"*Eigenvalues of the curl operator: variational formulation and numerical approximation*"

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Abstract

In electromagnetism, for linear isotropic media the relation between the magnetic induction $\boldsymbol{B}$ and the magnetic field $\boldsymbol{H}$ is given by $\boldsymbol{B} = \mu\boldsymbol{H}$, the scalar function μ being the magnetic permeability. If displacement currents are neglected, as in the case of eddy current problems, the current density $\boldsymbol{J}$ is given by $\boldsymbol{J} = \mathbf{curl}\boldsymbol{H}$. In this situation a magnetic field satisfying $\mathbf{curl}\,\boldsymbol{H} = \eta\boldsymbol{H}$, with η a scalar function, produces a vanishing magnetic force $\boldsymbol{J} \times \boldsymbol{B}$, and it is called a *force-free* field.

In fluid dynamics, a divergence-free field $\boldsymbol{u}$ satisfying $\mathbf{curl}\,\boldsymbol{u} = \eta\boldsymbol{u}$, with η a scalar function, is a steady solution of the Euler equations for incompressible inviscid flows (with pressure given by $p = -\frac{|\boldsymbol{u}|^2}{2}$), and it is called a *Beltrami* field.

Eigenfunctions of the curl operator are therefore force-free fields and Beltrami fields, and are of relevant physical interest. In particular, in plasma physics a magnetic field $\boldsymbol{H}$ which minimizes the magnetic energy with fixed helicity has to satisfy the equation $\mathbf{curl}\,\boldsymbol{H} = \lambda\boldsymbol{H}$ for some constant λ, thus it is an eigenfunction of the curl operator.

In this talk we are concerned with two topics: the formulation and analysis of the eigenvalue problem for the curl operator in a multiply-connected domain, and its numerical approximation by means of finite elements. Following the results in [1] and extending the previous ones in [2], [3], [4], we prove that the curl operator is self-adjoint on suitable Hilbert spaces, each of them being contained in the space of vector fields $\boldsymbol{v}$ for which $\mathbf{curl}\,\boldsymbol{v} \cdot \mathbf{n} = \mathbf{0}$ on the boundary. An important point is to note that additional conditions must be imposed when the physical domain is not topologically trivial: a viable choice is the vanishing of the line integrals of $\boldsymbol{v}$ on suitable homological cycles lying on the boundary.

We devise and analyze a saddle-point variational formulation, and propose a finite element numerical scheme, generalizing the results in [5], [6]. We prove that eigenvalues and eigenfunctions are efficiently approximated, and present some numerical results for testing the performance of the method.

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