Convergence analysis of Galerkin boundary element methods for Maxwell’s eigenvalue problems

Gerhard Unger

Institut für Numerische Mathematik
Technische Universität Graz

Special Semester on Computational Methods in Science and Engineering:
Analysis and Numerics of Acoustic and Electromagnetic Problems,
October 17-22, 2016
RICAM, Linz
Outline

Boundary integral formulation of Maxwell’s eigenvalue problems

Numerical analysis of a Galerkin approximation of the BI-formulation

Numerical solution of the discrete eigenvalue problem

Numerical examples

Conclusions
Outline

Boundary integral formulation of Maxwell’s eigenvalue problems

Numerical analysis of a Galerkin approximation of the BI-formulation

Numerical solution of the discrete eigenvalue problem

Numerical examples

Conclusions
Maxwell’s eigenvalue problems (1/2)

\[ \Omega \subset \mathbb{R}^3 \text{ bounded Lipschitz domain, } \Omega^\text{ext} := \mathbb{R}^3 \setminus \overline{\Omega} \text{ simply connected.} \]

Time-harmonic electromagnetic waves: \( \mathcal{E}(x, t) = \text{Re}\{e^{-i\omega t}E(x)\} \).

Material parameter: \( \varepsilon, \mu > 0 \).

- **Interior eigenvalue problem**

  Find \( \omega \in \mathbb{C} \) and \( E \in H(\text{curl}; \Omega), \ E \neq 0 \), such that:

  \[
  \text{curl curl } E - \omega^2 \varepsilon \mu E = 0, \quad \text{div}(\varepsilon E) = 0 \quad \text{in } \Omega; \quad n \times E = 0 \quad \text{on } \partial \Omega
  \]

  Property: Eigenvalues \( \omega \) are **real** and **isolated**.

- **Exterior eigenvalue problem (Scattering-resonance problem)**

  Find \( \omega \in \mathbb{C} \) and \( E \in H_{\text{loc}}(\text{curl}; \Omega^\text{ext}), \ E \neq 0 \), such that:

  \[
  \text{curl curl } E - \omega^2 \varepsilon \mu E = 0 \quad \text{in } \Omega^\text{ext}, \quad n \times E = 0 \quad \text{on } \partial \Omega, \quad E \text{ is outgoing}
  \]

  Properties: E-values are **non-real** with negative imag. part and **isolated**. Imaginary part of \( \omega \) describes damping in time:

  \[
  \mathcal{E}(x, t) = e^{\text{Im}(\omega)t} \text{Re}\{e^{-i\text{Re}(\omega)t}E(x)\}
  \]
Maxwell’s eigenvalue problems (2/2)

Transmission evp for the scattering at an interface

- $\mu_1, \varepsilon_1$ in $\Omega$
- $\mu_2, \varepsilon_2$ in $\Omega^{\text{ext}}$

Find $\omega \in \mathbb{C}$ and $0 \neq (E_1, E_2) \in H(\text{curl}; \Omega) \times H_{\text{loc}}(\text{curl}; \Omega^{\text{ext}})$ s.t.:

\[
\begin{align*}
\text{curl curl } E_1 - \omega^2 \varepsilon_1 \mu_1 E_1 &= 0, \quad \text{div}(\varepsilon_1 E_1) = 0 \quad \text{in } \Omega, \\
\text{curl curl } E_2 - \omega^2 \varepsilon_2 \mu_2 E_2 &= 0, \quad \text{div}(\varepsilon_2 E_2) = 0 \quad \text{in } \Omega^{\text{ext}}, \\
\mathbf{n} \times E_1 &= \mathbf{n} \times E_2 \quad \text{on } \Gamma, \\
\mu_1^{-1} \mathbf{n} \times \text{curl } E_1 &= \mu_2^{-2} \mathbf{n} \times \text{curl } E_2 \quad \text{on } \Gamma, \\
E_2 \text{ is outgoing}
\end{align*}
\]
Maxwell’s eigenvalue problems (2/2)

Transmission evp for the scattering at an interface

- $\mu_1, \varepsilon_1$ in $\Omega$
- $\mu_2, \varepsilon_2$ in $\Omega^{\text{ext}}$

Find $\omega \in \mathbb{C}$ and $0 \neq (E_1, E_2) \in H(\text{curl}; \Omega) \times H_{\text{loc}}(\text{curl}; \Omega^{\text{ext}})$ s. t.:

\[
\begin{align*}
\text{curl curl } E_1 - \omega^2 \varepsilon_1 \mu_1 E_1 &= 0, & \text{div}(\varepsilon_1 E_1) &= 0 & \text{in } \Omega, \\
\text{curl curl } E_2 - \omega^2 \varepsilon_2 \mu_2 E_2 &= 0, & \text{div}(\varepsilon_2 E_2) &= 0 & \text{in } \Omega^{\text{ext}}, \\
\mathbf{n} \times E_1 &= \mathbf{n} \times E_2 & & \text{on } \Gamma, \\
\mu_1^{-1} \mathbf{n} \times \text{curl } E_1 &= \mu_2^{-2} \mathbf{n} \times \text{curl } E_2 & & \text{on } \Gamma,
\end{align*}
\]

$E_2$ is outgoing

Application: Resonances of dielectric and plasmonic nanostructures (frequency depended and non-positive material parameter).
Representation of eigenfunctions in terms of their traces

**Interior/exterior eigenpair** \((\omega, E_{\text{int/ext}})\) [Stratton-Chu]

\[
E_{\text{int/ext}}(x) = \pm SL(\kappa)(\gamma_N E_{\text{int/ext}})(x) + DL(\kappa)(\gamma_D E_{\text{int/ext}})(x),
\]

for \(x \in \Omega / \Omega^\text{ext}\) and where \(\kappa := \omega \sqrt{\varepsilon \mu}\).

**Aim:** Compute Neumann trace \(\gamma_N E_{\text{int/ext}}\) on boundary \(\partial \Omega =: \Gamma\)

- **Single layer potential:** \(SL(\kappa) : \mathcal{X}(\Gamma) \rightarrow H_{\text{loc}}(\text{curl}; \Omega \cup \Omega^\text{ext})\)

\[
SL(\kappa)(\mu)(x) := \int_\Gamma \frac{e^{i\kappa|x-y|}}{4\pi|x-y|} \mu(y) ds_y + \frac{1}{\kappa^2} \text{grad} \int_\Gamma \frac{e^{i\kappa|x-y|}}{4\pi|x-y|} (\text{div}_\Gamma \mu)(y) ds_y
\]

- **Dirichlet / Neumann trace**

\[
\gamma_D E := (\gamma E) \times n, \quad \gamma_N E := n \times \gamma(\text{curl} E)(\in \mathcal{X}(\Gamma))
\]

- **\(\mathcal{X}(\Gamma) := \gamma_D (H(\text{curl}; \Omega)) = \{ \mu \in H^{-\frac{1}{2}}_x(\Gamma) : \text{div}_\Gamma \mu \in H^{-1/2}(\Gamma) \} \)**
Boundary integral formulation of Maxwell evps

**Interior/exterior eigenpair** \((\kappa, E^{\text{int/ext}})\) satisfies

\[
0 = \gamma D E^{\text{int/ext}} = \pm \gamma D SL(\kappa)(\gamma_N^{\text{int/ext}} E^{\text{int/ext}}) =: S(\kappa)(\gamma_N^{\text{int/ext}} E^{\text{int/ext}}) \text{ on } \Gamma
\]
Boundary integral formulation of Maxwell evps

**Interior/exterior eigenpair** \((\kappa, \mathbf{E}^{\text{int/ext}})\) satisfies

\[
0 = \gamma_D \mathbf{E}^{\text{int/ext}} = \pm \gamma_D SL(\kappa)(\gamma_N^{\text{int/ext}} \mathbf{E}^{\text{int/ext}}) =: S(\kappa)(\gamma_N^{\text{int/ext}} \mathbf{E}^{\text{int/ext}}) \text{ on } \Gamma
\]

- Single layer boundary integral operator \(S(\kappa) : \mathcal{X}(\Gamma) \rightarrow \mathcal{X}(\Gamma)\):

\[
S(\kappa) \mu(x) := \gamma_D SL(\kappa)(\mu)(x)
\]

\[
= \gamma_D \left[ \int_{\Gamma} \frac{e^{i\kappa|x-y|}}{4\pi|x-y|} \mu(y) ds_y + \frac{1}{\kappa^2} \nabla \int_{\Gamma} \frac{e^{i\kappa|x-y|}}{4\pi|x-y|} (\text{div}_\Gamma \mu)(y) ds_y \right]
\]
Boundary integral formulation of Maxwell evps

Interior/exterior eigenpair \((\kappa, \mathbf{E}^{\text{int/ext}})\) satisfies

\[
0 = \gamma D \mathbf{E}^{\text{int/ext}} = \pm \gamma D S\mathcal{L}(\kappa)\left(\mathbf{E}^{\text{int/ext}}\right) =: S(\kappa)\left(\mathbf{E}^{\text{int/ext}}\right) \text{ on } \Gamma
\]

Single layer boundary integral operator \(S(\kappa) : \mathcal{X}(\Gamma) \to \mathcal{X}(\Gamma)\):

\[
S(\kappa) \mu(x) := \gamma D S\mathcal{L}(\kappa)(\mu)(x)
\]

\[
= \gamma D \left[ \int_{\Gamma} \frac{e^{i\kappa|x-y|}}{4\pi|x-y|} \mu(y) \text{d}s_y + \frac{1}{\kappa^2} \nabla \int_{\Gamma} \frac{e^{i\kappa|x-y|}}{4\pi|x-y|} (\text{div}_\Gamma \mu)(y) \text{d}s_y \right]
\]

Eigenvalue problem formulation on \(\Gamma\)

Find \(\kappa \in \mathbb{C}\) and \(\mu \in \mathcal{X}(\Gamma) \setminus \{0\}\) such that:

\[S(\kappa) \mu = 0\]
Boundary integral formulation of Maxwell evps

**Interior/exterior eigenpair** \((\kappa, \mathbf{E}^{\text{int/ext}})\) satisfies

\[
0 = \gamma_D \mathbf{E}^{\text{int/ext}} = \pm \gamma_D S(L)(\gamma_N^{\text{int/ext}} \mathbf{E}^{\text{int/ext}}) =: S(\kappa)(\gamma_N^{\text{int/ext}} \mathbf{E}^{\text{int/ext}}) \text{ on } \Gamma
\]

- Single layer boundary integral operator \(S(\kappa) : \mathcal{X}(\Gamma) \rightarrow \mathcal{X}(\Gamma)\):

\[
S(\kappa) \mu(x) := \gamma_D S(L)(\kappa)(\mu)(x)
\]

\[
= \gamma_D \left[ \int_{\Gamma} \frac{e^{i \kappa|x-y|}}{4\pi|x-y|} \mu(y) ds_y + \frac{1}{\kappa^2} \nabla \int_{\Gamma} \frac{e^{i \kappa|x-y|}}{4\pi|x-y|} \left( \text{div}_\Gamma \mu \right)(y) ds_y \right]
\]

**Eigenvalue problem formulation on \(\Gamma\)**

Find \(\kappa \in \mathbb{C}\) and \(\mu \in \mathcal{X}(\Gamma) \setminus \{0\}\) such that:

\[
S(\kappa) \mu = 0
\]

- Eigenvalue problem is **nonlinear** w. r. t. the eigenvalue
- Eigenvalue problem is **linear** w. r. t. the eigenfunction \(\mu\)

On the discrete level nonlinear evp will be reduced to an equivalent linear evp
Equivalence of eigenvalue problem formulations

We have seen:
If \((\omega, \mathbf{E})\) with \(\mathbf{E} \in H_{(\text{loc})}(\mathbf{curl}; \Omega) \setminus \{0\}\), is an eigenpair of Maxwell’s interior or exterior eigenvalue problem, then for \(\kappa = \omega \sqrt{\varepsilon \mu}\) we have

\[ S(\kappa) \gamma_N \mathbf{E} = 0.\]

Assume that \((\kappa, \mu), \kappa \in \mathbb{C}\) and \(\mu \in \mathcal{X}(\Gamma) \setminus \{0\}\), satisfies

\[ S(\kappa) \mu = 0,\]

then:

- If \(\kappa \in \mathbb{R}\), then \((\frac{\kappa}{\sqrt{\varepsilon \mu}}, SL(\kappa)(\mu))\) is an e-pair of the interior evp.

- If \(\kappa \notin \mathbb{R}\), then \((\frac{\kappa}{\sqrt{\varepsilon \mu}}, SL(\kappa)(\mu))\) is an e-pair of the exterior evp.
Equivalence of eigenvalue problem formulations

We have seen:
If \((\omega, \mathbf{E})\) with \(\mathbf{E} \in H_{(\text{loc})}(\text{curl}; \Omega) \setminus \{0\}\), is an eigenpair of Maxwell’s interior or exterior eigenvalue problem, then for \(\kappa = \omega \sqrt{\varepsilon \mu}\) we have

\[
S(\kappa) \gamma_N \mathbf{E} = 0.
\]

Assume that \((\kappa, \mu), \kappa \in \mathbb{C}\) and \(\mu \in \mathcal{X}(\Gamma) \setminus \{0\}\), satisfies

\[
S(\kappa) \mu = 0,
\]

then:
- If \(\kappa \in \mathbb{R}\), then \((\frac{\kappa}{\sqrt{\varepsilon \mu}}, SL(\kappa)(\mu))\) is an e-pair of the interior evp.
- If \(\kappa \notin \mathbb{R}\), then \((\frac{\kappa}{\sqrt{\varepsilon \mu}}, SL(\kappa)(\mu))\) is an e-pair of the exterior evp.

Remark: Combined integral formulation as suggested in Buffa, Hiptmair (2004) for bvps leads to a perturbation of the interior e-values such that they are shifted to the upper complex half-plane.
Crucial properties of the BI formulation of the evp

Find $\kappa \in \mathbb{C}$ and $\mu \in \mathcal{X}(\Gamma) \setminus \{0\}$, such that:

$$S(\kappa)\mu = 0$$

- $S : \mathbb{C} \setminus \{0\} \to \mathcal{L}(\mathcal{X}(\Gamma), \mathcal{X}(\Gamma))$ is holomorphic

[Buffa, Christiansen, Costabel, Hiptmair, Schwab, ... (2001-2004)].

Conclusion: Evp can be treated in the framework of evps for holomorphic Fredholm operator functions which is a natural extension of evps of linear compact operators (same conceptional tools: Jordan chains, algebraic multiplicities, ...)

[Gohberg, Sigal, Keldesh, Wendland, M"oller, ...].

G. Unger

Convergence analysis of BEM for Maxwell’s eigenvalue problems

Crucial properties of the BI formulation of the evp

Find $\kappa \in \mathbb{C}$ and $\mu \in \mathcal{X}(\Gamma) \setminus \{0\}$, such that:

$$S(\kappa)\mu = 0$$

- $S : \mathbb{C} \setminus \{0\} \to \mathcal{L}(\mathcal{X}(\Gamma), \mathcal{X}(\Gamma))$ is holomorphic

- $S(\kappa)$ satisfies a generalized Gårding’s inequality, i.e.,
  $$\exists C(\kappa), T \in \mathcal{L}(\mathcal{X}(\Gamma), \mathcal{X}(\Gamma)), C(\kappa) \text{ compact and } T \text{ invertible}$$
  $$\text{Re} \left( \langle S(\kappa)\mu, T\overline{\mu} \rangle + \langle C(\kappa)\mu, \overline{\mu} \rangle \right) \geq c\|\mu\|_{\mathcal{X}(\Gamma)}^2 \quad \forall \mu \in \mathcal{X}(\Gamma),$$

[Buffa, Christiansen, Costabel, Hiptmair, Schwab, ...(2001-2004)].
Crucial properties of the BI formulation of the evp

Find \( \kappa \in \mathbb{C} \) and \( \mu \in \mathcal{X}(\Gamma) \setminus \{0\} \), such that:

\[
S(\kappa)\mu = 0
\]

- \( S : \mathbb{C} \setminus \{0\} \to \mathcal{L}(\mathcal{X}(\Gamma), \mathcal{X}(\Gamma)) \) is holomorphic
- \( S(\kappa) \) satisfies a generalized Gårding’s inequality, i.e.,
\[
\exists C(\kappa), T \in \mathcal{L}(\mathcal{X}(\Gamma), \mathcal{X}(\Gamma)), C(\kappa) \text{ compact and } T \text{ invertible}
\]
\[
\text{Re} \left( \langle S(\kappa)\mu, T\bar{\mu} \rangle + \langle C(\kappa)\mu, \bar{\mu} \rangle \right) \geq c\|\mu\|^2_{\mathcal{X}(\Gamma)} \quad \forall \mu \in \mathcal{X}(\Gamma),
\]

[Buffa, Christiansen, Costabel, Hiptmair, Schwab, ... (2001-2004)]. Here \( T \) is associated with the stable splitting \( \mathcal{X}(\Gamma) = \mathcal{V}(\Gamma) \oplus \mathcal{W}(\Gamma) \):

\[
T : \mathcal{V}(\Gamma) \oplus \mathcal{W}(\Gamma) \mapsto \mathcal{X}(\Gamma)
\]

\[
\nu + \omega \mapsto \nu - \omega
\]

where \( \mathcal{W}(\Gamma) = \ker(\text{div}_\Gamma) \cap \mathcal{X}(\Gamma) \) and \( \mathcal{V}(\Gamma) = \nabla_\Gamma H^{-\frac{1}{2}}(\Delta_\Gamma, \Gamma) \)
Crucial properties of the BI formulation of the evp

Find $\kappa \in \mathbb{C}$ and $\mu \in \mathcal{X}(\Gamma) \setminus \{0\}$, such that:

$$S(\kappa)\mu = 0$$

- $S : \mathbb{C} \setminus \{0\} \rightarrow \mathcal{L}(\mathcal{X}(\Gamma), \mathcal{X}(\Gamma))$ is holomorphic
- $S(\kappa)$ satisfies a generalized Gårding’s inequality, i.e.,
  $$\exists C(\kappa), T \in \mathcal{L}(\mathcal{X}(\Gamma), \mathcal{X}(\Gamma)), C(\kappa) \text{ compact and } T \text{ invertible}$$
  $$\Re \left( \langle S(\kappa)\mu, T\overline{\mu} \rangle + \langle C(\kappa)\mu, \overline{\mu} \rangle \right) \geq c \|\mu\|_{\mathcal{X}(\Gamma)}^2 \quad \forall \mu \in \mathcal{X}(\Gamma),$$

[Buffa, Christiansen, Costabel, Hiptmair, Schwab, ... (2001-2004)].

**Conclusion**: Evp can be treated in the framework of evps for holomorphic Fredholm operator functions which is a natural extension of evps of linear compact operators (same conceptional tools: Jordan chains, algebraic multiplicities, ...) [Gohberg, Sigal, Keldesh, Wendland, Möller, ...].
Outline

Boundary integral formulation of Maxwell’s eigenvalue problems

Numerical analysis of a Galerkin approximation of the BI-formulation

Numerical solution of the discrete eigenvalue problem

Numerical examples

Conclusions
Conforming Galerkin approximation of the evp

**Approximation space:** $\mathcal{RT}_p(\Gamma_h)$ Raviart-Thomas elements of order $p$

**Galerkin variational formulation**

Find $\kappa_h \in \mathbb{C} \setminus \{0\}$, and $\mu_h \in \mathcal{RT}_p(\Gamma_h) \setminus \{0\}$ s. t.

$$\langle S(\kappa_h)\mu_h, \chi_h \rangle = 0 \quad \forall \chi_h \in \mathcal{RT}_p(\Gamma_h).$$

This leads to a **nonlinear (holomorphic) matrix eigenvalue problem:**

$$S_h(\kappa_h)\mu = 0.$$
Conforming Galerkin approximation of the evp

Approximation space: $\mathcal{RT}_p(\Gamma_h)$ Raviart-Thomas elements of order $p$

**Galerkin variational formulation**

Find $\kappa_h \in \mathbb{C} \setminus \{0\}$, and $\mu_h \in \mathcal{RT}_p(\Gamma_h) \setminus \{0\}$ s. t.

$$\langle S(\kappa_h)\mu_h, \chi_h \rangle = 0 \quad \forall \chi_h \in \mathcal{RT}_p(\Gamma_h).$$

This leads to a **nonlinear (holomorphic) matrix eigenvalue problem**:

$$S_h(\kappa_h)\mu = 0.$$

Application of abstract convergence results of [KARMA (1996)] ([Stummel, Vainikko, Gregorieff, Jeggle, Wendland, ...]) requires

- a regular approximation of $S(\kappa)$ in $\mathcal{RT}_p(\Gamma_h)$,

i. e., for every bounded sequence $(\mu_h)_h \subset \mathcal{RT}_p(\Gamma_h)$ which induces a **compact** sequence $(P_h S(\kappa) \mu_h)_h$ it follows that $(\mu_h)_h$ is already **compact**.
Sufficient condition for regular approximation of T-Gårding operators for Galerkin schemes

Abstract result for sufficient condition for a regular approximation of T-Gårding operators for Galerkin schemes:


**Theorem**

*If there exists a $(T_h)_h$, $T_h : RT_p(\Gamma_h) \to RT_p(\Gamma_h)$, linear and continuous, such that

$$
\sup_{\mu_h \in RT_p(\Gamma_h) \{0\}} \frac{\| (T - T_h) \mu_h \| \chi(\Gamma)}{\| \mu_h \| \chi(\Gamma)} \to 0 \quad \text{as } h \to 0,
$$

then the Galerkin approximation of $S(\kappa)$ in $RT_p(\Gamma_h)$ is regular.*
Regular approximation of $S(\kappa)$ by $(\mathcal{R}\mathcal{T}_p(\Gamma_h))_h$

The regular approximation of $S(\kappa)$ by $(\mathcal{R}\mathcal{T}_p(\Gamma_h))_h$ follows directly from:

**Lemma**

The operator

$$T_h := P_h T : \mathcal{R}\mathcal{T}_p(\Gamma_h) \to \mathcal{R}\mathcal{T}_p(\Gamma_h),$$

satisfies

$$\sup_{\mu_h \in \mathcal{R}\mathcal{T}_p(\Gamma_h) \setminus \{0\}} \frac{\|(T - T_h)\mu_h\|_{\mathcal{X}(\Gamma)}}{\|\mu_h\|_{\mathcal{X}(\Gamma)}} \to 0 \quad \text{as} \quad h \to 0.$$

Proof is based on the gap property of the decomposition of the discrete space, i.e.,

- $\mathcal{R}\mathcal{T}_p(\Gamma_h) = \mathcal{V}_h \oplus \mathcal{W}_h$,
- $\max\{\delta(\mathcal{V}_h, \mathcal{V}), \delta(\mathcal{W}_h, \mathcal{W})\} \to 0$ as $h \to 0$,

and is done by combining proofs/results from

Convergence results (1/2)

Application of abstract convergence results of KARMA (1996):

**Theorem (Completeness of the discrete spectrum)**

For every e-value $\kappa$ of the evp $S(\kappa)\mu = 0$ there exists a sequence $\{\kappa_h\}_h$ of e-values and related e-functions $\{\mu_h\}_h$ of the Galerkin evp s. t.

$$\kappa_h \to \kappa \quad \text{as } h \to 0,$$

and there exists at least one limit point of $\{\mu_h\}_h$ and every limit point is an e-function of $\kappa$. 
Convergence results (1/2)

Application of abstract convergence results of Karma (1996):

**Theorem (Completeness of the discrete spectrum)**

For every eigenvalue $\kappa$ of the EVP $S(\kappa)\mu = 0$ there exists a sequence $\{\kappa_h\}_h$ of eigenvalues and related eigenfunctions $\{\mu_h\}_h$ of the Galerkin EVP such that

$$\kappa_h \to \kappa \quad \text{as } h \to 0,$$

and there exists at least one limit point of $\{\mu_h\}_h$ and every limit point is an eigenfunction of $\kappa$.

**Theorem (Non-pollution of the discrete spectrum)**

Let $K \subset \mathbb{C} \setminus \{0\}$ be a compact set such that in $K$ there are no eigenvalues of the EVP $S(\kappa)\mu = 0$.

Then for sufficiently fine discretizations there are no eigenvalues of the Galerkin problem in $K$. 
Theorem (Error estimates for semi-simple e-values)

Let \( \Lambda_0 \subset \mathbb{C} \setminus \{0\} \) be compact. Suppose that \( \kappa \) is in \( \Lambda_0 \) the only e-value of the evp \( S(\kappa)\mu = 0 \) and that it is semi-simple. Then, \( \exists h_0 > 0 \ \exists c > 0 \) such that for all \( 0 < h \leq h_0 \) we have

\[
|\kappa - \kappa_h| \leq c\delta_{h,p}(\kappa)^2 \quad \forall \kappa_h \in \Lambda_0,
\]

\[
\inf_{\mu \in \ker S(\kappa)} \|\mu - \mu_h\|_{X(\Gamma)} \leq c \left( |\kappa - \kappa_h| + \delta_{h,p}(\kappa) \right)
\]

for all e-pairs \((\kappa_h, \mu_h)\) of the Galerkin evp with \( \|\mu_h\|_{X(\Gamma)} = 1 \), where

\[
\delta_{h,p}(\kappa) = \delta \left( \ker S(\kappa), \mathcal{RT}_p(\Gamma_h) \right).
\]

- Convergence order for non-semisimple e-values:

\[
|\kappa - \kappa_h| = \mathcal{O}(\varepsilon_{h,p}(\kappa)^{2/\alpha}).
\]
Convergence results (2/2)

Theorem (Error estimates for semi-simple e-values)

Let $\Lambda_0 \subset \mathbb{C} \setminus \{0\}$ be compact.
Suppose that $\kappa$ is in $\Lambda_0$ the only e-value of the evp $S(\kappa)\mu = 0$ and that it is semi-simple. Then, $\exists h_0 > 0 \ \exists c > 0$ such that for all $0 < h \leq h_0$ we have

$$|\kappa - \kappa_h| \leq c \delta_{h,p}(\kappa)^2 \quad \forall \kappa_h \in \Lambda_0,$$

$$\inf_{\mu \in \ker S(\kappa)} \|\mu - \mu_h\|_{X(\Gamma)} \leq c (|\kappa - \kappa_h| + \delta_{h,p}(\kappa))$$

for all e-pairs $(\kappa_h, \mu_h)$ of the Galerkin evp with $\|\mu_h\|_{X(\Gamma)} = 1$, where

$$\delta_{h,p}(\kappa) = \delta (\ker S(\kappa), \mathcal{RT}_p(\Gamma_h)).$$

- Convergence order for non-semisimple e-values:
  $$|\kappa - \kappa_h| = \mathcal{O}(\varepsilon_{h,p}(\kappa)^2/\alpha).$$
- For sufficiently smooth e- functions: $|\kappa - \kappa_h| = \mathcal{O}(h^{2p+3})!$
Convergence results (2/2)

**Theorem (Error estimates for semi-simple e-values)**

Let $\Lambda_0 \subset \mathbb{C} \setminus \{0\}$ be compact. Suppose that $\kappa$ is in $\Lambda_0$ the only e-value of the evp $S(\kappa)\mu = 0$ and that it is semi-simple. Then, $\exists h_0 > 0 \exists c > 0$ such that for all $0 < h \leq h_0$ we have

$$|\kappa - \kappa_h| \leq c\delta_{h,p}(\kappa)^2, \quad \forall \kappa_h \in \Lambda_0,$$

$$\inf_{\mu \in \ker S(\kappa)} \|\mu - \mu_h\|_{X(\Gamma)} \leq c (|\kappa - \kappa_h| + \delta_{h,p}(\kappa))$$

for all e-pairs $(\kappa_h, \mu_h)$ of the Galerkin evp with $\|\mu_h\|_{X(\Gamma)} = 1$, where

$$\delta_{h,p}(\kappa) = \delta (\ker S(\kappa), \mathcal{RT}_{p}(\Gamma_h)).$$

- Convergence order for non-semisimple e-values:
  $$|\kappa - \kappa_h| = \mathcal{O}(\varepsilon_{h,p}(\kappa)^2/\alpha).$$

- For sufficiently smooth e-functions: $|\kappa - \kappa_h| = \mathcal{O}(h^{2p+3})!$

- Interior e-values are all semi-simple.
Transmission evp for the scattering at a surface

**Boundary integral formulation**

\[
\begin{pmatrix}
M_1(\omega) + M_2(\omega) & \mu_1 S_1(\omega) + \mu_2 S_2(\omega) \\
\omega^2 \varepsilon_1 S_1(\omega) + \omega^2 \varepsilon_2 S_2(\omega) & M_1(\omega) + M_2(\omega)
\end{pmatrix}
\begin{pmatrix}
\gamma_D E \\
\gamma_N E
\end{pmatrix} = 0, \\
:= A(\omega)
\]

\[M_i(\omega) := \frac{1}{2} \left( \gamma_N^{\text{int}} + \gamma_N^{\text{ext}} \right) \omega \sqrt{\varepsilon_i \mu_i} SL(\omega \sqrt{\varepsilon_i \mu_i}), \quad S_i(\omega) := S(\omega \sqrt{\varepsilon_i \mu_i}).\]
Transmission evp for the scattering at a surface

**Boundary integral formulation**

\[
\begin{pmatrix}
M_1(\omega) + M_2(\omega) & \mu_1 S_1(\omega) + \mu_2 S_2(\omega) \\
\omega^2 \varepsilon_1 S_1(\omega) + \omega^2 \varepsilon_2 S_2(\omega) & M_1(\omega) + M_2(\omega)
\end{pmatrix} \begin{pmatrix}
\gamma_D E \\
\gamma_N E
\end{pmatrix} = 0,
\]

\[M_i(\omega) := \frac{1}{2} \left( \gamma_N^{int} + \gamma_N^{ext} \right) \omega \sqrt{\varepsilon_i \mu_i} S L(\omega \sqrt{\varepsilon_i \mu_i}), \quad S_i(\omega) := S(\omega \sqrt{\varepsilon_i \mu_i}).\]

- **A(\omega)** satisfies a generalized Gårding’s-inequality with \(T \leftrightarrow \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}\)
  
  [Buffa, Hiptmair, von Petersdorff, Schwab (2003)]

- Analogous convergence analysis for Galerkin approximation of above evp is possible as for \(S(\kappa) \mu = 0\).
Transmission evp for the scattering at a surface

**Boundary integral formulation**

\[
\begin{pmatrix}
M_1(\omega) + M_2(\omega) & \mu_1 S_1(\omega) + \mu_2 S_2(\omega) \\
\omega^2 \varepsilon_1 S_1(\omega) + \omega^2 \varepsilon_2 S_2(\omega) & M_1(\omega) + M_2(\omega)
\end{pmatrix}
\begin{pmatrix}
\gamma_D E \\
\gamma_N E
\end{pmatrix}
= 0,
\]

\[:=A(\omega)\]

\[M_i(\omega) := \frac{1}{2} \left( \gamma_N^{\text{int}} + \gamma_N^{\text{ext}} \right) \omega \sqrt{\varepsilon_i \mu_i} SL(\omega \sqrt{\varepsilon_i \mu_i}), \quad S_i(\omega) := S(\omega \sqrt{\varepsilon_i \mu_i}).\]

- \(A(\omega)\) satisfies a generalized Gårding’s-inequality with \(T \longleftrightarrow \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}\)

[Buffa, Hiptmair, von Petersdorff, Schwab (2003)]

- Analogous convergence analysis for Galerkin approximation of above evp is possible as for \(S(\kappa) \mu = 0\).

**Remark:** The same analysis is valid if \(\varepsilon\) depends holomorphically on \(\omega\).
Outline

Boundary integral formulation of Maxwell’s eigenvalue problems

Numerical analysis of a Galerkin approximation of the BI-formulation

Numerical solution of the discrete eigenvalue problem

Numerical examples

Conclusions
Contour integral method for nonlinear eigenvalue problems

Introduced by Sakurai et al. 2009, Beyn 2012
Contour integral method for nonlinear eigenvalue problems

Introduced by Sakurai et al. 2009, Beyn 2012

- **Property:** Finds all e-values (and e-vectors) which lie inside of a given contour without any initial approximations in a reliable way.

![Diagram of contour integral method](image)
Contour integral method for nonlinear eigenvalue problems

Introduced by Sakurai et al. 2009, Beyn 2012

- **Property:** Finds all e-values (and e-vectors) which lie inside of a given contour without any initial approximations in a reliable way.

- **Idea:** Reduction to an equivalent *linear* evp which has the same e-values as the nonlinear evp inside the contour by contour int. of the resolvent:

\[
\int_C f(\xi)S_h(\xi)^{-1} d\xi = \sum_{j=1}^{M} f(\kappa(j))\mu^{(j)}\nu^{(j)H}.
\]
Contour integral method for nonlinear eigenvalue problems

Introduced by Sakurai et al. 2009, Beyn 2012

- **Property:** Finds all e-values (and e-vectors) which lie inside of a given contour without any initial approximations in a reliable way.

- **Idea:** Reduction to an equivalent linear evp which has the same e-values as the nonlinear evp inside the contour by contour int. of the resolvent:

\[
\int_C f(\xi)S_h(\xi)^{-1}d\xi = \sum_{j=1}^{M} f(\kappa^{(j)})\mu^{(j)}\nu^{(j)H}.
\]

- **Main computational cost:** Solution of several linear systems.
Contour integral method for nonlinear eigenvalue problems

Introduced by Sakurai et al. 2009, Beyn 2012

- Property: Finds all e-values (and e-vectors) which lie inside of a given contour without any initial approximations in a reliable way.

- Idea: Reduction to an equivalent linear evp which has the same e-values as the nonlinear evp inside the contour by contour int. of the resolvent:

\[
\int_C f(\xi)S_h(\xi)^{-1} d\xi = \sum_{j=1}^{M} f(\kappa^{(j)})\mu^{(j)}\nu^{(j)H}.
\]

- Main computational cost: Solution of several linear systems.

- Contour int. method approximates the poles of the resolvent $S_h(\kappa)^{-1}$. 

G. Unger

Convergence analysis of BEM for Maxwell’s eigenvalue problems 21.10.2016 18 / 25
Reasons for particular choice of BI formulation of evp

The evps for the following operator functions are equivalent in $\mathbb{C} \setminus \{0\}$:

i)

$$S(\kappa) = A(\kappa) + \frac{1}{\kappa^2} \text{curl}_\Gamma \circ V(\kappa) \circ \text{div}_\Gamma$$

ii)

$$\kappa S(\kappa) = \kappa A(\kappa) + \frac{1}{\kappa} \text{curl}_\Gamma \circ V(\kappa) \circ \text{div}_\Gamma$$

iii)

$$\kappa^2 S(\kappa) = \kappa^2 A(\kappa) + \text{curl}_\Gamma \circ V(\kappa) \circ \text{div}_\Gamma$$
Reasons for particular choice of BI formulation of evp

The evps for the following operator functions are equivalent in $\mathbb{C} \setminus \{0\}$:

i) 

\[ S(\kappa) = A(\kappa) + \frac{1}{\kappa^2} \text{curl}_\Gamma \circ V(\kappa) \circ \text{div}_\Gamma \]

- has a pole at $\kappa = 0$

ii) 

\[ \kappa S(\kappa) = \kappa A(\kappa) + \frac{1}{\kappa} \text{curl}_\Gamma \circ V(\kappa) \circ \text{div}_\Gamma \]

- has pole at $\kappa = 0$

iii) 

\[ \kappa^2 S(\kappa) = \kappa^2 A(\kappa) + \text{curl}_\Gamma \circ V(\kappa) \circ \text{div}_\Gamma \]

- has no pole at $\kappa = 0$
Reasons for particular choice of BI formulation of evp

The evps for the following operator functions are equivalent in $\mathbb{C} \setminus \{0\}$:

i) $S(\kappa) = A(\kappa) + \frac{1}{\kappa^2} \text{curl}_\Gamma \circ V(\kappa) \circ \text{div}_\Gamma$

- has a pole at $\kappa = 0$
- resolvent has no pole at $\kappa = 0$ $\rightarrow$ contour int. method is not effected

ii) $\kappa S(\kappa) = \kappa A(\kappa) + \frac{1}{\kappa} \text{curl}_\Gamma \circ V(\kappa) \circ \text{div}_\Gamma$

- has pole at $\kappa = 0$
- resolvent has a pole at $\kappa = 0$; eigenspace $= \ker \text{div}_\Gamma$

iii) $\kappa^2 S(\kappa) = \kappa^2 A(\kappa) + \text{curl}_\Gamma \circ V(\kappa) \circ \text{div}_\Gamma$

- has no pole at $\kappa = 0$
- resolvent has a pole at $\kappa = 0$; eigenspace $= \ker \text{div}_\Gamma$
Outline

Boundary integral formulation of Maxwell’s eigenvalue problems

Numerical analysis of a Galerkin approximation of the BI-formulation

Numerical solution of the discrete eigenvalue problem

Numerical examples

Conclusions
Interior eigenvalue problem

We refer to


- Several numerical examples for different geometries.
- Comparison with FEM-computations.
- Computations of photonic band gaps.

Numerical examples confirm the theoretical results.
Transmission evp for the scattering at a dielectric sphere

- Software BEM++ ($\mathcal{RT}_0(\Gamma_h)$), contour integral method
- Reference solutions by Mie theory ($\varepsilon_1 = 4.0$, $\varepsilon_2 = \mu_1 = \mu_2 = 1.0$)

| $h$  | dof | $|\kappa_1 - \kappa_{1,h}|$ | eoc | $|\kappa_2 - \kappa_{2,h}|$ | eoc |
|------|-----|-----------------|-----|-----------------|-----|
| 0.4  | 144 | 6.38e-2         | -   | 8.67e-2         | -   |
| 0.2  | 906 | 9.96e-3         | 2.7 | 1.40e-2         | 2.6 |
| 0.1  | 3855| 2.24e-3         | 2.2 | 3.18e-3         | 2.1 |
| 0.05 | 16620| 5.15e-4        | 2.1 | 7.37e-4         | 2.1 |
Transmission e-values for the scattering at a dielectric cube

- $\varepsilon_1 = 4.0$, $\varepsilon_2 = \mu_1 = \mu_2 = 1.0$.
- Reference solutions are computed solutions for $h = 0.012$.

| $h$  | dof | $|\kappa_1 - \tilde{\kappa}_1,h|$ | eoc | $|\kappa_2 - \tilde{\kappa}_2,h|$ | eoc |
|------|-----|-------------------------------|-----|-------------------------------|-----|
| 0.707 | 72  | 7.94e-2                       |     | 1.24e-1                       |     |
| 0.354 | 288 | 6.14e-3                       | 3.7 | 1.21e-2                       | 3.4 |
| 0.177 | 1152| 6.05e-4                       | 3.3 | 1.16e-3                       | 3.4 |
| 0.088 | 4608| 6.21e-5                       | 3.3 | 1.17e-4                       | 3.4 |

G. Unger
Convergence analysis of BEM for Maxwell's eigenvalue problems
23 / 25
Outline

Boundary integral formulation of Maxwell’s eigenvalue problems

Numerical analysis of a Galerkin approximation of the BI-formulation

Numerical solution of the discrete eigenvalue problem

Numerical examples

Conclusions
Conclusions

- Rigorous numerical analysis of conforming Galerkin approximations of BI formulations of Maxwell’s interior, exterior and transmission evp with Raviart-Thomas elements.
- Theoretical a-priori convergence orders for e-value approximation are consistent with observed convergence orders in numerical examples.
- Reliable numerical solution of discretized evp with contour integral method.