) \nabla \varphi) + \text{div}_x [\delta f(\hat{u})] = 0$$

in $\hat{K}_i$. 

$G(\hat{u}) \equiv \hat{U}$
Problem description

Find \( \hat{u} : \hat{K}_i \to \mathbb{R}^n \) such that

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Problem description

Find \( \hat{U} : \hat{K}_i \to \mathbb{R}^n \) such that

\[
\partial_t \hat{U} + \text{div}_x [\delta f(G^{-1}(\hat{U}))] = 0
\]

in \( \hat{K}_i \),

\[
\hat{U}(\cdot, 0) = \hat{U}_{i-1}(\cdot, 1)
\]

in \( \Omega_v(i) \).
Mapped conservation laws

Problem description

Find \( \hat{u} : \hat{K}_i \rightarrow \mathbb{R}^n \) such that

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- conservation law in new variable \( \hat{U} \)
- inverse transformation \( G^{-1}(\hat{U}) \) needed
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- conservation law in new variable $\hat{U}$
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Inverse transformation $G^{-1}(\hat{U})$

$$\hat{U} \equiv G(\hat{u}) := \hat{u} - f(\hat{u}) \nabla \varphi$$ (3)
Inverse transformation $G^{-1}(\hat{U})$

\[
\hat{U} \equiv G(\hat{u}) := \hat{u} - f(\hat{u}) \nabla \varphi
\]  \hspace{1cm} (3)

Convection equation

Flux function: $f(u) := bu, \quad b \in \mathbb{R}^2$

\[
\hat{U} = (1 - b \cdot \nabla \varphi)\hat{u}
\]
Inverse transformation $G^{-1}(\hat{U})$

$$\hat{U} \equiv G(\hat{u}) := \hat{u} - f(\hat{u}) \nabla \varphi$$  \hspace{1cm} (3)

**Convection equation**

Flux function: $f(u) := bu, \quad b \in \mathbb{R}^2$

$$\hat{U} = (1 - b \cdot \nabla \varphi) \hat{u} \iff \hat{u} = \frac{\hat{U}}{1 - b \cdot \nabla \varphi}$$

solvable if $|\nabla \varphi| < \frac{1}{|b|}$
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solvable if $|\nabla \varphi| < \frac{1}{|b|}$ (known CFL-condition)

Theorem

If there holds $|\nabla \varphi| < \frac{1}{c}$, then (3) has a unique solution $\hat{u}$.

$c \ldots$ maximal speed
Inverse transformation \( G^{-1}(\hat{U}) \)

Find \((\rho, m, E) : \Omega \times (0, T] \rightarrow \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}\) s.t.

\[
\partial_t \begin{pmatrix} \rho \\ m \\ E \end{pmatrix} + \text{div} \begin{pmatrix} m \\ \frac{1}{\rho} m \otimes m + pl \\ \frac{m}{\rho} (E + p) \end{pmatrix} = 0
\]
Inverse transformation $G^{-1}(\hat{U})$

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\[\hat{U} \equiv G(\hat{u}) := \hat{u} - f(\hat{u}) \nabla \varphi\]  \hspace{1cm} (3)

\[\hat{u} = (\hat{\rho}, \hat{m}, \hat{E})\]

\[\hat{U} = (\hat{R}, \hat{M}, \hat{F})\]

\[(\hat{\rho}, \hat{m}, \hat{E}) = \hat{G}^{-1}(\hat{R}, \hat{M}, \hat{F})\]
Inverse transformation $G^{-1}(\hat{U})$

Find $(\rho, m, E) : \Omega \times (0, T] \to \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$ s.t.

$$
\partial_t \left( \begin{array}{c}
\rho \\
m \\
E
\end{array} \right) + \text{div} \left( \frac{1}{\rho} m \otimes m + pl \right) = 0
$$

$$
\hat{\rho} = \frac{\hat{R}^2}{a_1 - \frac{2}{d} |\nabla \varphi|^2 a_3}
$$

$$
\hat{m} = \frac{\hat{\rho}}{\hat{R}} (\hat{M} + \frac{2}{d} a_3 \nabla \varphi)
$$

$$
\hat{E} = \frac{\hat{\rho}}{\hat{R}} (\hat{F} + \frac{2}{d} a_3 \nabla \varphi \cdot \hat{m})
$$

where

\[ a_1 = \hat{R} - \hat{M} \cdot \nabla \varphi, \quad a_2 = 2 \hat{F} \hat{R} - |\hat{M}|^2, \quad a_3 = \frac{a_2}{a_1 + \sqrt{a_1^2 - \frac{4(d+1)}{d^2} |\nabla \varphi|^2 a_2}}. \]
Conservation

After time step $\hat{U}(x, 1) \rightarrow \hat{U}(x, 0)$ there holds

\[ \int_{\gamma_i} f(u) u \cdot \nu_i \, ds = \int_{\gamma_{i-1}} f(u) u \cdot \nu_{i-1} \, ds. \]
Conservation

\[ \hat{K}_i - \hat{U}(x, 0) \rightarrow \hat{K}_i - \hat{U}(x, 1) \]

**Parametrizations**

\[ \gamma_{i-1} : x \mapsto (x, \tau_{i-1}(x)) \], \quad \gamma_i : x \mapsto (x, \tau_i(x)) \]

and space-time unit normal vectors \( \nu_{i-1}, \nu_i \).
After time step \( \hat{U}(x, 1) \rightarrow \hat{U}(x, 1) \) there holds

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### Parametrizations

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After time step $\hat{U}(x, 0) \rightarrow \hat{U}(x, 1)$ there holds

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$$\nu_i \approx \begin{pmatrix} -\nabla \tau_i(x) \\ 1 \end{pmatrix}, \quad \nu_{i-1} \approx \begin{pmatrix} -\nabla \tau_{i-1}(x) \\ 1 \end{pmatrix}$$
After time step $\hat{U}(x, 0) \rightarrow \hat{U}(x, 1)$ there holds

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$$\nu_i \approx \begin{pmatrix} -\nabla \tau_i(x) \\ 1 \end{pmatrix}, \quad \nu_{i-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
Conservation

After time step $\hat{U}(x, 0) \rightarrow \hat{U}(x, 1)$ there holds

$$\int_{\gamma_i} \left( \begin{array}{c} f(u) \\ u \end{array} \right) \cdot \nu_i \, ds = \int_{\gamma_{i-1}} u \, ds.$$  

$$\nu_i \approx \left( \begin{array}{c} -\nabla \tau_i(x) \\ 1 \end{array} \right), \quad \nu_{i-1} = \left( \begin{array}{c} 0 \\ 1 \end{array} \right)$$
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Tent pitching algorithm in 1D

\[ \nabla \varphi = 0 \]

\[ \varphi(x, \hat{t}) := (1 - \hat{t}) \tau_{i-1}(x) + \hat{t} \tau_i(x) \]
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\[ \varphi(x, \hat{t}) := (1 - \hat{t}) \tau_{i-1}(x) + \hat{t} \tau_i(x) \]

\[ \hat{U} = \hat{u} - f(\hat{u}) \nabla \varphi \quad \xrightarrow{\nabla \varphi = 0} \quad \hat{U} = \hat{u} \]
Find $\psi : \Omega \times (0, T] \to \mathbb{R}$ s.t. 

$$\partial_{tt}\psi - \text{div}(\nabla \psi) = 0 \quad \text{in } \Omega \times (0, T].$$
The wave equation

Find $\psi : \Omega \times (0, T] \to \mathbb{R}$ s.t.

$$\partial_{tt}\psi - \text{div}(\nabla \psi) = 0 \quad \text{in } \Omega \times (0, T].$$

With

$$\begin{pmatrix} q \\ \mu \end{pmatrix} := \begin{pmatrix} -\nabla \psi \\ \partial_t \psi \end{pmatrix},$$
The wave equation

Find $\psi : \Omega \times (0, T] \to \mathbb{R}$ s.t.

$$\partial_{tt} \psi - \text{div}(\nabla \psi) = 0 \quad \text{in} \ \Omega \times (0, T].$$

With $(q_\mu) := (-\nabla \psi \partial_t \psi)$,

we obtain

$$\partial_t \begin{pmatrix} q \\ \mu \end{pmatrix} + \text{div} \begin{pmatrix} I_\mu \\ q^T \end{pmatrix} = 0.$$
The wave equation

Find \( \psi : \Omega \times (0, T] \to \mathbb{R} \) s.t.

\[
\partial_{tt} \psi - \text{div}(\nabla \psi) = 0 \quad \text{in } \Omega \times (0, T].
\]

With

\[
\begin{pmatrix} q \\ \mu \end{pmatrix} := \begin{pmatrix} -\nabla \psi \\ \partial_t \psi \end{pmatrix},
\]

we obtain

\[
\partial_t \begin{pmatrix} q \\ \mu \end{pmatrix} + \text{div} \begin{pmatrix} I \mu \\ q^T \end{pmatrix} = 0.
\]

Mapping to space-time cylinder leads to

\[
\partial_t \left[ \begin{pmatrix} \hat{q} \\ \hat{\mu} \end{pmatrix} - \begin{pmatrix} I \hat{\mu} \\ \hat{q}^T \end{pmatrix} \nabla \varphi \right] + \text{div} \left[ \delta \begin{pmatrix} I \hat{\mu} \\ \hat{q}^T \end{pmatrix} \right] = 0.
\]
The wave equation

Find $\psi : \Omega \times (0, T] \to \mathbb{R}$ s.t.

$$\partial_{tt} \psi - \text{div}(\nabla \psi) = 0 \quad \text{in} \; \Omega \times (0, T].$$

With

$$\begin{pmatrix} q \\ \mu \end{pmatrix} := \begin{pmatrix} -\nabla \psi \\ \partial_t \psi \end{pmatrix},$$

we obtain

$$\partial_t \begin{pmatrix} q \\ \mu \end{pmatrix} + \text{div} \begin{pmatrix} l \mu \\ q^T \end{pmatrix} = 0.$$

Mapping to space-time cylinder leads to

$$\partial_t \left[ \begin{pmatrix} I \\ -\nabla \varphi^T \\ 1 \end{pmatrix} \begin{pmatrix} \hat{q} \\ \hat{\mu} \end{pmatrix} \right] + \text{div} \begin{pmatrix} l \delta \hat{\mu} \\ \delta \hat{q}^T \end{pmatrix} = 0.$$
The wave equation

\[ \partial_{\hat{t}} \left[ \begin{pmatrix} I & -\nabla \varphi \\ -\nabla \varphi^T & 1 \end{pmatrix} \begin{pmatrix} \hat{q} \\ \hat{\mu} \end{pmatrix} \right] + \text{div} \left( \begin{pmatrix} I \delta \hat{\mu} \\ \delta \hat{q}^T \end{pmatrix} \right) = 0. \] (4)
The wave equation

\[
\partial_t \left[ \begin{pmatrix} I & -\nabla \varphi \\ -\nabla \varphi^\top & 1 \end{pmatrix} \begin{pmatrix} \hat{q} \\ \hat{\mu} \end{pmatrix} \right] + \text{div} \left( \begin{pmatrix} I \delta \hat{\mu} \\ \delta \hat{q}^\top \end{pmatrix} \right) = 0. \tag{4}
\]

Space-discretization by DG leads to time-dependent mass matrix

\[
\partial_t \hat{M} \hat{u} + A \hat{u} = 0.
\]
The wave equation

\[ \partial_t \left[ \begin{pmatrix} I & -\nabla \varphi \\ -\nabla \varphi^T & 1 \end{pmatrix} \begin{pmatrix} \hat{q} \\ \hat{\mu} \end{pmatrix} \right] + \text{div} \left( I \delta \hat{\mu} \overline{\delta \hat{q}^T} \right) = 0. \tag{4} \]

Space-discretization by DG leads to time-dependent mass matrix

\[ \partial_t M\hat{u} + A\hat{u} = 0. \]

Introduce a new variable \( y = M\hat{u} \) and discretize transformed system

\[ \partial_t y + AM^{-1}y = 0 \]

by a Runge-Kutta method.
The wave equation

Find $\psi : \Omega \times (0, T] \rightarrow \mathbb{R}$ s.t.

$$\partial_{tt}\psi - \text{div}(\nabla \psi) = 0 \quad \text{in } \Omega \times (0, T].$$

With

$$\begin{pmatrix} q \\ \mu \end{pmatrix} := \begin{pmatrix} -\nabla \psi \\ \partial_t \psi \end{pmatrix},$$

we obtain

$$\partial_t \begin{pmatrix} q \\ \mu \end{pmatrix} + \text{div} \begin{pmatrix} I_{\mu} \\ q^T \end{pmatrix} = 0.$$

Domain $\Omega = [0, \pi]^2$, $T = \sqrt{2}\pi$,

$$\psi(x, t) = \frac{1}{\sqrt{2}} \cos(x_1) \cos(x_2) \sin(\sqrt{2}t)$$
The wave equation, 2+1 dimensions

**Figure 1:** Convergence rates in two space dimensions with RK2 for various spatial polynomial degrees $p$ of approximation, with

$$e^2 = \|q(\cdot, T) - q_h\|_{L^2(\Omega)}^2 + \|\mu(\cdot, T) - \mu_h\|_{L^2(\Omega)}^2.$$
Instead of

\[ \partial_t \hat{M} \hat{u} + A \hat{u} = 0, \]

consider the system:

\[ \partial_t \hat{U} + A \hat{u} = 0, \]

(5a)

\[ \hat{U} = M \hat{u}. \]

(5b)

Expansion of \( \hat{u} = \sum_i \hat{t}_i \hat{u}_i \) and \( \hat{U} = \sum_i \hat{t}_i \hat{U}_i \), with

\[ \hat{U}_{n+1} = \hat{U}_n + 1 \]

\[ M \hat{u}_{n+1} = \hat{U}_{n+1} - M' \hat{u}_n. \]
Instead of
\[ \partial_t M\hat{u} + A\hat{u} = 0, \]
consider the system
\[ \partial_t \hat{U} + A\hat{u} = 0, \]  \hspace{1cm} (5a)
\[ \hat{U} = M\hat{u}. \]  \hspace{1cm} (5b)
Instead of
\[ \partial_\hat{t} M\hat{u} + A\hat{u} = 0, \]
consider the system
\[ \partial_\hat{t} \hat{U} + A\hat{u} = 0, \] \hfill (5a)
\[ \hat{U} = M\hat{u}. \] \hfill (5b)

Expansion of \( \hat{u} = \sum_i \hat{\imath}^i \hat{u}_i \) and \( \hat{U} = \sum_i \hat{\imath}^i \hat{U}_i \), with
Instead of\[
\frac{\partial}{\partial t} M \hat{u} + A \hat{u} = 0, \]
consider the system\[
\frac{\partial}{\partial t} \hat{U} + A \hat{u} = 0, \tag{5a}
\]
\[
\hat{U} = M \hat{u}. \tag{5b}
\]

Expansion of \( \hat{u} = \sum_i \hat{t}^i \hat{u}_i \) and \( \hat{U} = \sum_i \hat{t}^i \hat{U}_i \), with\[
\hat{U}_{n+1} = \frac{1}{n+1} A \hat{u}_n, \]
\[
M \hat{u}_{n+1} = \hat{U}_{n+1} - M' \hat{u}_n.
\]
Figure 2: Convergence rates in two space dimensions with 2 Taylor steps for various spatial polynomial degrees $p$ of approximation, with $e^2 = \|q(\cdot, T) - q_h\|_{L^2(\Omega)}^2 + \|\mu(\cdot, T) - \mu_h\|_{L^2(\Omega)}^2$. 

- $p = 1$: $O(h)$
- $p = 2$: $O(h^2)$
- $p = 3$: $O(h^3)$
- $p = 4$: $O(h^4)$
Figure 3: Convergence rates in two space dimensions with 4 Taylor steps for various spatial polynomial degrees $p$ of approximation, with
\[ e^2 = \|q(\cdot, T) - q_h\|_{L^2(\Omega)}^2 + \|\mu(\cdot, T) - \mu_h\|_{L^2(\Omega)}^2. \]
Find $\psi : \Omega \times (0, T] \rightarrow \mathbb{R}$ s.t.

$$\partial_{tt}\psi - \text{div}(\alpha \nabla \psi) = 0 \quad \text{in } \Omega \times (0, T].$$

With

$$\begin{pmatrix} q \\ \mu \end{pmatrix} := \begin{pmatrix} -\nabla \psi \\ \partial_t \psi \end{pmatrix},$$

we obtain

$$\partial_t \begin{pmatrix} q \\ \mu \end{pmatrix} + \text{div} \left( \begin{pmatrix} I \mu \\ q^T \end{pmatrix} \right) = 0$$

Domain $\Omega = [0, \pi]^3$, $T = \frac{2\pi}{\sqrt{3}}$,

$$\psi(x, t) = \frac{1}{\sqrt{3}} \cos(x_1) \cos(x_2) \cos(x_3) \sin(\sqrt{3}t)$$
The wave equation, 3+1 dimensions

Figure 4: Convergence rates in three space dimensions for various spatial polynomial degrees \( p \) of approximation and \( p \) Taylor steps, with

\[
e^2 = \| q(\cdot, T) - q_h \|_{L^2(\Omega)}^2 + \| \mu(\cdot, T) - \mu_h \|_{L^2(\Omega)}^2.
\]
The Maxwell equations

\[ \partial_t \begin{pmatrix} \varepsilon E \\ \mu H \end{pmatrix} = \begin{pmatrix} \text{curl } H \\ - \text{curl } E \end{pmatrix} \]
The Maxwell equations

\[ \partial_t \begin{pmatrix} \varepsilon E \\ \mu H \end{pmatrix} = \begin{pmatrix} \text{curl } H \\ - \text{curl } E \end{pmatrix} \]

can be written as

\[ \partial_t \begin{pmatrix} \varepsilon E \\ \mu H \end{pmatrix} + \text{div} \begin{pmatrix} - \text{skew } H \\ \text{skew } E \end{pmatrix} = 0, \]

with \((\text{skew } E)_{ij} := \varepsilon_{ijk}E_k\).
Figure 5: Resonator, 489k curved elements, largest to smallest element: 5:1
The Maxwell equations

Figure 6: $H_y$ at $t=260$, 260 time slabs, 148k tents per slab, $p^2$ local Taylor time-steps

Shared memory server, 4 E7-8867 CPUs with 16 cores each.
The Maxwell equations

**Figure 6:** $H_y$ at $t=260$, 260 time slabs, 148k tents per slab, $p^2$ local Taylor time-steps

Shared memory server, 4 E7-8867 CPUs with 16 cores each.

$p=2$: 29 374 980 dofs, 20 min
Figure 6: $H_y$ at $t=260$, 260 time slabs, 148k tents per slab, $p^2$ local Taylor time-steps

Shared memory server, 4 E7-8867 CPUs with 16 cores each.

$p=2$: 29 374 980 dofs, 20 min

$p=3$: 58 751 160 dofs, 3 h 33 min
Figure 7: Resonator with sharp edges, 224k curved elements, largest to smallest element: 10:1
Figure 8: $H_y$ at $t=260$, 260 time slabs, 66k tents per slab, $p^2$ local Taylor time-steps

Shared memory server, 4 E7-8867 CPUs with 16 cores each.
The Maxwell equations

Figure 8: $H_y$ at $t=260$, 260 time slabs, 66k tents per slab, $p^2$ local Taylor time-steps

Shared memory server, 4 E7-8867 CPUs with 16 cores each.

$p=2$: 13 452 000 dofs, 8 min
Figure 8: $H_y$ at $t=260$, 260 time slabs, 66k tents per slab, $p^2$ local Taylor time-steps

Shared memory server, 4 E7-8867 CPUs with 16 cores each.

$p=2$: 13 452 000 dofs, 8 min

$p=3$: 26 904 000 dofs, 1 h 27 min
Euler equations

Find \((\rho, m, E) : \Omega \times (0, T] \rightarrow \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}\) s.t.

\[
\partial_t \begin{pmatrix} \rho \\ m \\ E \end{pmatrix} + \text{div} \begin{pmatrix} m \\ \frac{1}{\rho} m \otimes m + pl \\ \frac{m}{\rho} (E + p) \end{pmatrix} = 0
\]
### Entropy admissibility condition

| $\mathcal{E}(u) \in \mathbb{R}$ ...entropy, $\mathcal{F}(u) \in \mathbb{R}^N$ ...entropy flux |

The pair $(\mathcal{E}, \mathcal{F})$ is called the entropy pair.

**Mapped entropy admissibility condition**

\[ \partial_t \hat{\mathcal{E}}(\hat{u}) + \text{div} \hat{\mathcal{F}}(\hat{u}) = \delta(\partial_t \mathcal{E}(u) + \text{div} \mathcal{F}(u)) \circ \Phi \leq 0 \]

$\Rightarrow$ entropy viscosity regularization on space-time cylinder
Entropy admissibility condition

\[ \mathcal{E}(u) \in \mathbb{R} \ldots \text{entropy}, \quad \mathcal{F}(u) \in \mathbb{R}^N \ldots \text{entropy flux} \]

\[ \Rightarrow \partial_t \mathcal{E}(u) + \text{div} \mathcal{F}(u) \leq 0 \]
Entropy admissibility condition

\[ \mathcal{E}(u) \in \mathbb{R} \text{ ...entropy, } \mathcal{F}(u) \in \mathbb{R}^N \text{ ...entropy flux} \]

\[ \Rightarrow \partial_t \mathcal{E}(u) + \text{div } \mathcal{F}(u) \leq 0 \]

The pair \((\mathcal{E}, \mathcal{F})\) is called the entropy pair.
Entropy admissibility condition

\[ \mathcal{E}(u) \in \mathbb{R} \ldots \text{entropy, } \mathcal{F}(u) \in \mathbb{R}^N \ldots \text{entropy flux} \]

\[ \Rightarrow \partial_t \mathcal{E}(u) + \text{div} \mathcal{F}(u) \leq 0 \]

The pair \((\mathcal{E}, \mathcal{F})\) is called the entropy pair.

\[
\begin{align*}
\hat{\mathcal{E}}(w) &= \mathcal{E}(w) - \mathcal{F}(w) \nabla \varphi, \\
\hat{\mathcal{F}}(w) &= \delta \mathcal{F}(w).
\end{align*}
\]
Entropy admissibility condition

\[ \mathcal{E}(u) \in \mathbb{R} \text{ ...entropy, } \mathcal{F}(u) \in \mathbb{R}^N \text{ ...entropy flux} \]

\[ \Rightarrow \partial_t \mathcal{E}(u) + \text{div} \mathcal{F}(u) \leq 0 \]

The pair \((\mathcal{E}, \mathcal{F})\) is called the entropy pair.

\[ \hat{\mathcal{E}}(w) = \mathcal{E}(w) - \mathcal{F}(w) \nabla \varphi, \]

\[ \hat{\mathcal{F}}(w) = \delta \mathcal{F}(w). \]

Mapped entropy admissibility condition

\[ \partial_t \hat{\mathcal{E}}(\hat{u}) + \text{div} \hat{\mathcal{F}}(\hat{u}) = \delta (\partial_t \mathcal{E}(u) + \text{div} \mathcal{F}(u)) \circ \Phi \leq 0 \]
Entropy admissibility condition

\[ \mathcal{E}(u) \in \mathbb{R} \ldots \text{entropy}, \quad \mathcal{F}(u) \in \mathbb{R}^N \ldots \text{entropy flux} \]

\[ \Rightarrow \partial_t \mathcal{E}(u) + \text{div} \mathcal{F}(u) \leq 0 \]

The pair \((\mathcal{E}, \mathcal{F})\) is called the entropy pair.

\[ \hat{\mathcal{E}}(w) = \mathcal{E}(w) - \mathcal{F}(w) \nabla \varphi, \]

\[ \hat{\mathcal{F}}(w) = \delta \mathcal{F}(w). \]

Mapped entropy admissibility condition

\[ \partial_t \hat{\mathcal{E}}(\hat{u}) + \text{div} \hat{\mathcal{F}}(\hat{u}) = \delta (\partial_t \mathcal{E}(u) + \text{div} \mathcal{F}(u)) \circ \Phi \leq 0 \]

\[ \Longrightarrow \text{entropy viscosity regularization on space-time cylinder} \]
Problem description

Find $\hat{U} : \hat{K}_i \rightarrow \mathbb{R}^n$ such that

$$\partial_t \hat{U} + \text{div}_x \left[ \delta f(G^{-1}(\hat{U})) \right] = 0$$
Problem description

Find $\hat{U} : \hat{K}_i \rightarrow \mathbb{R}^n$ such that

$$\partial_t \hat{U} + \text{div}_x [\delta f(G^{-1}(\hat{U}))] = 0$$

Recall that $\hat{U} \equiv G(\hat{u}) := \hat{u} - f(\hat{u}) \nabla \varphi$ is discontinuous.
Entropy viscosity regularization

**Problem description**

Find \( \hat{U} : \hat{K}_i \rightarrow \mathbb{R}^n \) such that

\[
\partial_t \hat{U} + \text{div}_x \left[ \delta f \left( G^{-1}(\hat{U}) \right) \right] = 0
\]

Recall that \( \hat{U} \equiv G(\hat{u}) := \hat{u} - f(\hat{u}) \nabla \varphi \) is discontinuous.

\[\implies\text{artificial viscosity on } \hat{u} = G^{-1}(\hat{U})\]
**Problem description**

Find $\hat{U} : \hat{K}_i \rightarrow \mathbb{R}^n$ such that

$$\partial_t \hat{U} + \text{div}_x [\delta f(G^{-1}(\hat{U}))] = \frac{\nu_i}{k_i} \text{div}_x [\delta \nabla_x G^{-1}(\hat{U})]$$

Recall that $\hat{U} \equiv G(\hat{u}) := \hat{u} - f(\hat{u}) \nabla \varphi$ is discontinuous.

$\Rightarrow$ artificial viscosity on $\hat{u} = G^{-1}(\hat{U})$

$\nu_i$ ... entropy viscosity coefficient

$k_i = \delta(v^{(i)})$ ... tent height
Euler equations

Find \((\rho, m, E) : \Omega \times (0, T] \rightarrow \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}\) s.t.

\[
\partial_t \begin{pmatrix} \rho \\ m \\ E \end{pmatrix} + \text{div} \begin{pmatrix} m \\ \frac{1}{\rho} m \otimes m + pI_m \\ \frac{m}{\rho} (E + p) \end{pmatrix} = 0
\]
Find \((\rho, m, E) : \Omega \times (0, T] \to \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}\) s.t.

\[
\partial_t \begin{pmatrix} \rho \\ m \\ E \end{pmatrix} + \text{div} \begin{pmatrix} m \\ \frac{1}{\rho} m \otimes m + pl \\ \frac{m}{\rho} (E + p) \end{pmatrix} = 0
\]

\[
\rho = 1.4, \quad m = \rho(3, 0)^T, \quad p = 1
\]
Euler equations

Find \((\rho, m, E) : \Omega \times (0, T] \rightarrow \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}\) s.t.

\[
\partial_t \begin{pmatrix} \rho \\ m \\ E \end{pmatrix} + \text{div} \left( \begin{pmatrix} m \\ \frac{1}{\rho} m \otimes m + pl \\ \frac{m}{\rho} (E + p) \end{pmatrix} \right) = 0
\]

\(\rho = 1.4\), \quad m = \rho (3, 0)^T, \quad p = 1
Figure 9: Tent pitched time slab
Figure 9: Tent pitched time slab
**Figure 10:** Solution of Mach 3 wind tunnel at $t = 4$, $P^4$ discontinuous finite elements on 3951 triangles, 59 265 dofs

Implementation based on NGSolve, NGS-Py
Summary

Mapped Tent Pitching (MTP) method
Mapped Tent Pitching (MTP) method

+ high order in space and time
Mapped Tent Pitching (MTP) method

+ high order in space and time

+ local instead of global CFL-condition
Summary

Mapped Tent Pitching (MTP) method

+ high order in space and time

+ local instead of global CFL-condition

+ reduced size of local problems in contrast to SDG schemes
Mapped Tent Pitching (MTP) method

+ high order in space and time
+ local instead of global CFL-condition
+ reduced size of local problems in contrast to SDG schemes

− inverse transformation $G^{-1}(\hat{U})$ needed for non-linear equations
Outlook & references

Outlook

- adaptive tent pitching strategies
- convergence theory for explicit methods
- operator splitting for viscous terms

Thank you for your attention!
Outlook & references

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- adaptive tent pitching strategies
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Thank you for your attention!
### Outlook

- adaptive tent pitching strategies
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Outlook & references

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Thank you for your attention!