lecture 5 14.11.16

Ω (open) $\subset \mathbb{R}^3$, recall:

* $\Lambda := \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}$

* $A : \mathring{R} \times R \subset L^2 \times L^2 \to L^2 \times L^2$ defined by

$$A x := \begin{pmatrix} 0 & -\mathring{\mathrm{rot}}^* \\ \mathring{\mathrm{rot}} & 0 \end{pmatrix} x = \begin{pmatrix} 0 & -\mathrm{rot} \\ \mathring{\mathrm{rot}} & 0 \end{pmatrix} x$$

is skew-selfadjoint.

* $M : \mathring{R} \times R \subset L^2_\Lambda := L^2_\varepsilon \times L^2_\mu \to L^2_\Lambda$ defined by

$$M x := i \Lambda^{-1} A x = i \begin{pmatrix} 0 & -\varepsilon^{-1}\mathrm{rot} \\ \mu^{-1}\mathring{\mathrm{rot}} & 0 \end{pmatrix} x$$

is selfadjoint.

① <u>time dependent</u>:

$(\partial_t - iM)\, x = f$ & initial conditions.

boundary conditions are already included.

$\hookrightarrow$ M selfadjoint $\Rightarrow$ solution theory by

(i) spectral theory

$$x(t) = e^{itM} x_0 + \int_0^t e^{i(s-t)M} f(s)\, ds$$

(ii) semi-group theory

(iii) Picard - theory.

② <u>time-harmonic</u>:

$$(\mathcal{M} - \omega)x = f \quad \rightsquigarrow \text{ Fredholm Alternative}$$

③ <u>static case</u>:

$$\mathcal{M}x = f \quad \& \quad \text{div-conditions}$$

$\rightsquigarrow$ Toolbox

<u>For case ③</u>:

Is $R(\mathcal{M})$ closed? (central point)

for ② we need: $\boxed{D(\mathfrak{M}) \hookrightarrow\hookrightarrow L^2_\Lambda}$

$\Rightarrow R(\mathfrak{M}) = R(\mathcal{M})$ is closed.

$$\mathcal{M} : D(\mathcal{M}) \subset L^2_\Lambda \to L^2_\Lambda, \quad D(\mathcal{M}) = \mathring{R} \times R$$

$$\mathfrak{M} : D(\mathfrak{M}) = D(\mathcal{M}) \cap \overline{R(\mathcal{M})} \subset \overline{R(\mathcal{M})} \to \overline{R(\mathcal{M})}$$

with

$$\begin{aligned} D(\mathfrak{M}) &= (\mathring{R} \times R) \cap (\varepsilon^{-1}\overline{\operatorname{rot} R} \times \mu^{-1}\overline{\mathring{\operatorname{rot}}\, \mathring{R}}) \\ &= (\mathring{R} \cap \varepsilon^{-1}\overline{\operatorname{rot} R}) \times (R \cap \mu^{-1}\overline{\mathring{\operatorname{rot}}\, \mathring{R}}) \end{aligned}$$

(clearly: $\overline{\varepsilon^{-1}\operatorname{rot} R} = \varepsilon^{-1}\overline{\operatorname{rot} R}$)

so what we <u>need to show</u>:

$$D(\mathcal{A}_1) = \mathring{R} \cap \varepsilon^{-1}\overline{\operatorname{rot} R} \hookrightarrow\hookrightarrow L^2_\varepsilon$$

$$\Updownarrow \text{ cf. exercises}$$

$$D(\mathcal{A}_1^*) = R \cap \mu^{-1}\overline{\mathring{\operatorname{rot}}\, \mathring{R}} \hookrightarrow\hookrightarrow L^2_\mu$$

These indeed follow from:

$$\boxed{\mathring{R}\cap\varepsilon^{-1}D \hookrightarrow L^2_\varepsilon \;\wedge\; R\cap\mu^{-1}\mathring{D}\hookrightarrow L^2_\mu}$$

(This we will show in the last lecture)

<u>The static problems:</u>

„$n\times E|_\Gamma = 0$"

$$\begin{aligned}
&(\text{ESP}) \quad && A_1 E = \mathring{\mathrm{rot}}\,\mathring{E} = F \in R(\mathring{\mathrm{rot}}) = R(A_1)\\
&\mu := 1 && A_0^* E = -\operatorname{div}\varepsilon E = f \in R(\operatorname{div}) = R(A_0^*)\\
& && \pi_1 E = \pi_D E = D \in \mathcal{D}_\varepsilon = K_1
\end{aligned}$$

$$\begin{aligned}
&(\text{MSP}) \quad && A_1^* H = \mathrm{rot}\,H = G \in R(\mathrm{rot}) = R(A_1^*)\\
&\varepsilon := 1 && A_2 H = \mathring{\operatorname{div}}\,\mu H = g \in R(\mathring{\operatorname{div}}) = R(A_2)\\
& && \pi_2 H = \pi_N \hat{H} = N \in \mathcal{N}_\mu = K_2
\end{aligned}$$

„$n\cdot H|_\Gamma = 0$"

Remember:

$$\begin{aligned}
A_0 &= \mathring{\nabla} : \mathring{H}^1 \subset L^2 \to L^2_\varepsilon, \; u \mapsto \mathring{\nabla} u\\
A_1 &= \mu^{-1}\mathring{\mathrm{rot}} : \mathring{R} \subset L^2_\varepsilon \to L^2_\mu, \; E \mapsto \mu^{-1}\mathring{\mathrm{rot}}\,E\\
A_2 &= \mathring{\operatorname{div}}\,\mu : \mu^{-1}\mathring{D} \subset L^2_\mu \to L^2, \; H \mapsto \mathring{\operatorname{div}}\,\mu H\\
A_0^* &= -\operatorname{div}\varepsilon : \varepsilon^{-1} D \subset L^2_\varepsilon \to L^2, \; H \mapsto -\operatorname{div}\varepsilon H\\
A_1^* &= \varepsilon^{-1}\mathrm{rot} : R \subset L^2_\mu \to L^2_\varepsilon, \; E \mapsto \varepsilon^{-1}\mathrm{rot}\,E\\
A_2^* &= -\nabla : H^1 \subset L^2 \to L^2_\mu, \; u \mapsto -\nabla u
\end{aligned}$$

<u>Theorem 8:</u>

Let A_i, A_{i+1} be linear, densely defined, closed and $A_{i+1}A_i \subset 0$. Assume $R(A_i)$, $R(A_{i+1})$ are closed, then

$$A_{i+1} x = f \,, \quad A_i^* x = g \,, \quad \Pi_{i+1} x = h$$

is uniquely solvable in $D_{i+1} := D(A_{i+1}) \cap D(A_i^*)$, iff $f \in R(A_{i+1})$, $g \in R(A_i^*)$, $h \in K_{i+1}$. The unique solution is given by

$$x = x_f + x_g + h \in D(\mathcal{A}_{i+1}) \oplus D(\mathcal{A}_i^*) \oplus K_{i+1},$$

where

$$x_f = \mathcal{A}_{i+1}^{-1} f \in D(\mathcal{A}_{i+1}) = D(A_{i+1}) \cap R(A_{i+1}^*),$$
$$x_g = (\mathcal{A}_i^*)^{-1} g \in D(\mathcal{A}_i^*) = D(A_i^*) \cap R(A_i),$$

depends continuously on the data, i.e.

$$|x|^2_{H_{i+1}} \leq c_{i+1}^2 |x_f|^2_{H_{i+1}} + c_i^2 |x_g|^2_{H_{i-1}} + |h|^2_{H_i}$$

and

$$x_f = \Pi_{A_{i+1}} x \,, \quad x_g = \Pi_{A_i} x \,.$$

Remark:

① x_f, x_g solve

$$\begin{aligned} A_{i+1} x_f &= f & \qquad A_{i+1} x_g &= 0 \\ A_i^* x_f &= 0 & A_i^* x_g &= g \\ \Pi_{i+1} x_f &= 0 & \Pi_{i+1} x_g &= 0 \end{aligned}$$

② $D_{i+1} = D(A_{i+1}) \cap D(A_i^*) \hookrightarrow H_{i+1}$

$\Rightarrow$ $R(A_i)$, $R(A_{i+1})$ are closed.

& $K_{i+1} = N(A_{i+1}) \cap N(A_i^*)$ is finite dimensional

& ... Poincaré-type estimates,

$\mathcal{A}_{i+1}^{-1}$ is continuous / compact...

③ In (ESP): $i=0$; in (MSP): $i=1$.

<u>Proof of Theorem 8</u>:

„$\Rightarrow$": clear!

„$\Leftarrow$": $f \in R(A_{i+1}) = R(\mathcal{A}_{i+1})$ closed

$\overset{\text{toolbox}}{\Rightarrow}$ $\mathcal{A}_{i+1}^{-1} : R(A_{i+1}) \to D(\mathcal{A}_{i+1})$

is continuous.

We put:

$$x_f := \mathcal{A}_{i+1}^{-1} f \in D(\mathcal{A}_{i+1}) = D(A_{i+1}) \cap R(A_{i+1}^*)$$
$$\subset N(A_i^*) \cap K_{i+1}^{\perp}$$

⌜ Helmholtz-decomposition:

$$H_{i+1} = R(A_i) \oplus K_{i+1} \oplus R(A_{i+1}^*)$$
$$\Rightarrow N(A_i^*) = R(A_{i+1}^*) \oplus K_{i+1} \quad ⌟$$

Then: $A_{i+1} x_f = f$, $x_f \in N(A_i^*) \cap K_{i+1}^{\perp}$

$g \in R(A_i^*) = R(\mathcal{A}_i^*)$ is closed

$\overset{\text{toolbox}}{\Rightarrow}$ $(\mathcal{A}_i^*)^{-1} : R(A_i^*) \to D(\mathcal{A}_i^*)$

is continuous.

We put:

$$x_g := (\mathcal{A}_i^*)^{-1} g \in D(\mathcal{A}_i^*) = D(A_i^*) \cap R(A_i)$$
$$\subset N(A_{i+1}) \cap K_{i+1}^{\perp}$$

Then: $A_i^* x_g = g$, $x_g \in N(A_{i+1}) \cap K_{i+1}^{\perp}$

Furthermore: $x := x_f + x_g + h$

$$\Rightarrow |x|^2 = |x_f|^2 + |x_g|^2 + |h|^2$$
$$\leq c_{i+1}^2 |f|^2 + c_i^2 |g|^2 + |h|^2$$

$$\forall y \in D(\mathcal{A}_{i+1}) : \ |y| \leq c_{i+1} |A_{i+1} y|$$
$$\forall z \in D(\mathcal{A}_i^*) : \ |z| \leq c_i |A_i^* z|$$

$\square$

Numerical application:

"Variational formulations" to find x_g, x_f?

(interested in second order formulations; good for coercivity, symmetry, ...)

$$x_f := \mathcal{A}_{i+1}^{-1} f \in D(\mathcal{A}_{i+1}) = D(A_{i+1}) \cap R(A_{i+1}^*)$$

(first order potential for f)

$$y_f := (\mathcal{A}_{i+1}^*)^{-1} x_f = (\mathcal{A}_{i+1}^*)^{-1} \mathcal{A}_{i+1}^{-1} f$$

(second order potential for f)

$$\Rightarrow A_{i+1}^* y_f = x_f$$

Let $\phi \in D(\mathcal{A}_{i+1}^*)$: $\leftarrow$ ($\mathcal{A}_{i+1}^*$ for coercivity)

$$\langle A_{i+1}^* y_f , A_{i+1}^* \phi \rangle = \langle x_f , A_{i+1}^* \phi \rangle = \langle f, \phi \rangle$$

Observe:

* $\varphi \mapsto \langle f, \varphi \rangle \in D(\mathcal{A}_{i+1}^*)' \,/\, H_{i+2}' \cong H_{i+2}$
* $(\psi, \varphi) \mapsto \langle A_{i+1}^* \psi, A_{i+1}^* \varphi \rangle$ is a continuous bilinearform on $D(\mathcal{A}_{i+1})^*$ and coercive, since:

$$\forall \varphi \in D(\mathcal{A}_{i+1}^*) : |\varphi| \leq c_{i+1} |A_{i+1}^* \varphi|$$

Formulation of the problem:

Find $\tilde{y}_F \in D(\mathcal{A}_{i+1}^*)$ such that $\forall \phi \in D(\mathcal{A}_{i+1}^*)$

(*)
$$< \mathcal{A}_{i+1}^* \tilde{y}_F , \mathcal{A}_{i+1}^* \phi > = < f, \phi >$$

Riesz / Lax-Milgram $\Rightarrow$ $\exists! \ \tilde{y}_F \in D(\mathcal{A}_{i+1}^*)$

still unclear: $x_F \overset{?}{=} \mathcal{A}_{i+1}^* y_F$

$\hookrightarrow$ numerically the space $D(\mathcal{A}_{i+1}^*)$ is ugly and should be replaced:

1. observation: (*) holds for all $\phi \in D(A_{i+1}^*)$, since

$$D(A_{i+1}^*) = N(A_{i+1}^*) \oplus D(\mathcal{A}_{i+1}^*)$$
$$\Downarrow$$
$$\phi = \phi_N + \hat{\phi}$$

$\Rightarrow$ for $\phi \in D(A_{i+1}^*)$:

$$< A_{i+1}^* \tilde{y}_F , A_{i+1}^* \phi >$$
$$= < \mathcal{A}_{i+1}^* \hat{y}_F , A_{i+1}^* \hat{\phi} >$$
$$= < f, \hat{\phi} >$$
$$= < f, \phi > - \underbrace{< f, \phi_N >}_{\substack{=0, \text{ since} \\ f \in R(A_{i+1}) = N(A_{i+1}^*)^\perp}} = < f, \phi >$$

2. observation:

$D(\mathcal{A}_{i+1}^*) = D(A_{i+1}^*) \cap R(A_{i+1})$;
put the second condition in a second equation.

↳ Reformulate $(*)$:

$(*) \iff$ Find $\widehat{y}_F \in D(A_{i+1}^*)$ such that
$\forall\, \phi \in D(A_{i+1}^*)$:

$y_F \in R(A_{i+1}) = N(A_{i+1}^*)^\perp$ ↘

$\langle A_{i+1}^* \widehat{y}_F, A_{i+1}^* \phi \rangle = \langle f, \phi \rangle$
$\forall\, \psi \in N(A_{i+1}^*)$:
$\langle \widetilde{y}_F, \psi \rangle = 0$

Now: $N(A_{i+1}^*) = R(A_{i+2}^*) \oplus K_{i+2}$, so
if we assume: $K_{i+2} = \{0\}$, we get
$N(A_{i+1}^*) = R(A_{i+2}^*) = R(\mathcal{A}_{i+2}^*)$

↳ Reformulate $(*)$:

$(*) \iff$ Find $\widehat{y}_F \in D(A_{i+1}^*)$ such that
$\forall\, \phi \in D(A_{i+1}^*)$:
$\langle A_{i+1}^* \widehat{y}_F, A_{i+1}^* \phi \rangle = \langle f, \phi \rangle$

$(*_2)$

$\forall\, \Theta \in D(A_{i+2}^*)$:
$\langle \widetilde{y}_F, A_{i+2}^* \Theta \rangle = 0$ Here it has to be $\mathcal{A}_{i+2}^*$.

$(*) \iff$ Find $(\widehat{y}_F, v_f) \in D(A_{i+1}^*) \times D(\mathcal{A}_{i+2}^*)$ s.t.
$\forall\, \phi \in D(A_{i+1}^*)$:
$\langle A_{i+1}^* \widetilde{y}_F, A_{i+1}^* \phi \rangle$
$+ \langle \phi, A_{i+2}^* \phi \rangle = \langle f, \phi \rangle$

$(*_3)$

$\forall\, \Theta \in D(A_{i+2}^*)$: ← Here we can also put $\mathcal{A}_{i+2}^*$
$\langle \widehat{y}_F, A_{i+2}^* \Theta \rangle = 0$

Remark:

$\widetilde{y}_F$ solution to $(*_2) \Rightarrow (\widetilde{y}_F, 0)$ solution to $(*_3)$

We will show:

If $(\tilde{y}_F, v_F)$ is a solution to $(*_3)$, then:

$$v_f = 0$$

Proof:

Put $\phi := A_{i+2}^* v_F$

$\Rightarrow \| A_{i+2}^* v_f \|^2 = 0$

$\Rightarrow A_{i+2}^* v_F = 0$

$\Rightarrow v_f = 0$ (Here you see why $D(\mathcal{A}_{i+2}^*)$ is needed)

$v_F \in D(\mathcal{A}_{i+2}^*)$, $\mathcal{A}_{i+2}^*$ is injective

$\boxed{4}$

For numerics one additionally needs

① coercivity:

$\langle A_{i+1}^* \cdot , A_{i+1}^* \cdot \rangle$ is coercive on

$$\underbrace{N(A_{i+2})}_{= R(A_{i+1}) \oplus K_{i+2} = R(A_{i+1}) \oplus \{0\}} \cap D(A_{i+1}^*) = D(\mathcal{A}_{i+1}^*)$$

$\rightsquigarrow$ clear, according to our formulations.

② inf-sup-condition:

$$\inf_{0 \neq \psi \in D(\mathcal{A}_{i+2}^*)} \sup_{0 \neq \phi \in D(A_{i+1}^*)} \frac{\langle \phi, A_{i+2}^* \psi \rangle}{|\phi|_{D(A_{i+1}^*)} |\psi|_{D(A_{i+2}^*)}}$$

$$\geq \inf_{0 \neq \psi \in D(\mathcal{A}_{i+2}^*)} \frac{|A_{i+2}\psi|}{|\psi|_{D(A_{i+2}^*)}} \geq (1 + c_{i+2}^2)^{-1/2}$$

$\phi := A_{i+2}^* \psi \in N(A_{i+1}^*)$