INVERSE SCATTERING IN ACOUSTICS AND ELASTICITY USING HIGH-ORDER TOPOLOGICAL DERIVATIVES

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Many practical (e.g. design or identification) problems of physics or engineering involve the optimization of some objective function, whose evaluation often involves solving a forward problem which is typically involves partial differential equations (PDEs). The resulting need for efficient computational procedures for such PDE-constrained optimization problems has in turn prompted the appearance and implementation of many treatments, involving e.g. parameter or shape sensitivity, model reduction, or surrogate approximate models based on asymptotic methods. The latter category in particular includes the concept of topological derivative, which quantifies the perturbation induced to a cost functional J by the virtual creation of an object occupying a region $B_a(\boldsymbol{z})$ with prescribed center $\boldsymbol{z}$ inside the solid and vanishingly small characteristic radius a. For the case of a small elastic inhomogeneity embedded in a three-dimensional elastic solid, considered in this work, the well known expansion

$$J\big(B_a(\boldsymbol{z})\big) = J(\emptyset) + a^3\mathcal{T}(\boldsymbol{z}) + o(a^3)$$

holds [**9, 10, 14**], where $J\big(B_a(\boldsymbol{z})\big)$ denotes the value taken by the objective function of interest when the elastic solid contains an inhomogeneity of support $B_a(\boldsymbol{z})$ and prescribed material characteristics. The field $\mathcal{T}(\boldsymbol{z})$ can then be used e.g. to direct structural optimization algorithm towards optimal topologies, see e.g. [**12, 4, 1**], or as a means for quatitative flaw identification, see e.g. [**3, 5, 6, 11**].

A natural extension of the concept of topological derivative consists in expanding $J\big(B_a(\boldsymbol{z})\big)$ to higher orders in a, in order to formulate more accurate asymptotic approximations of J. Previous efforts in this direction include [**8**] for inclusions or cracks in 2D conductive media, [**7**] for sound-hard obstacles in 3D acoustic media, and [**13**] for expansions of the potential energy of 2D elastic solids. Higher-order topological derivatives were in particular shown in [**7, 8**] to allow computationally fast and quantitatively accurate flaw identification while avoiding the need for initial guesses.

In this work, expansions of the form

$$J\big(B_a(\boldsymbol{z})\big) = J(\emptyset) + a^3\mathcal{T}_3(\boldsymbol{z}) + a^4\mathcal{T}_4(\boldsymbol{z}) + a^5\mathcal{T}_5(\boldsymbol{z}) + a^6\mathcal{T}_6(\boldsymbol{z}) + o(a^6) \tag{1}$$

are established and justified for a large class of objective functionals, in the case of a small inhomogeneity embedded in a three-dimensional acoustic or elastic medium. The chosen order $O(a^6)$ results from the fact that, for misfit functions J of least-squares type, the perturbations of the residuals entering the definition of J are of order $O(a^3)$ under the present conditions. Like in [**7, 8**], we introduce the adjoint solution associated with J prior to performing the expansion in powers of a, allowing to establish the above expansion on the basis of (i) the expansion of the solution to the underlying elastic transmission problem in B_a (inner expansion) and (ii) the leading-order contribution to the solution on the support of the objective function density (leading outer expansion). A volume integral equation (VIE) formulation is used for deriving the necessary solution expansions, rather than coupled boundary integral equations in previous studies on solution asymptotics for elastic transmission problems [**2**]; this choice is motivated by the fact that the geometrical support of the VIE is B_a, which facilitates the implementation and use of coordinate rescaling commonly used in the derivation of such asymptotic models. Moreover, unlike in the previously-mentioned investigations where expansions were derived and implemented, we both establish the relevant high-order topological expansion (i.e. give complete expressions for functions $\mathcal{T}_4(\boldsymbol{z})$, $\mathcal{T}_5(\boldsymbol{z})$, $\mathcal{T}_6(\boldsymbol{z})$ in addition to the already-known ordinary topological derivative $\mathcal{T}_3(\boldsymbol{z})$) and provide its justification.

The main focus of the presentation will be in the acoustic case. Corresponding results for the elastodynamic case (obtained with Rmi Cornaggia as part of his PhD thesis) will then be summarized, and results of numerical experiments presented for both cases.

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