Transformation of corner singularities in presence of small or large parameters

Monique Dauge

IRMAR, Université de Rennes 1, FRANCE

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http://perso.univ-rennes1.fr/monique.dauge

Outline

1. Joint works and discussions
2. Example 1: Self-similar perturbations of a 2D corner
3. Example 2: Neumann-Robin boundary conditions
4. Example 3: Thin layers
5. Example 4: Conducting material in 2D eddy current formulation
6. Profiles and corner layers

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   - Conclusion

Joint works

M. COSTABEL, M. DAUGE, Example 2

A singularly perturbed mixed boundary value problem,
Comm. Partial Differential Equations 21, 11 & 12 (1996), 1919–1949.

G. CALOZ, M. COSTABEL, M. DAUGE, G. VIAL Example 3

Asymptotic expansion of the solution of an interface problem in a polygonal domain with thin layer *Asymptotic Analysis* 50 (1/2) (2006), 121–173.

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M. DAUGE P. DULAR, L. KRÄHENBÜHL, V. PÉRON, R. PERRUSSEL, C. POIGNARD Example 4

Corner asymptotics of the magnetic potential in the eddy-current model
Mathematical Methods in the Applied Sciences 37, 13 (2014) 1924–1955

M. COSTABEL, M. DALLA RIVA, M. DAUGE, P. MUSOLINO Example 1

Converging expansions for Lipschitz self-similar perforations of a plane sector,
In preparation, (2016).

2D eddy current formulation for a conductor surrounded by a dielectric medium
Recent discussions with R. HIPTMAIR, R. CASAGRANDE, K. SCHMIDT, A. SEMIN

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Example 1: Self-similar perturbations of a 2D corner

See references

1. Let Γ be the infinite plane sector

$$\Gamma = \{x = (x_1, x_2) \in \mathbb{R}^2, \quad r > 0,\ \theta \in (0, \omega)\}$$

with polar coordinates (r, θ) and the opening $\omega \in (0, \pi) \cup (\pi, 2\pi)$.

2. Bounded un-perturbed domain $\Omega \equiv \Omega_0$, with $\Omega \subset \Gamma$. Assume
   - Ω bounded connected with curvilinear polygonal boundary,
   - $\exists r_0 > 0$ such that

$$\mathscr{B}(0, r_0) \cap \Omega = \mathscr{B}(0, r_0) \cap \Gamma$$

3. Bounded perturbation pattern P, with $\mathrm{P} \subset \Gamma$.
   - P connected with curvilinear polygonal boundary,
   - $\exists R_0 > 0$ such that

$$\mathscr{B}(0, R_0)^{\complement} \cap \mathrm{P} = \mathscr{B}(0, R_0)^{\complement} \cap \Gamma$$

4. Family of perturbed domains, with $\varepsilon \in (0, \varepsilon_0)$

$$\Omega_\varepsilon = \Omega \cap \varepsilon \mathrm{P}$$

Dirichlet Problems for Example 1

Simplifying assumption

- $f \in L^2(\Omega)$ with support outside a ball $\mathscr{B}(0, r_1)$

General assumption

- $f \in L^2(\Omega)$ analytic or $\mathscr{C}^\infty$ inside a ball $\mathscr{B}(0, r_1)$

We choose the simplifying assumptions for the talk because

- A zero Taylor expansion of f at the corners allows to discard polynomials from corner asymptotics, so to avoid log terms and convergence issues.

The family of problems under consideration is the family of Dirichlet problems for $\varepsilon \in [0, \varepsilon_0)$

$$(\mathscr{P}_\varepsilon) \qquad \begin{cases} -\Delta u_\varepsilon = f & \text{in } \Omega_\varepsilon \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

Unique solution in $u_\varepsilon \in H_0^1(\Omega_\varepsilon)$

Two particular cases of Example 1

Question

- Behavior of u_ε close to 0 (in the region $\mathscr{B}(0, r_0) \cap \Omega_\varepsilon$) as $\varepsilon \to 0$.

Let us consider two examples

1a A rounded corner: The boundary of P is smooth!

1b A cracked corner: The boundary of P has a crack Σ abutting at 0 ($P = \Gamma \setminus \Sigma$, with a straight segment Σ).

Set

$$\boxed{\alpha = \frac{\pi}{\omega}}$$

In both examples, $\Omega = \Omega_0$ has a corner of opening ω at 0. Therefore [Kondrat'ev 1967] u_0 has an expansion as

$$u_0 = u_{0,\mathrm{reg}}^K + \sum_{k=1}^{K} \gamma_k r^{k\alpha} \sin k\alpha\theta, \quad u_{0,\mathrm{reg}}^K \in H^{m+1}(\Omega \cap \mathscr{B}(0, r_0)), \quad m = [(K+1)\alpha]$$

Paradoxes on singularities for Example 1

On the one hand

$$u_0 = \gamma_1 r^\alpha \sin \alpha\theta + u_{0,\text{reg}}, \quad u_{0,\text{reg}} \in H^2(\Omega \cap \mathscr{B}(0, r_0))$$

On the other hand

1a Rounded corner: u_ε is smooth in $\Omega_\varepsilon \cap \mathscr{B}(0, r_0)$

1b A cracked corner: The first singularity of u_ε behaves like $r^{\frac{1}{2}} \sin \frac{\theta}{2}$

Thus

1a u_ε has no singularity but u_0 has one

1b u_ε has a stronger singularity than u_0

How can the singularity $r^\alpha \sin \alpha\theta$ associated with $\varepsilon = 0$

1a Disappear as soon as $\varepsilon > 0$

1b Transform into a stronger singularity as soon as $\varepsilon > 0$

?

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Example 2: Neumann-Robin boundary conditions

See references

- Take a polygon Ω with straight sides
- Choose a side, denoted by Σ_R. Denote $\partial\Omega \setminus \Sigma_R$ by Σ_N
- Assume for simplicity that the openings of Ω at the two ends of Σ_R are the same, say ω
- Assume for simplicity that $\frac{\pi}{\omega}$ *is not* a rational number
- Assume for simplicity that $f \in L^2(\Omega)$ with support outside neighborhoods of the ends of Σ_R

Then the Neumann-Robin problem is for $\varepsilon \in [0, \varepsilon_0)$

$$(\mathscr{P}_\varepsilon) \qquad \begin{cases} -\Delta u_\varepsilon = f & \text{in } \Omega \\ \partial_n u_\varepsilon = 0 & \text{on } \Sigma_N \\ \varepsilon \partial_n u_\varepsilon + u_\varepsilon = 0 & \text{on } \Sigma_R \end{cases}$$

When $\varepsilon = 0$, we obtain a mixed Neumann-Dirichlet problem

$$(\mathscr{P}_0) \qquad \begin{cases} -\Delta u_0 = f & \text{in } \Omega \\ \partial_n u_0 = 0 & \text{on } \Sigma_N \\ u_0 = 0 & \text{on } \Sigma_R \end{cases}$$

Example 2: Variational formulations and singularities

Variational spaces:

$$V_\varepsilon = H^1(\Omega),\ \varepsilon > 0, \quad \text{and} \quad V_0 = \{u \in H^1(\Omega), \quad u\big|_{\Sigma_R} = 0\}.$$

Variational formulations

$(\mathscr{P}_\varepsilon)$

$$u_\varepsilon \in H^1(\Omega),\ \ \forall v \in H^1(\Omega),\ \ \int_\Omega \nabla u_\varepsilon \cdot \nabla v \,\mathrm{d}x + \frac{1}{\varepsilon}\int_{\Sigma_R} u_\varepsilon\, v \,\mathrm{d}\sigma = \int_\Omega f\, v \,\mathrm{d}x$$

$(\mathscr{P}_0)$

$$u_0 \in V_0, \quad \forall v \in V_0, \quad \int_\Omega \nabla u_0 \cdot \nabla v \,\mathrm{d}x = \int_\Omega f\, v \,\mathrm{d}x$$

Singularities of u_0 (for a Neumann-Dirichlet corner) *recall that* $\alpha = \frac{\pi}{\omega}$

$$r^{\alpha/2}\cos\tfrac{1}{2}\alpha\theta, \quad r^{3\alpha/2}\cos\tfrac{3}{2}\alpha\theta, \quad r^{5\alpha/2}\cos\tfrac{5}{2}\alpha\theta, \ldots$$

Singularities of u_ε (for a Neumann-Robin corner):

$$r^\alpha\cos\alpha\theta + \varepsilon^{-1}\gamma_1 r^{\alpha+1}\cos(\alpha+1)\theta + \varepsilon^{-2}\gamma_2 r^{\alpha+2}\cos(\alpha+2)\theta + \ldots$$
$$r^{2\alpha}\cos 2\alpha\theta + \varepsilon^{-1}\gamma_1' r^{2\alpha+1}\cos(2\alpha+1)\theta + \varepsilon^{-2}\gamma_2' r^{2\alpha+2}\cos(2\alpha+2)\theta + \ldots$$

i.e. Neumann singularity plus successive shadows. Negative powers of ε

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Example 3: Thin layer

See references

- Take a polygon Ω with straight sides
- Consider the union Σ of two consecutive sides σ_1 and σ_2. Let $O = \overline{\sigma}_1 \cap \overline{\sigma}_2$ be the corner inside Σ.
- Assume for simplicity that the opening ω at O is such that $\frac{\pi}{\omega} \notin \mathbb{Q}$
- Assume for simplicity that the openings of Ω at the two exterior corners c_1 and c_2 in $\overline{\Sigma}$ equal $\frac{\pi}{2}$
- Define the layer L_ε of width ε as the polygon formed by Σ, the segments $\sigma_{\ell,\varepsilon}$, $\ell = 1, 2$, parallel to σ_ℓ at the distance ε outside Ω, and two perpendicular segments of length ε issued from c_1 and c_2.
- Assume for simplicity that $f \in L^2(\Omega)$ with support outside a neighb. of O

Set

$$\Omega_\varepsilon = \Omega \cup \Sigma \cup \mathsf{L}_\varepsilon$$

and consider the transmission problem, with a positive number $\beta \neq 1$

$$u_\varepsilon \in H^1_0(\Omega_\varepsilon), \quad \forall v \in H^1_0(\Omega_\varepsilon), \quad \int_\Omega \nabla u_\varepsilon \cdot \nabla v \, \mathrm{d}x + \beta \int_{\mathsf{L}_\varepsilon} \nabla u_\varepsilon \cdot \nabla v \, \mathrm{d}x = \int_\Omega f \, v \, \mathrm{d}x.$$

Thus u_0 is solution of the Dirichlet pb on $\Omega = \Omega_0$.

Example 3: Singularities

Singularities of u_0 at 0 (Dirichlet) *recall that* $\alpha = \frac{\pi}{\omega}$

$$r^{\alpha} \sin \alpha\theta, \quad r^{2\alpha} \sin 2\alpha\theta, \quad r^{3\alpha} \sin 3\alpha\theta, \ldots$$

Singularities of u_ε at 0: Those of the transmission problem

$$\begin{cases} -\Delta U_\Omega = F & \text{in } \Gamma_\Omega \\ -\Delta U_L = 0 & \text{in } \Gamma_L \\ [U] = 0 & \text{on } \partial\Gamma_\Omega \cap \partial\Gamma_L \\ \partial_n U_\Omega + \beta\, \partial_n U_L = 0 & \text{on } \partial\Gamma_\Omega \cap \partial\Gamma_L \end{cases}$$

Here

- Γ_Ω is the sector of opening ω and Γ_L is (the interior of) its complement
- U_Ω and U_L are the restrictions of U to Γ_Ω and Γ_L, respectively.
- n exterior derivative.

The singularities of the transmission problems are different from the Dirichlet singularities.

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Example 4: Conducting material in 2D eddy current formulation

Take a conductor body M (conductivity σ) surrounded by a dielectric body N. Harmonic Maxwell equations

$$\mathbf{curl}\, \boldsymbol{E} = -i\kappa\mu \boldsymbol{H} \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{H} = (i\kappa\epsilon + \sigma \mathbf{1}_{\mathrm{M}})\boldsymbol{E} + \mathbf{j}_0$$

The eddy current formulation consists in neglecting $\kappa\epsilon$ with respect to σ:

$$\mathbf{curl}\, \boldsymbol{E} = -i\kappa\mu \boldsymbol{H} \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{H} = \sigma \mathbf{1}_{\mathrm{M}} \boldsymbol{E} + \mathbf{j}_0$$

and add the gauge condition

$$\operatorname{div} \boldsymbol{E} = 0 \quad \text{in} \quad \mathrm{N}.$$

Assume that the domains and the fields do not depend on the variable x_3 and that $\boldsymbol{H} = (0, 0, u)^\top$. With, now,

$$\Omega \subset \mathbb{R}^2, \quad \overline{\mathrm{M}} \subset \Omega, \quad \mathrm{N} = \Omega \setminus \overline{\mathrm{M}}$$

the 2D eddy current formulation for a magnetic potential u is

$$(\mathscr{P}_\delta) \qquad u_\delta \in H^1_0(\Omega), \ \ \forall v \in H^1_0(\Omega), \ \ \int_\Omega \nabla u_\delta \cdot \nabla v \, \mathrm{d}x + \frac{i}{\delta^2} \int_{\mathrm{M}} u_\delta \, v \, \mathrm{d}x = \int_{\mathrm{N}} f \, v \, \mathrm{d}x$$

with

$$\boxed{\delta^{-2} = \kappa\mu\sigma}$$

Example 4: Singularities

The parameter δ is (proportional to) the skin depth. It tends to 0 when σ tends to infinity.

- Assume that Ω is smooth.
- Assume for simplicity that the conductor M has only one corner in O. Let $2\pi - \omega$ be its opening, i.e. ω is the opening of N at O.
- Assume for simplicity that the opening ω is such that $\frac{\pi}{\omega} \notin \mathbb{Q}$
- Assume for simplicity that $f \in L^2(\mathrm{N})$ with support outside a neighb. of O

In strong form $(\mathscr{P}_\delta)$ is

$$-\Delta u_\delta + \frac{i}{\delta^2}\, \mathbf{1}_{\mathrm{M}}\, u_\delta = f \quad \text{in} \quad \Omega, \quad \text{with} \quad u_\delta \in H_0^1(\Omega)$$

and $(\mathscr{P}_0)$ is

$$-\Delta u_0 = f \quad \text{in} \quad \mathrm{N}, \quad \text{with} \quad u_0 \in H_0^1(\mathrm{N})$$

Singularities of u_0 at 0 (Dirichlet) *recall that* $\alpha = \frac{\pi}{\omega}$

$$r^\alpha \sin \alpha\theta, \quad r^{2\alpha} \sin 2\alpha\theta, \quad r^{3\alpha} \sin 3\alpha\theta, \ldots$$

Singularities of u_δ at 0 ?

Example 4: Singularities, continued

See references

- Singularities at a corner are determined by the principal part of the operator.
- The principal part of $-\Delta + \frac{i}{\delta^2}\,\mathbf{1}_M$ is $-\Delta$. The corner O is inside Ω. Hence no principal singularity!
- By the way the solution u belongs to $H^2(\Omega)$.

But, still, singularities appear as shadows of harmonic polynomials. Indeed (here * = wildcard)

$$1 + \delta^{-2} r^2 \left(\log^2 r\ \varphi_2^*(\theta) + \log r\ \varphi_1^*(\theta) + \varphi_0^* \right) + \delta^{-4} r^4 \sum_{\ell=0}^{4} \log^\ell r\ \varphi_\ell^*(\theta) + \dots$$

$$r\cos\theta + \delta^{-2} r^3 \sum_{\ell=0}^{3} \log^\ell r\ \varphi_\ell^*(\theta) + \delta^{-4} r^5 \sum_{\ell=0}^{5} \log^\ell r\ \varphi_\ell^*(\theta) + \dots$$

$$r\sin\theta + \delta^{-2} r^3 \sum_{\ell=0}^{3} \log^\ell r\ \varphi_\ell^*(\theta) + \delta^{-4} r^5 \sum_{\ell=0}^{5} \log^\ell r\ \varphi_\ell^*(\theta) + \dots$$

The logarithmic terms of highest degree are present!

- The strongest singularity of u_δ is $r^2 \log^2 r$,
- Versus the strongest singularity of u_0 that is $r^{\frac{\pi}{\omega}}$

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Two strategies

We have a family of singularly perturbed problems $(\mathscr{P}_\varepsilon)$ with solutions u_ε and want to understand

- How u_ε tends to u_0
- How singularities transform as $\varepsilon \to 0$

2 strategies, in the framework of multiscale analysis:

A Convert the singularities s^λ (homogeneous of degree λ) of u_0 into profiles K^λ solutions of a model problem with $\varepsilon = 1$. Roughly

$$A_0 s^\lambda = 0, \quad A_1 K^\lambda = 0 \quad \text{and} \quad K^\lambda - s^\lambda = \mathcal{O}(r^\lambda)$$

B Leave the singular part as is, but introduce a corrector Y^λ (a corner layer) to reduce the distance to u_ε.

Example 1: Profiles and corner layers

$u_0 \simeq \sum_{\lambda\in\Lambda} c_\lambda s^\lambda$, with $\Lambda = \mathbb{N}_* \frac{\pi}{\omega}$ and $s^\lambda = r^\lambda \sin\lambda\theta$. We have, on the sector Γ:

$$\begin{cases} \Delta s^\lambda &= 0 & \text{in } \Gamma \\ s^\lambda &= 0 & \text{on } \partial\Gamma. \end{cases}$$

Lemma

There exists a unique solution K^λ in the perturbation pattern P of the problem

$$\begin{cases} \Delta K^\lambda &= 0 & \text{in P} \\ K^\lambda &= 0 & \text{on } \partial\text{P} \\ K^\lambda &= s^\lambda + o(1) & \text{as } R\to\infty \end{cases}$$

Proof: Set $K^1_{-1}(\text{P}) = \{u \in L^2_{\text{loc}}(\text{P}),\ \nabla u,\ r^{-1}u \in L^2(\text{P})\}$. Define Y^λ as the variational solution in $K^1_{-1}(\text{P})$ of the problem

$$\begin{cases} \Delta Y^\lambda &= 0 & \text{in P} \\ Y^\lambda &= -s^\lambda & \text{on } \partial\text{P}\setminus\partial\Gamma \end{cases}$$

(note that $\partial\text{P}\setminus\partial\Gamma$ is compact and $s^\lambda = 0$ on $\partial\text{P}\cap\partial\Gamma$). Then set

$$K^\lambda = s^\lambda + Y^\lambda$$

Example 1: Profiles and corner layers, asymptotics at infinity

Then $K^\lambda\left(\frac{x}{\varepsilon}\right)$ is harmonic and satisfies the Dirichlet condition of εP.

- Formally $u'_\varepsilon :\simeq \sum_{\lambda\in\Lambda} c_\lambda\, \varepsilon^\lambda K^\lambda\left(\frac{x}{\varepsilon}\right)$ satisfies the Dirichlet condition of εP.
- But may differ from u_ε on $\partial\Omega \setminus \partial\Gamma$. This error is far from O.

To analyze it, one needs

The complete asymptotic at infinity of K^λ

as negative powers of r.

Lemma

There exists a sequence of numbers $\gamma_{\lambda,\lambda'}$ so that

$$K^\lambda(X) \simeq s^\lambda(X) + \sum_{\lambda'\in\Lambda} \gamma_{\lambda,\lambda'} R^{-\lambda'} \sin\lambda'\theta, \quad R\to\infty$$

Then

$$\varepsilon^\lambda K^\lambda\left(\frac{x}{\varepsilon}\right) \simeq s^\lambda(x) + \sum_{\lambda'\in\Lambda} \varepsilon^{\lambda+\lambda'} \gamma_{\lambda,\lambda'} r^{-\lambda'} \sin\lambda'\theta, \quad |x|\geq \varepsilon R_0$$

Example 1: Finite expansions

A Choose $m \in \mathbb{N}$. Using the profiles K^λ

$$u_\varepsilon = u_0^{[m]}(x) + \sum_{\lambda \leq m} \varepsilon^\lambda u_\lambda^{[m]}(x) + \sum_{\lambda+\lambda' \leq m} c_{\lambda,\lambda'} \varepsilon^{\lambda+\lambda'} K^\lambda\left(\frac{x}{\varepsilon}\right) + \mathcal{O}(\varepsilon^m)$$

The terms $u_\lambda^{[m]}$ are more or less regular: $u_0^{[m]} \in H^m$ (in fact in K^m_{-m}). This is a moving asymptotics. But it allows to capture the transformation of singularities through the profiles K^λ

B Choose $m \in \mathbb{N}$. Using the corner layers Y^λ

$$u_\varepsilon = u_0(x) + \sum_{\lambda \leq m} \varepsilon^\lambda u_\lambda(x) + \sum_{\lambda+\lambda' \leq m} c_{\lambda,\lambda'} \varepsilon^{\lambda+\lambda'} Y^\lambda\left(\frac{x}{\varepsilon}\right) + \mathcal{O}(\varepsilon^m)$$

- In certain configurations (essentially when the perturbation consists of small holes dug in Ω), it is possible to prove that the series converge. Here the assumption that $f \equiv 0$ close to 0 is important. When this assumption is relaxed to analyticity, we can only prove *conditional convergence*.
- The conditions that $\Omega \subset \Gamma$ and $P \subset \Gamma$ can be relaxed, with the help of cut-offs: χ_0 localizes near 0 and multiply rapid terms, whereas χ_∞ localizes near infinity and $\chi_\infty\left(\frac{x}{\varepsilon}\right)$ multiply slow terms.

Example 2: Profiles

Denote $s^\lambda = r^\lambda \cos \lambda\theta$, for $\lambda \in \Lambda = (\mathbb{N}_* - \frac{1}{2})\frac{\pi}{\omega}$, the singularities of the mixed Neumann-Dirichlet problem. They satisfies on the sector Γ

$$\begin{cases} \Delta s^\lambda &= 0 & \text{in } \Gamma \\ \partial_n s^\lambda &= 0 & \text{if } \theta = 0, \\ s^\lambda &= 0 & \text{if } \theta = \omega. \end{cases}$$

Denote by $\mathscr{S}^\lambda(\Gamma)$ the space of homogeneous functions of degree λ on Γ.

Lemma

There exists a unique solution K^λ in the sector Γ to the problem

$$\begin{cases} \Delta K^\lambda &= 0 & \text{in } \Gamma \\ \partial_n K^\lambda &= 0 & \text{if } \theta = 0, \\ \partial_n K^\lambda + K^\lambda &= 0 & \text{if } \theta = \omega, \\ K^\lambda &= s^\lambda + \mathcal{O}(R^\lambda) & \text{as } R \to \infty \end{cases}$$

that has the form

$$K^\lambda = s^\lambda + \sum_{\ell=1}^{\mathcal{L}} s_\ell^\lambda + \mathcal{O}(R^{\frac{1}{2}}), \quad \text{with} \quad s_\ell^\lambda \in \mathscr{S}^{\lambda-\ell}(\Gamma), \quad \mathcal{L} = [\lambda + \tfrac{1}{2}].$$

Example 2: Profiles

Proof. Denote $s_0^\lambda := s^\lambda$. For $\ell = 1, \dots,$ there exists unique $s_\ell^\lambda \in \mathscr{S}^{\lambda-\ell}(\Gamma)$ such that

$$\begin{cases} \Delta s_\ell^\lambda &= 0 & \text{in } \Gamma \\ \partial_n s_\ell^\lambda &= 0 & \text{if } \theta = 0, \\ s_\ell^\lambda &= -\partial_n s_{\ell-1}^\lambda & \text{if } \theta = \omega, \end{cases}$$

The existence as homogeneous solutions and the uniqueness is due to the assumption $\frac{\pi}{\omega} \notin \mathbb{Q}$ that implies that $\lambda - \ell$ does not hit the Mellin spectrum $\Lambda = (\mathbb{Z}_* - \frac{1}{2})\frac{\pi}{\omega}$ of the mixed Neumann-Dirichlet problem. Introduce the sum

$$X^\lambda = \chi_\infty \sum_{\ell=0}^{\mathcal{L}} s_\ell^\lambda$$

with a radial smooth cut-off, $\chi_\infty \equiv 1$ for $|x| \geq 2$ and $\chi_\infty \equiv 0$ for $|x| \leq 1$ Then, on the Robin side $\theta = \omega$ and by construction

$$\partial_n X^\lambda + X^\lambda = \chi_\infty \, \partial_n s_{\mathcal{L}}^\lambda$$

The trace $\partial_n s_{\mathcal{L}}^\lambda$ is homogeneous of degree $\lambda - \mathcal{L} - 1$ that is $< -\frac{1}{2}$. Thus $\chi_\infty \, \partial_n s_{\mathcal{L}}^\lambda$ belongs to $L^2(\mathbb{R}_+)$. We enter the variational space $\mathfrak{V}$ $\cdots/\cdots$

Example 2: Profiles

The variational space for the mixed Neumann-Robin problem on the sector Γ

$$\begin{cases} \Delta U &= F & \text{in } \Gamma \\ \partial_n U &= 0 & \text{if } \theta = 0, \\ \partial_n U + U &= G & \text{if } \theta = \omega, \end{cases}$$

is

$$\mathfrak{V} = \left\{ U \in L^2_{\text{loc}}(\Gamma), \quad \nabla U \in L^2(\Gamma), \quad U\big|_{\theta=\omega} \in L^2(\mathbb{R}_+) \right\}$$

and the variational formulation is

$$U \in \mathfrak{V}, \ \ \forall V \in \mathfrak{V}, \ \ \int_\Gamma \nabla U \cdot \nabla V \, \mathrm{d}X + \int_{\theta=\omega} U \, V \, \mathrm{d}R = \int_\Gamma F \, V \, \mathrm{d}x + \int_{\theta=\omega} G \, V \, \mathrm{d}R$$

Thus we find $Y^\lambda \in \mathfrak{V}$ solution of

$$\begin{cases} \Delta Y^\lambda &= 0 & \text{in } \Gamma \\ \partial_n Y^\lambda &= 0 & \text{if } \theta = 0, \\ \partial_n Y^\lambda + Y^\lambda &= -X^\lambda & \text{if } \theta = \omega, \end{cases}$$

and our profile is finally

$$K^\lambda = X^\lambda + Y^\lambda = \chi_\infty \left(s^\lambda + \sum\nolimits_{\ell=1}^{\mathcal{L}} s_\ell^\lambda \right) + Y^\lambda, \quad \text{with} \quad s_\ell^\lambda \in \mathscr{S}^{\lambda-\ell}, \ Y^\lambda \in \mathfrak{V}$$

Example 2: Full expansion at infinity

The constructive approach can be carried on, but will never account for the whole asymptotics at infinity, because part of it is hiding in a variat. solution.

- Need for an analytic tool to find this asymptotics: The Mellin transform.
- Since the mixed Robin-Neumann is not an homogeneous operator, this is a non-classical setting.

Let us analyze Y^λ. Set $u = \chi_\infty\, Y^\lambda$ and $g = -\chi_\infty^2\, \partial_n s_{\mathcal{L}}^\lambda$, so that

$$\begin{cases} \Delta u &= 0 & \text{in } \Gamma \\ \partial_n u &= 0 & \text{if } \theta = 0, \\ \partial_n u + u &= g & \text{if } \theta = \omega, \end{cases}$$

Introduce for $\mu \in \mathbb{C}$ and the Mellin transform of u and g:

$$U[\mu] = \int_0^\infty r^{-\mu}\, u(r,\cdot)\, \frac{\mathrm{d}r}{r}, \quad G[\mu] = \int_0^\infty r^{-\mu}\, g(r)\, \frac{\mathrm{d}r}{r}$$

- $U[\mu]$ is holomorphic for $\mathrm{Re}\,\mu > 0$
- $G[\mu]$ is meromorphic with a unique pole of order 1 at

$$\mu_0 = \lambda - [\lambda + \tfrac{1}{2}] - 1 \qquad \left[\, \mu_0 < -\tfrac{1}{2} \,\right]$$

Example 2: Full expansion at infinity

The mixed Neumann-Robin becomes after Mellin transform

$$(*) \qquad \begin{cases} (\partial_\theta^2 + \mu^2) U[\mu] = 0 & \text{in } (0,\omega) \\ \partial_\theta U[\mu] = 0 & \text{if } \theta = 0, \\ U[\mu] = -\partial_\theta U[\mu+1] + G[\mu] & \text{if } \theta = \omega, \end{cases}$$

The Mellin symbol $\mathfrak{A}[\mu] : H^2(0,\omega) \to L^2(0,\omega) \times \mathbb{R} \times \mathbb{R}$ of the mixed NR is the operator on the left of $(*)$. The poles of $\mathfrak{A}[\mu]^{-1}$ are simple and situated on $\Lambda = (\mathbb{Z}_* - \frac{1}{2})\frac{\pi}{\omega}$. Re-write $(*)$ as

$$U[\mu] = \mathfrak{A}[\mu]^{-1}\Big(0,\ 0,\ G[\mu] - \partial_\theta U[\mu+1](\omega)\Big)$$

- Since $U[\mu]$ is holomorphic for $\operatorname{Re}\mu \geq 0$, the rhs above is meromorphic for $\operatorname{Re}\mu > -1$.
- Hence meromorphic extension of $U[\mu]$ for $\operatorname{Re}\mu > -1$.
- Can be continued step by step up to $\operatorname{Re}\mu \to -\infty$.
- The poles are

$$\mu_0 - \ell \ \ (\forall \ell \in \mathbb{N}) \quad \text{and} \quad \lambda' - \ell \ \ (\forall \lambda' \in \Lambda \cap \mathbb{R}_-,\ \forall \ell \in \mathbb{N})$$

Example 2: Full expansion at infinity of profiles

Recall that the profile is K^λ and the corner layer is Y^λ. They are associated with the singularity $s^\lambda = r^\lambda \cos\lambda\theta$, for $\lambda \in \Lambda \cap \mathbb{R}_+$

Lemma

There exist homogeneous functions $s^\lambda_{\lambda',\ell} \in \mathscr{S}^{\lambda'-\ell}$ so that

$$K^\lambda \simeq s^\lambda + \sum_{\ell\geq 1} s^\lambda_\ell + \sum_{\lambda'\in\Lambda,\ \lambda<0} \sum_{\ell\geq 0} s^\lambda_{\lambda',\ell} \qquad r\to\infty$$

$$Y^\lambda \simeq \sum_{\ell>\mathcal{L}} s^\lambda_\ell + \sum_{\lambda'\in\Lambda,\ \lambda<0} \sum_{\ell\geq 0} s^\lambda_{\lambda',\ell} \qquad r\to\infty$$

The material is ready for the description of the solutions u_ε and u_0, along the lines of Example 1. The analysis is more difficult due the singular perturbational character of the operator $\varepsilon\partial_n + \mathbb{I}$

Conclusion

What about examples 3 and 4?

The new difficulty there is the presence of a further anisotropic scale (thin layer for Example 3 and boundary layer – skin effect – for Example 4). This was solved 10 years ago for Example 3, and is still under investigation for Example 4.

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Thank you for your attention