Initiation into corner singularities

Monique Dauge
IRMAR, Université de Rennes 1, FRANCE

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http://perso.univ-rennes1.fr/monique.dauge
Outline

1. Brief history of corner problems
2. Planned – Typical examples
3. Standard regularity in smooth domains for coercive forms
4. 2D polygon – Corner localization
5. 2D sector: $\Delta$ Dirichlet
6. 2D – $\Delta$ Dirichlet in weighted spaces
7. 2D – $\Delta$, miscellaneous in weighted spaces
8. 2D, general
9. 3D regular cones
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   - 3D corners
   - Regular cones in $\mathbb{R}^n$ and unweighted spaces
   - Injectivity modulo polynomials
What it is about

We address special combinations of classes of

1. Domains
2. Partial Differential Equations
3. Functional spaces
4. Questions

General framework of corner studies

1. Domains with general and combined corner types
2. Elliptic boundary value problems
3. Weighted and standard Sobolev spaces or Hölder classes
4. Existence, regularity, expansions of solutions

A sort of maximal framework would include

1. Cones in $\mathbb{R}^n$, conical manifolds, edges, manifolds with corners, polyhedra, recursively defined corner domains, ...
2. Multi-order Agmon-Douglis-Nirenberg systems with variable coefficients, possibly non-smooth with asymptotics, possibly piecewise smooth on compatible partitions
3. Sobolev spaces based on $L^2$ (Hilbert) or $L^p$ ($p > 1$), including (or not) weights based on distances to singular sets. Possible analytic-type control of derivatives.
4. Fredholm, semi-Fredholm, regularity shift, asymptotics of various types
A brief history of elliptic BVP with corners: Russian school

V. A. Kondrat’ev
Boundary-value problems for elliptic equations in domains with conical or angular points.

1. Domains with conical points
2. Scalar elliptic BVP
3. Hilbert Sobolev spaces with or without weights
4. Fredholm, regularity, asymptotics

V. G. Maz’ya, B. A. Plamenevskii
Elliptic boundary value problems on manifolds with singularities.

V. A. Kozlov, V. G. Maz’ya, J. Rossmann
*Elliptic boundary value problems in domains with point singularities.*

V. Maz’ya and J. Rossmann
*Elliptic equations in polyhedral domains.*

1. Hierarchy of singular sets
2. Elliptic systems
3. $L^p$ Sobolev spaces, Schauder classes, with weights
4. Fredholm, regularity, asymptotics
A brief history of elliptic BVP with corners: $\Psi$-do calculus

**S. Rempe, B. W. Schulze**

*Asymptotics for Elliptic Mixed Boundary Problems.*

Akademie-Verlag, 1989.

**B. W. Schulze**

*Pseudo-differential operators on manifolds with singularities.*


**B.-W. Schulze**

*Boundary value problems and singular pseudo-differential operators.*


**R. B. Melrose**

Pseudodifferential operators, corners and singular limits,


**R. B. Melrose**

Calculus of conormal distributions on manifolds with corners,


**R. B. Melrose**

*Differential analysis on manifolds with corners,*

A brief history of elliptic BVP with corners: In France & Belgium

P. Grisvard
Problèmes aux limites dans les polygones. Mode d’emploi.

P. Grisvard
Singularités en élasticité.

P. Grisvard
*Boundary Value Problems in Non-Smooth Domains.*

1 Polygonal domains

2 Elliptic BVP (Laplace, Lamé, $\Delta^2$)

S. Nicaise
Le laplacien sur les réseaux deux-dimensionnels polygonaux topologiques.

S. Nicaise
*Polygonal interface problems.*

1 Polygonal domains

2 Elliptic transmission problems (piecewise constant on polygonal subdomains)
A brief history of elliptic BVP with corners: Us, before Maxwell

**M. Dauge**

*Elliptic Boundary Value Problems in Corner Domains.*

1. Hierarchy of singular sets
2. Elliptic systems
3. Hilbert Sobolev spaces without weights (interaction with polynomials)
4. Semi-Fredholm, Fredholm, regularity, corner-edge asymptotics

**M. Costabel, M. Dauge**

General edge asymptotics of solutions of second order elliptic boundary value problems.

**M. Costabel, M. Dauge**

Construction of corner singularities for Agmon-Douglis-Nirenberg elliptic systems.

**M. Costabel, M. Dauge**

Stable asymptotics for elliptic systems on plane domains with corners.

1. Edges
2. Elliptic systems
3. Hilbert Sobolev spaces
4. Structure of singularities
A brief history of elliptic BVP with corners: Maxwell

M. Costabel, M. Dauge
Maxwell and Lamé eigenvalues on polyhedra.

M. Costabel, M. Dauge
Singularities of electromagnetic fields in polyhedral domains.

M. Costabel, M. Dauge, S. Nicaise
Singularities of Maxwell interface problems.
I. Babuška, B. Guo

Regularity of the solution of elliptic problems with piecewise analytic data. I. Boundary value problems for linear elliptic equation of second order,

I. Babuška, B. Guo

Regularity of the solution of elliptic problems with piecewise analytic data. II. The trace spaces and application to the boundary value problems with nonhomogeneous boundary conditions,

2D Polygonal domains

M. Costabel, M. Dauge, and S. Nicaise

Analytic regularity for linear elliptic systems in polygons and polyhedra,

M. Costabel, M. Dauge, and S. Nicaise

Weighted analytic regularity in polyhedra,

3D Polyhedral domains

M. Costabel, M. Dauge, and S. Nicaise

GLC project *(Grand Livre des Coins)*,
?? (20??)
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Aim: Introduce corner singularities in acoustic & electromagnetic problems in 2D and 3D domains.

Two main steps:

A  All (almost all) details in a minimal framework
   1  2D: Infinite plane sectors and bounded polygons
   2  Model operators (homogeneous with constant coefficients) of acoustics & electromagnetics: Laplace operator $\Delta$, magnetostatic Maxwell system.
   3  Weighted $L^2$ Sobolev spaces
   4  Regularity and singularities, estimates

B  Hints of results and proofs in an extended framework
   1  3D: Infinite regular cones, infinite edges, infinite polyhedral cones, bounded domains with conical points, edges or corners.
   2  More operators of acoustics & electromagnetics (with lower order terms, variable coefficients, transmission conditions): Helmholtz operator, harmonic Maxwell system, several materials with distinct electromagnetic properties.
   3  Ordinary non-weighted $L^2$ Sobolev spaces (“non-weighted” is not a particular case of “weighted”)
   4  Regularity and singularities, estimates
Problems in variational form

We consider problems in simple *(coercive, not saddle point)* variational form posed on a domain $\Omega \subset \mathbb{R}^n$. This requires two ingredients:

- A real bilinear *(or complex sesquilinear)* form $a$ of order 1 defined on a space $X(\Omega)$

$$a(u, v) = \sum_{i=1}^{d} \sum_{j=1}^{d} \sum_{|\alpha| \leq 1} \sum_{|\beta| \leq 1} \int_{\Omega} a_{ij}^{\alpha\beta}(x) \partial_x^{\alpha} u_i \partial_x^{\beta} \bar{v}_j \, dx, \quad u, v \in X(\Omega)$$

- $x = (x_1, \ldots, x_n)$ is the variable in $\Omega \subset \mathbb{R}^n$ and for $\alpha = (\alpha_1, \ldots, \alpha_n)$

$$\partial_x^{\alpha} = \frac{\partial^{\alpha_1}}{\partial x_1} \cdots \frac{\partial^{\alpha_n}}{\partial x_n}$$

- $d$ is the dimension of the system: $u = (u_1, \ldots, u_d), v = (v_1, \ldots, v_d)$
- the coefficients $a_{ij}^{\alpha\beta}$ are smooth *(or piecewise smooth)* in $\Omega$
- for acoustics, elasticity and in many cases, the space $X(\Omega)$ is defined as

$$X(\Omega) = H^1(\Omega)^d \quad \text{for } \Delta, \text{ Lamé, etc...}$$

- for electromagnetism, $X(\Omega)$ is defined as

$$X(\Omega) = H(\text{curl}; \Omega) \cap H(\text{div}; \Omega) \quad \text{for Maxwell}$$

- A variational space $V \subset X(\Omega)$ that determines essential (or Dirichlet) boundary conditions

$$V = \{ u \in X(\Omega), \quad \Pi^D u = 0 \text{ on } \partial \Omega \}$$

where $\Pi^D$ is a chosen projection operator (examples follow).
Typical examples [The Laplace operator of acoustics]

Scalar $\nabla \cdot \nabla$ form (here $d = 1$)

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx = \sum_{|\alpha| = 1} \int_\Omega \partial_\alpha^x u \partial_\alpha^x v \, dx = \sum_{\ell=1}^n \int_\Omega \partial_{x_\ell} u \partial_{x_\ell} v \, dx$$

The space $X(\Omega)$ is the Sobolev space $H^1(\Omega) = \{u \in L^2(\Omega), \nabla u \in L^2(\Omega)^n\}$.

Two main cases plus a generalization:

1. **Dirichlet:** The projection operator $\Pi^D$ is the identity on $\partial\Omega$, so

$$V = H_0^1(\Omega) = \overset{\circ}{H}^1(\Omega) = \{u \in H^1(\Omega), \ u = 0 \ on \ \partial\Omega\}$$

2. **Neumann:** The projection operator $\Pi^D$ is zero on $\partial\Omega$, so $V = H^1(\Omega)$

2’. **Mixed:** $\partial\Omega$ has a partition in two pieces $\partial D\Omega$ and $\partial \Omega \setminus \partial D\Omega$:

$$\Pi^D(x) = \begin{cases} 1 & x \in \partial D\Omega \\ 0 & x \in \partial \Omega \setminus \partial D\Omega \end{cases}$$
**Typical vector examples with \( d = n \) [Lamé and Maxwell]**

**Lamé elastic bilinear form** defined for \( \mathbf{u}, \mathbf{v} \in H^1(\Omega)^n \)

\[
a(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \left( 2\mu \sum_{i=1}^{n} \sum_{j=1}^{n} e_{ij}(\mathbf{u}) e_{ij}(\mathbf{v}) + \lambda \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} \right) \, dx
\]

- \( \lambda \geq 0 \) and \( \mu > 0 \) are the Lamé coefficients
- \( e_{ij}(\mathbf{u}) = \frac{1}{2} (\partial_i u_j + \partial_j u_i) \) are the components of the strain tensor.

To define essential boundary conditions, introduce \( \mathbf{n} \) as the exterior unit normal to \( \partial \Omega \).

**3** **Simply supported:** The projection operator \( \Pi^D \) is defined as \( \Pi^D \mathbf{u} = \mathbf{u} - (\mathbf{u} \cdot \mathbf{n}) \mathbf{n} \), so \( \Pi^D \mathbf{u} = 0 \) means that the tangential component of \( \mathbf{u} \) is 0 on the boundary \( \partial \Omega \).

\[
V = H_N := \{ \mathbf{u} \in H^1(\Omega)^n, \quad \mathbf{u} - (\mathbf{u} \cdot \mathbf{n}) \mathbf{n} = 0 \text{ on } \partial \Omega \}
\]

**Maxwell regularized electric form** defined for \( \mathbf{u}, \mathbf{v} \in H(\operatorname{curl}; \Omega) \cap H(\operatorname{div}; \Omega) = \mathbb{R} \cap D \)

\[
a(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \left( \mu^{-1} \operatorname{curl} \mathbf{u} \, \operatorname{curl} \mathbf{v} + s \operatorname{div} \mathbf{\varepsilon} \mathbf{u} \, \operatorname{div} \mathbf{\varepsilon} \mathbf{v} \right) \, dx
\]

with electric permittivity \( \varepsilon > 0 \) and magnetic permeability \( \mu > 0 \). Real parameter \( s > 0 \).

**4** **Electric perfect conductor condition:** The projector \( \Pi^D \) is defined as above.

\[
V = X_N := \{ \mathbf{u} \in H(\operatorname{curl}; \Omega) \cap H(\operatorname{div}; \Omega), \quad \mathbf{u} - (\mathbf{u} \cdot \mathbf{n}) \mathbf{n} = 0 \text{ on } \partial \Omega \}
\]

NB: If \( n = 3 \), \( \mathbf{u} - (\mathbf{u} \cdot \mathbf{n}) \mathbf{n} = 0 \) if and only if \( \mathbf{u} \times \mathbf{n} = 0 \).
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Coercivity

Definition

Let \( a \) be a sesquilinear form defined on \( V \subset X(\Omega) \) with compact embedding in \( L^2(\Omega) \).

- \( a \) is said continuous if
  \[
  \exists C > 0, \quad \forall u, v \in V, \quad |a(u, v)| \leq C\|u\|_{X(\Omega)} \|v\|_{X(\Omega)}
  \]

- \( a \) is said \( V \)-coercive if
  \[
  \exists C > 0, \quad \forall u \in V, \quad \text{Re} \ a(u, u) \geq c\|u\|^2_{X(\Omega)} - C\|u\|^2_{L^2(\Omega)}
  \]

- \( a \) is said strongly \( V \)-coercive if \( C \) can be taken to 0 in (2).

A continuous sesquilinear form \( a \) defined on \( V \) is associated with an operator

\[
A : V \rightarrow V' \quad u \mapsto (v \mapsto a(u, v))
\]

Theorem S.1

- If \( a \) is continuous and strongly \( V \)-coercive, then \( A \) is an isomorphism \( V \rightarrow V' \)
- If \( a \) is continuous and \( V \)-coercive, then \( A \) is Fredholm of index 0 from \( V \) into \( V' \)
Integration by parts

Assumption: $Au = f$ with $f \in L^2(\Omega)^d$

$$u \in V, \forall v \in V, \quad a(u, v) = \int_{\Omega} f \bar{v} \, dx$$

Therefore, in the distributional sense (we take $v \in \mathcal{D}(\Omega)$)

$$Lu = f \quad \text{with} \quad (Lu)_j = \sum_{i=1}^{d} \sum_{|\alpha| \leq 1} \sum_{|\beta| \leq 1} (-1)^{|\beta|} \partial_{x}^{\beta} a_{ij}^{\alpha \beta}(x) \partial_{x}^{\alpha} u_{i}, \quad j = 1, \ldots, d$$

Integrating by parts gives a sense to “natural boundary conditions” that complete the essential boundary conditions. Revisit examples.

Here $A$ is the second order operator associated with the general $a$. It is distinct from generic first order operators by Dirk
Integration by parts: The four examples

1. **Dirichlet for scalar Laplacian** (strongly coercive):
   \[
   \begin{align*}
   -\Delta u &= f \quad \text{in } \Omega \\
   u &= 0 \quad \text{on } \partial \Omega.
   \end{align*}
   \]

2. **Neumann for scalar Laplacian** (coercive):
   \[
   \begin{align*}
   -\Delta u &= f \quad \text{in } \Omega \\
   \partial_n u &= 0 \quad \text{on } \partial \Omega.
   \end{align*}
   \]

3. **Simply supported for Lamé** (strongly coercive): Setting \( u_{\tau} = u - (u \cdot n)n \)
   \[
   \begin{align*}
   -(\mu \Delta + (\lambda + \mu) \nabla \text{div})u &= f \quad \text{in } \Omega \\
   u_{\tau} &= 0 \quad \text{on } \partial \Omega \\
   \mu(\partial_n u) \cdot n + (\lambda + \mu) \text{div } u &= 0 \quad \text{on } \partial \Omega
   \end{align*}
   \]

4. **Electric perfect conductor condition for Maxwell** (strongly coercive): \( (\mu = \varepsilon = 1) \)
   \[
   \begin{align*}
   (\text{curl curl} - s \nabla \text{div})u &= f \quad \text{in } \Omega \\
   u_{\tau} &= 0 \quad \text{on } \partial \Omega \\
   \text{div } u &= 0 \quad \text{on } \partial \Omega
   \end{align*}
   \]

NB: In any open set where \( \partial \Omega \) is flat and \( u_{\tau} = 0 \), we have \( \text{div } u = \partial_n (u \cdot n) = (\partial_n u) \cdot n \).
The regular case [general]

Assume
- $\Omega$ is a bounded open set with a smooth boundary in $\mathbb{R}^n$
- The coefficients $a_{ij}^{\alpha\beta}$ of the form $a$ are smooth functions on $\overline{\Omega}$
- The variational space is a subset of $H^1(\Omega)$, i.e. $X(\Omega) = H^1(\Omega)$
- The form $a$ is continuous and $V$-coercive
- $u \in V$ is solution of $Au = f$ with $f \in L^2(\Omega)$, \[\text{i.e. } \forall v \in V, \quad a(u, v) = \int_\Omega f \overline{v} \, dx\]

**Theorem S.2**

Under the previous assumptions:
- $u$ belongs to the Sobolev space $H^2(\Omega)$
- There exists a constant $C$ independent of $u$ such that
  \[
  \|u\|_{H^2(\Omega)} \leq C \left( \|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)} \right)
  \]
- Let $\mathcal{U}$ and $\mathcal{U}'$ two open sets in $\mathbb{R}^n$ such that $\overline{\mathcal{U}} \subset \mathcal{U}'$; Set $\mathcal{V} = \Omega \cap \mathcal{U}$ and $\mathcal{V}' = \Omega \cap \mathcal{V}'$. We have the local a priori estimates with a constant independent of $u$
  \[
  \|u\|_{H^2(\mathcal{V})} \leq C \left( \|f\|_{L^2(\mathcal{V}')} + \|u\|_{H^1(\mathcal{V}')} \right)
  \]
- If $f$ belongs to $H^{m-2}(\Omega)$ with an integer $m > 2$, then $u$ belongs to $H^m(\Omega)$ and the estimates (1) and (2) hold with $H^m$ and $H^{m-2}$ instead of $H^2$ and $L^2$, respectively.
The regular case [Maxwell]

Assume

- $\Omega$ is a bounded open set with a smooth boundary in $\mathbb{R}^n$
- The coefficients $\mu$ and $\varepsilon$ of the regularized Maxwell form $a$ are smooth positive functions on $\overline{\Omega}$. Recall

$$a(u, v) = \int_{\Omega} \left( \mu^{-1} \text{curl } u \cdot \text{curl } v + s \text{div } \varepsilon u \cdot \text{div } \varepsilon v \right) \, dx$$

**Theorem S.3**  
See Martin’s course
Convex polygons

Under the previous assumptions:

- The variational space $X_N(\Omega)$ is a subset of $H^1(\Omega)^d$, which means that $X_N(\Omega)$ coincides with the variational space $H_N(\Omega)$ of elasticity.
- The same holds with the “magnetic” spaces $X_T(\Omega)$ and $H_T(\Omega)$ for which the essential boundary condition is $u \cdot n = 0$ on $\partial \Omega$.
- There exists a constant $C$ such that for all $u \in X_N(\Omega)$ (or $u \in H_T(\Omega)$)

$$\|u\|_{H^1(\Omega)} \leq C \left( \| \text{curl } u \|_{L^2(\Omega)} + \| \text{div } u \|_{L^2(\Omega)} + \| u \|_{L^2(\Omega)} \right)$$

As a consequence of this theorem, the regularized Maxwell form satisfies the assumptions of the general case.

So the conclusions of Theorem S.2 apply.
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Dirichlet Laplacian on a polygon – Corner localization

We are going to study in detail solutions of

$$-\Delta u = f \quad \text{in} \quad \Omega, \quad u \in H^1_0(\Omega)$$

when $\Omega$ is a polygon in the plane $\mathbb{R}^2$, i.e. $\Omega$ is a bounded open set and its boundary is a finite union of segments. The ends of these segments are the corners of $\Omega$.

- Denote by $C$ the set of corners $c$.
- For each $c \in C$, there exists an infinite plane sector $\Gamma_c$ that coincides with $\Omega$ in a neighborhood $B(c, R_c)$ of $c$.
- Denote by $\omega_c$ the opening of $\Gamma_c$, which defines the opening of $\Omega$ at the corner $c$.
- Introduce local polar coordinates $(r_c, \theta_c)$ such that

$$\Gamma_c = \{ x \in \mathbb{R}^2, \ r_c > 0, \ \theta_c \in (0, \omega_c) \}$$

Introduce a smooth cut-off

$$\chi_c(x) = \begin{cases} 
0 & \text{if } x \notin B(c, R_c) \\
1 & \text{if } x \in B(c, R_c/2) 
\end{cases}$$

We have $-\Delta(\chi_c u) = \chi_c f + 2\nabla \chi_c \cdot \nabla u + (\Delta \chi_c) u$. Therefore, by extension by 0

$$\Delta(\chi_c u) \in L^2(\Gamma_c), \quad \chi_c u \in H^1_0(\Gamma_c)$$

Regularity outside corners

For any cut-off $\chi$ with support $U'$ disjoint from the corners we find by Theorem S.2 that

$$\chi u \in H^2(U \cap \Omega) \quad \text{with} \quad U = \chi^{-1}(1).$$
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We have reduced our problem on a polygon to problems of the type (we re-baptize as \(u\) the localized function \(\chi_c u\) and we drop the index \(c\))

\[
\begin{align*}
\Delta u &= f \text{ in } \Gamma, \quad f \in L^2(\Gamma), \quad \text{supp } f \subset B(0, R) \\
u &\in H^1_0(\Gamma), \quad \text{supp } u \subset B(0, R)
\end{align*}
\]

Use polar coordinates \((r, \theta)\). We have

\[
x_1 = r \cos \theta, \quad x_2 = r \sin \theta \quad \text{and} \quad \begin{cases} 
    r \partial_r &= r \cos \theta \partial_{x_1} + r \sin \theta \partial_{x_2} \\
    \partial_\theta &= -r \sin \theta \partial_{x_1} + r \cos \theta \partial_{x_2}
\end{cases}
\]

Set \(\tilde{u}(r, \theta) = u(x)\) and \(\tilde{g}(r, \theta) = r^2 f(x)\). The equation \(\Delta u = f\) is equivalent to

\[
((r \partial_r)^2 + \partial_\theta^2) \tilde{u} = \tilde{g} \quad \text{in } \mathbb{R}_+ \times I \quad \text{with } I = (0, \omega).
\]

Then set \(t = \log r\) (i.e. \(r = e^t\)) – Euler change of variables –

\[
r \partial_r = \partial_t \quad \text{and} \quad dx = r \, dr \, d\theta = e^{2t} \, dt \, d\theta
\]

Set \(\bar{u}(t, \theta) = \tilde{u}(r, \theta)\) and \(\bar{g}(t, \theta) = \tilde{g}(r, \theta)\). The equation \(\Delta u = f\) is equivalent to

\[
(\partial_t^2 + \partial_\theta^2) \bar{u} = \bar{g} \quad \text{in } \mathbb{R} \times I \quad \text{with } I = (0, \omega).
\]
Dirichlet Laplacian on a sector – Exponential weights

**Lemma**

(1) If $f \in L^2(\Gamma)$ with supp $f \in B(0, R)$, then $\tilde{g} = e^{2t\tilde{f}}$ satisfies

$$\forall \eta \leq 1, \quad e^{-\eta t} \tilde{g} \in L^2(\mathbb{R} \times \mathcal{I}) \quad \text{and} \quad \|e^{-\eta t} \tilde{g}\|_{L^2(\mathbb{R} \times \mathcal{I})} \leq C\|f\|_{L^2(\Gamma)}$$

(2) If $u \in H^1_0(\Gamma)$ with supp $u \in B(0, R)$, then $\tilde{u}$ satisfies

$$\forall \eta \leq 0, \quad e^{-\eta t} \tilde{u} \in H^1_0(\mathbb{R} \times \mathcal{I}) \quad \text{and} \quad \|e^{-\eta t} \tilde{u}\|_{H^1(\mathbb{R} \times \mathcal{I})} \leq C\|u\|_{H^1(\Gamma)}$$

The constant $C$ is independent of $\eta$, of $u$ and of $f$.

**Proof of (1)**

$$\int_{\mathbb{R}} \int_{\mathcal{I}} |e^{-\eta t} \tilde{g}(t, \theta)|^2 \, dt \, d\theta = \int_{\mathbb{R}} \int_{\mathcal{I}} |e^{(-\eta + 1)t} \tilde{f}(t, \theta)|^2 \, e^{2t} \, dt \, d\theta$$

$$= \int_{\Gamma} |r^{-\eta + 1} f(x)|^2 \, dx$$

$$\leq R^{1-\eta} \int_{\Gamma} |f(x)|^2 \, dx$$

because of the support condition and $1 - \eta \geq 0$. 

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**Proof of (2)**

\[
\int_{\mathbb{R}} \int_{\mathcal{I}} |e^{-\eta t} \partial_t \tilde{u}(t, \theta)|^2 \ dt \ d\theta = \int_{\mathbb{R}} \int_{\mathcal{I}} |e^{-\eta t} e^{-t} \partial_t \tilde{u}(t, \theta)|^2 \ e^{2t} \ dt \ d\theta \\
\leq \int_{\Gamma} |r^{-\eta} \partial_x u(x)|^2 + |r^{-\eta} \partial_x u(x)|^2 \ dx \\
\leq R^{-\eta} \int_{\Gamma} |\nabla u(x)|^2 \ dx
\]

because of the support condition and \( -\eta \geq 0 \). The same for \( \partial_\theta \tilde{u} \).

- For \( \tilde{u} \), we find

\[
\int_{\mathbb{R}} \int_{\mathcal{I}} |e^{-\eta t} \tilde{u}(t, \theta)|^2 \ dt \ d\theta = \int_{\Gamma} |r^{-\eta-1} u(x)|^2 \ dx \\
\leq R^{-\eta} \int_{\Gamma} |r^{-1} u(x)|^2 \ dx
\]

and may conclude if we know that \( r^{-1} u \) belongs to \( L^2(\Gamma) \).

- The latter is true thanks to Poincaré’s inequality (due to boundary conditions)

\[
\|\tilde{u}(r, \cdot)\|_{L^2(\mathcal{I})} \leq C \|\partial_\theta \tilde{u}(r, \cdot)\|_{L^2(\mathcal{I})}
\]

that implies \( \|r^{-1} u\|_{L^2(\Gamma)} \leq C \|\nabla u\|_{L^2(\Gamma)} \).
**Definition**

- If \( r^{-\eta-1} \nu \) belongs to \( L^2(\Gamma) \), we define the Mellin transform \( \mathcal{M}\nu \) as
  \[
  \mathcal{M}\nu[\lambda](\theta) = \int_0^{\infty} r^{-\lambda} \tilde{\nu}(r, \theta) \frac{dr}{r}, \quad \lambda \in \mathbb{C}, \text{ Re } \lambda = \eta
  \]

- If \( e^{-\eta t} \tilde{\nu} \) belongs to \( L^2(\mathbb{R} \times I) \), we define the Fourier-Laplace transform \( \mathcal{L}(\tilde{\nu}) \) as
  \[
  \mathcal{L}(\tilde{\nu})[\lambda](\theta) = \int_{\mathbb{R}} e^{-\lambda t} \tilde{\nu}(t, \theta) \ dt, \quad \lambda \in \mathbb{C}, \text{ Re } \lambda = \eta
  \]

Note the equivalence

\[
  r^{-\eta-1} \nu \in L^2(\Gamma) \iff e^{-\eta t} \tilde{\nu} \in L^2(\mathbb{R} \times I)
  \]

and the identities, with \( \mathcal{F} \) the standard Fourier transform,

\[
  \lambda = \eta + i \xi : \quad \mathcal{M}\nu[\lambda] = \mathcal{L}(\tilde{\nu})[\lambda] = \mathcal{F}(e^{-\eta t} \tilde{\nu})[\xi]
  \]

We mainly use it in its Mellin form.
Properties

Let \( \mathcal{X} \) be a Hilbert space of functions on the interval \( \mathcal{I} \), for example \( \mathcal{X} = H^m(\mathcal{I}) \).

- The function \( e^{-\eta t}\mathcal{V} \) belongs to \( L^2(\mathbb{R}, \mathcal{X}) \), if and only if the function
  \[
  \xi \mapsto \mathcal{M}\mathcal{V}[\eta + i\xi] \quad \text{belongs to} \quad L^2(\mathbb{R}, \mathcal{X})
  \]

- Let \( \eta_0 < \eta_1 \). If \( e^{-\eta t}\mathcal{V} \in L^2(\mathbb{R}, \mathcal{X}) \) for \( \eta = \eta_0 \) and \( \eta = \eta_1 \), then \( e^{-\eta t}\mathcal{V} \in L^2(\mathbb{R}, \mathcal{X}) \) for all \( \eta \in [\eta_0, \eta_1] \) and the function
  \[
  \lambda \mapsto \mathcal{M}\mathcal{V}[\lambda] \quad \text{is holomorphic with values in} \quad \mathcal{X}
  \]

Recall that

\[
\begin{align*}
\mathcal{U} & \in H^1_0(\Gamma), \\
\text{supp} \mathcal{U} & \subset B(0, R), \\
\Delta \mathcal{U} & = f \in L^2(\Gamma).
\end{align*}
\]

With \( g := r^2f \) and \( \mathcal{I} = (0, \omega) \) we have

- \( \mathcal{M}\mathcal{U} \) is **holomorphic** in the complex half-plane \( \text{Re} \lambda < 0 \) with values in \( H^1_0(\mathcal{I}) \)
- \( \mathcal{M}g \) is **holomorphic** in the complex half-plane \( \text{Re} \lambda < 1 \) with values in \( L^2(\mathcal{I}) \) and
  \[
  \xi \mapsto \mathcal{M}g[\eta + i\xi] \quad \text{belongs to} \quad L^2(\mathbb{R} \times \mathcal{I}) \quad \text{for all} \quad \eta \leq 1.
  \]
- Owing to formulas \( \mathcal{L}(\partial_t \mathcal{U})[\lambda] = \lambda \mathcal{L}(\mathcal{U})[\lambda] \) and \( \mathcal{L}(\partial_{\omega} \mathcal{U})[\lambda] = \partial_{\omega} \mathcal{L}(\mathcal{U})[\lambda] \) we have
  \[
  (\lambda^2 + \partial_{\omega}^2) \mathcal{M}\mathcal{U}[\lambda] = \mathcal{M}g[\lambda], \quad \forall \lambda, \quad \text{Re} \lambda \leq 0.
  \]
Dirichlet Laplacian on a sector – Mellin symbol

**Definition**

The **Mellin symbol** of $\Delta$ with Dirichlet conditions is defined for any $\lambda \in \mathbb{C}$ as

$$\mathcal{A}[\lambda] : \quad H^1_0(\mathcal{I}) \longrightarrow H^{-1}(\mathcal{I})$$

$$U \longmapsto (\lambda^2 + \partial_\theta^2) U$$

The symbol $\mathcal{A}[\lambda]$ can also be considered as acting from $(H^2 \cap H^1_0)(\mathcal{I})$ into $L^2(\mathcal{I})$.

Denote by $\mathfrak{P}$ the positive Laplace operator $-\partial_\theta^2$ from $(H^2 \cap H^1_0)(\mathcal{I})$ into $L^2(\mathcal{I})$.

- The operator $\mathfrak{P}$ is self-adjoint on $L^2(\mathcal{I})$, positive, with compact resolvent. Its spectrum $\sigma(\mathfrak{P})$ is a discrete subset of $(0, +\infty)$. Explicit calculation shows that

$$\sigma(\mathfrak{P}) = \{ \mu_\ell := \left( \frac{\ell \pi}{\omega} \right)^2, \quad \ell \in \mathbb{N}_* \}$$

Since

$$\mathcal{A}[\lambda] = -\mathfrak{P} + \lambda^2$$

we deduce that the spectrum $\sigma(\mathcal{A})$ of the Mellin symbol $\mathcal{A}$ is

$$\sigma(\mathcal{A}) = \{ \lambda_\ell := \frac{\ell \pi}{\omega}, \quad \ell \in \mathbb{Z}_* \}$$

- The symbol $\lambda \mapsto \mathcal{A}[\lambda]$ is **holomorphic** and its inverse $\lambda \mapsto \mathcal{A}[\lambda]^{-1}$ is **meromorphic**.
Recall that
\[ \forall \lambda \text{ s.t. } \Re \lambda \leq 0, \quad A[\lambda] M u[\lambda] = M g[\lambda]. \]

We define a meromorphic extension \( U[\lambda] \) of \( M u[\lambda] \) by setting
\[ \forall \lambda \text{ s.t. } \Re \lambda \in (0, 1], \quad U[\lambda] = A[\lambda]^{-1} (M g[\lambda]) \]

We are going to prove

**Proposition**

1. Exists \( u_1 \) such that \( e^{-t} \ddot{u}_1 \in H^2(\mathbb{R} \times I) \) and \( M u_1[\lambda] = U[\lambda] \) for \( \lambda \) s.t. \( \Re \lambda = 1 \)
2. The difference between \( u \) and \( u_1 \) is given by the residue formula
\[ u_1 - u = \sum_{\lambda_0 \in \sigma(A) \atop \Re \lambda_0 \in (0,1)} \text{Res} r^\lambda U[\lambda] \]

Using that \( \sigma(A) = \frac{\pi}{\omega} \mathbb{Z}_* \), we obtain as a corollary
\[
\begin{cases} 
  u_1 = u & \text{if } \omega < \pi \\
  u_1 - u = \text{Res} r^\lambda U[\lambda] & \text{if } \omega > \pi
\end{cases}
\]
The proof of the proposition is based on estimates of $\mathcal{A}[\lambda]^{-1}$ in operator norm. Define the parameter norm $\|U\|_{H^m(I;\lambda)}$ for $m \in \mathbb{N}$ as

$$\|U\|_{H^m(I;\lambda)}^2 = \sum_{k=0}^{m} (|\lambda| + 1)^{2k} \|U\|_{H^{m-k}(I)}^2$$

**Lemma**

Let $\eta_0 \leq \eta_1$ and $\delta > 0$. Define the set

$$\Lambda = \left\{ \lambda \in \mathbb{C}, \; \text{Re} \lambda \in [\eta_0, \eta_1] \right\} \setminus \bigcup_{\lambda_0 \in \sigma(\mathcal{A})} \mathcal{B}(\lambda_0, \delta).$$

Let $m \geq 2$. Exists a constant $C$ such that

$$\forall \lambda \in \Lambda, \; \forall G \in H^{m-2}(I), \quad \|\mathcal{A}[\lambda]^{-1}G\|_{H^m(I;\lambda)} \leq C \|G\|_{H^{m-2}(I;\lambda)}$$

**Proof:** Based on two steps.

**Step 1** If $K$ is a compact set in $\mathbb{C}$ disjoint from the spectrum of $\mathcal{A}$ the resolvent estimate (1) holds with a constant $C$ depending on $K$ by continuity of $\mathcal{A}[\lambda]^{-1}$ with respect to $\lambda$.

**Step 2** It remains to prove (1) for a set $\Lambda$ that is the intersection of the strip

$$\{ \lambda \in \mathbb{C}, \; \text{Re} \lambda \in [\eta_0, \eta_1] \}$$

when $|\lambda|$ is large enough...
Dirichlet Laplacian on a sector – Resolvent estimates, continued

\[ (1) \quad \forall \lambda \in \Lambda, \quad \forall G \in L^2(I), \quad \| \mathfrak{A}[\lambda]^{-1} G \|_{H^m(I;\lambda)} \leq C \| G \|_{H^{m-2}(I;\lambda)} \]

To perform Step 2 of the proof of (1), we choose \( \lambda, G \), and set \( V = \mathfrak{A}[\lambda]^{-1} G \). Then introduce on \( \mathbb{R} \times I \)

\[
\tilde{\nu}(t, \theta) = e^{\lambda t} V(\theta) \quad \text{and} \quad \tilde{g}(t, \theta) = e^{\lambda t} G(\theta).
\]

We can check:
- \( \tilde{\nu} \in H^1((-2, 2) \times I) \), \( \tilde{\nu} = 0 \) on \( \mathbb{R} \times \partial I \)
- \( \Delta \tilde{\nu} = \tilde{g} \)
- Local estimates (Theorem S.2) give

\[
\| \tilde{\nu} \|_{H^m((-1, 1) \times I)} \leq C \left( \| \tilde{g} \|_{H^{m-2}((-2, 2) \times I)} + \| \tilde{\nu} \|_{H^1((-2, 2) \times I)} \right)
\]

- Coming back to \( V \) and \( G \) —here \( C \) depends on \( \eta_0 \) and \( \eta_1 \),

\[
\| V \|_{H^m(I;\lambda)} \leq C \left( \| G \|_{H^{m-2}(I;\lambda)} + \| V \|_{H^1(I;\lambda)} \right)
\]

- We conclude because for \( |\lambda| \) large enough, \( C \| V \|_{H^1(I;\lambda)} \leq \frac{1}{2} \| V \|_{H^m(I;\lambda)} \).
**Proposition, point (1), more precise**

Let \( u_1 \) be defined as the inverse Mellin transform

\[
u_1(x) = \frac{1}{2i\pi} \int_{\text{Re } \lambda = 1} r^\lambda U[\lambda](\theta) \, d\lambda
\]

Then \( e^{-t} \tilde{u}_1 \in H^2(\mathbb{R} \times I) \), which means for \( u_1 \):

\[r^{-2+|\alpha|} \partial_x^\alpha u_1 \in L^2(\Gamma), \quad |\alpha| \leq 2.\]

- \( U[\lambda] = A[\lambda]^{-1} Mg[\lambda] \)
- The Mellin transform of \( g = r^2 f \) satisfies \( \xi \mapsto Mg[1 + i\xi] \) belongs to \( L^2(\mathbb{R}, L^2(I)) \)
- The resolvent estimate with \( m = 2 \) on the line \( \text{Re } \lambda = 1 \) (disjoint from \( \sigma(A)! \) yields

\[
\xi \mapsto |1 + i\xi|^k U[1 + i\xi] \quad \text{belongs to} \quad L^2(\mathbb{R}, H^{2-k}(I)), \quad k = 0, 1, 2.
\]

- Since

\[
\frac{1}{2i\pi} \int_{\text{Re } \lambda = 1} r^\lambda U[\lambda] \, d\lambda = \frac{1}{2\pi} \int_\mathbb{R} e^{it\xi} U[1 + i\xi] \, d\xi
\]

we find by inverse Fourier transform that \( e^{-t} \tilde{u}_1 \in H^2(\mathbb{R} \times I) \).

- Back to Cartesian coordinates with \( r = e^t \) gives the weighted regularity for \( u_1 \).
Proposition, point (2)

\[ u_1 - u = \sum_{\lambda_0 \in \sigma(\mathfrak{A})} \text{Res } r^\lambda U[\lambda] \]

- \( U[\lambda] = \mathfrak{A}[\lambda]^{-1} \mathfrak{M} g[\lambda] \)
- The poles of \( U[\lambda] \) are in the set of poles of \( \mathfrak{A}[\lambda]^{-1} \) since \( \mathfrak{M} g[\lambda] \) is holomorphic
- For any simple rectifiable curve \( \gamma \) that is contained in the open strip \( \text{Re } \lambda \in (-\infty, 1) \) and surrounds the pole \( \lambda_0 = \frac{\pi}{\omega} \) when \( \omega > \pi \), we have

\[
\frac{1}{2i\pi} \int_{\gamma} r^\lambda U[\lambda](\theta) \, d\lambda = \sum_{\lambda_0 \in \sigma(\mathfrak{A})} \text{Res } r^\lambda U[\lambda] \]

- Take \( \gamma \) as the rectangle \( \text{Re } \lambda = 0, \text{Im } \lambda = -\xi \), \( \text{Re } \lambda = 1 - \delta \), \( \text{Re } \lambda = \xi \), with \( \delta > 0 \) small enough and \( \xi > 0 \). We may push \( \xi \) to infinity and \( \delta \) to 0 using the resolvent estimates to bound \( U[\lambda] \).
Dirichlet Laplacian on a sector – The residue

If $\omega > \pi$, we have one residue in the relevant region $\text{Re } \lambda \in [0, 1]$.

- Recall that $A[\lambda] = -\mathcal{P} + \lambda^2$ with $\mathcal{P} = -\partial_\theta^2$ on $H_0^1(I)$.
- Let $\phi_\ell$ be an orthonormal spectral basis for $\mathcal{P}$ associated with eigenvalues $\mu_\ell$
  \[
  \mu_\ell = \lambda_\ell^2 \quad \text{with} \quad \lambda_\ell = \frac{\ell \pi}{\omega}, \quad \text{and} \quad \phi_\ell(\theta) = \sqrt{\frac{2}{\omega}} \sin \frac{\ell \pi}{\omega} \theta, \quad [\ell \geq 1]
  \]
- Then
  \[
  \mathcal{P} U = \sum_{\ell \in \mathbb{N}_*} \mu_\ell \langle \phi_\ell, U \rangle \phi_\ell \quad \text{hence} \quad (-\mathcal{P} + \lambda^2)^{-1} G = \sum_{\ell \in \mathbb{N}_*} \frac{1}{\lambda^2 - \lambda_\ell^2} \langle \phi_\ell, G \rangle \phi_\ell
  \]
- With $G[\lambda]$ the Mellin transform of $g = r^2 f$, we find
  \[
  \text{Res } r^\lambda U[\lambda] = \text{Res } r^\lambda (-\mathcal{P} + \lambda^2)^{-1} G[\lambda]
  \]
  \[
  = r^{\lambda_1} \frac{1}{2 \lambda_1} \langle \phi_1, G[\lambda_1] \rangle \phi_1
  \]
  \[
  = \gamma_1 r^{\frac{\pi}{\omega}} \sin \frac{\pi}{\omega} \theta \quad \text{with} \quad \gamma_1 = \frac{1}{\pi} \int_{\Gamma} r^{-\frac{\pi}{\omega}} \sin \frac{\pi}{\omega} \theta f(x) \, dx
  \]
- Finally $u = u_1 - \gamma_1 r^{\frac{\pi}{\omega}} \sin \frac{\pi}{\omega} \theta$ and the proposition is proved and the theorem follows...
Dirichlet Laplacian on a sector – Theorem with rhs in $L^2$

**Theorem L.1**

Let $\Gamma$ be a plane sector of opening $\omega \in (0, \pi) \cup (\pi, 2\pi)$ and $u \in H^1_0(\Gamma)$ with compact support such that

$$\Delta u = f \text{ in } \Gamma, \quad f \in L^2(\Gamma).$$

- If $\omega < \pi$, then $u$ belongs to $H^2(\Gamma)$ and moreover satisfies $r^{-2}u$, $r^{-1}\nabla u$ in $L^2(\Gamma)$
- If $\omega > \pi$, then

$$(L.1) \quad u = u_1 - \gamma_1 r^{\frac{\pi}{\omega}} \sin \frac{\pi}{\omega} \theta \quad \text{with} \quad \gamma_1 = \frac{1}{\pi} \int_{\Gamma} r^{-\frac{\pi}{\omega}} \sin \frac{\pi}{\omega} \theta f(x) \, dx$$

and $u_1$ satisfying

$$r^{-2+|\alpha|} \partial^\alpha_x u_1 \in L^2(\Gamma), \quad |\alpha| \leq 2$$

**Case of a crack**

If $\omega = 2\pi$, Theorem L.1 does not apply because the Mellin symbol has a pole in $\lambda = 1$. Nevertheless it is possible to prove that (L.1) holds with the weaker regularity for $u_1$

$$\partial^\alpha_x u_1 \in L^2(\Gamma), \quad |\alpha| = 2, \quad \text{and} \quad r^{-2+\delta+|\alpha|} \partial^\alpha_x u_1 \in L^2(\Gamma), \quad |\alpha| \leq 1, \, \delta > 0.$$
**Outline**

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Dirichlet Laplacian on a polygon – Theorem with rhs in $L^2$

**Theorem L.2**

Let $\Omega$ be a plane polygon with corners set $\mathcal{C} \ni c$ associated with openings $\omega_c \in (0, \pi) \cup (\pi, 2\pi)$. Let $\chi_c$ smooth cut-off functions separating the corners.

- Let $u \in H^1_0(\Omega)$ be solution of
  \[
  \Delta u = f \quad \text{in} \quad \Omega, \quad f \in L^2(\Omega).
  \]

- Then
  \[
  u = u_{1,\text{reg}} + \sum_{c \in \mathcal{C}, \omega_c > \pi} \chi_c(x) \gamma_c r_c^{\frac{\pi}{\omega_c}} \sin \frac{\pi}{\omega_c} \theta_c
  \]

with
  - constants $\gamma_c$ depending continuously on $f \in L^2(\Omega)$
  - a regular part $u_{1,\text{reg}} \in H^2(\Omega)$ satisfying
    \[
    r_c^{-2+|\alpha|} \partial_x^\alpha u_1 \in L^2(\Omega), \quad |\alpha| \leq 2, \quad c \in \mathcal{C}
    \]

**Proof:** Localize around each corner as specified in Section 4. Then apply Theorem L.1 at each corner, multiply the expansion (L.1) by $\chi_c$ and sum on $c \in \mathcal{C}$. 
The same tools allow to prove results in a general class of weighted Sobolev spaces (those introduced by Kondrat’ev, with a different notation).

**Definition**

Let \( m \in \mathbb{N} \) and \( \beta \in \mathbb{R} \).

- Let \( \Gamma \) be a plane sector, with \( r \) the distance to its vertex.
  \[
  K^m_{\beta}(\Gamma) = \{ u \in L^2_{\text{loc}}(\Gamma), \quad r^{\beta + |\alpha|} \partial^\alpha_x u \in L^2(\Gamma), \ \forall \alpha, \ |\alpha| \leq m \}
  \]

- Let \( \Omega \) be a polygon with corner set \( \mathcal{C} \), and \( r_c = r_c(x) \) the distance of \( x \) to \( c \). Let \( \rho \) be the minimum of the \( r_c, c \in \mathcal{C}, \) i.e. the distance function to the set of corners.
  \[
  K^m_{\beta}(\Omega) = \{ u \in L^2_{\text{loc}}(\Omega), \quad \rho^{\beta + |\alpha|} \partial^\alpha_x u \in L^2(\Omega), \ \forall \alpha, \ |\alpha| \leq m \}
  \]

Straightforward properties (or already seen):

- If \( u \in K^m_{\beta}(\Omega) \), then \( \chi_c u \in K^m_{\beta}(\Gamma_c) \).
- \( \Delta \) is continuous from \( K^m_{\beta}(\Gamma) \) into \( K^{m-2}_{\beta+2}(\Gamma) \)
- \( K^m_{\beta}(\Gamma) \subset K^{m-1}_{\beta}(\Gamma) \subset \cdots \subset K^0_{\beta}(\Gamma) \)
- \( K^m_{\beta}(\Omega) \subset K^{m'}_{\beta'}(\Omega) \) for \( m' \leq m \) and \( \beta' \geq \beta \)
- \( K_{-m}(\Omega) \subset H^m(\Omega) \subset K^m_0(\Omega) \)
- \( H^1_0(\Omega) \subset K^1_{-1}(\Omega) \)
Like in the particular case \( m = 2, \beta = -2 \), we can prove

**Lemma**

Let \( m \in \mathbb{N} \) and \( \beta \in \mathbb{R} \). Let \( \Gamma \) be a plane sector of opening \( \omega \), and \( \mathcal{I} = (0, \omega) \). Set \( \eta = -\beta - 1 \). The following three propositions are equivalent

1. \( u \in K_{\beta}^m(\Gamma) \)
2. \( e^{-\eta t} \bar{u} \in H^m(\mathbb{R} \times \mathcal{I}) \), with \( u(x) = \bar{u}(t, \theta) \).
3. The Mellin transform \( U[\lambda] := \mathcal{M}u[\lambda] \) is well defined for \( \Re \lambda = \eta \) and the function

\[
(1) \quad \xi \mapsto \| U[\eta + i\xi] \|_{H^m(\mathcal{I}; \eta + i\xi)} \quad \text{belongs to} \quad L^2(\mathbb{R})
\]

Moreover, if the function \( U \) satisfies (1), then the inverse Mellin transform

\[
u(x) = \frac{1}{2i\pi} \int_{\Re \lambda = \eta} r^\lambda U[\lambda](\theta) \, d\lambda
\]

defines an element of \( K_{\beta}^m(\Gamma) \).
Dirichlet Laplacian in weighted Sobolev spaces on a sector

The same tools as used for Theorem L.1, including the general form of our resolvent estimates, allows to prove:

**Theorem L.3**

Let $\Gamma$ be a plane sector of opening $\omega$. Let $m \geq 2$ and $\beta < -1$. Set $\eta = -\beta - 1$. We assume that

The line $\text{Re } \lambda = \eta$ is disjoint from the Mellin spectrum $\sigma(\mathcal{A})$

(here this means that $\eta \notin \frac{\pi}{\omega} \mathbb{Z}_*$)

- Let $u \in H^1_0(\Gamma)$ with compact support such that

  $$\Delta u = f \text{ in } \Gamma, \quad f \in K^{m-2}_{\beta+2}(\Gamma).$$

- Then

  $$(L.3) \quad u = u_\eta + \sum_{\ell \in \mathbb{N}^*, \ell \frac{\pi}{\omega} < \eta} \gamma_\ell \ r^{\ell \frac{\pi}{\omega}} \ \sin \ell \frac{\pi}{\omega} \theta$$

  with $u_\eta \in K^m_\beta(\Gamma)$ and $\gamma_\ell = \frac{1}{\pi} \int_{\Gamma} r^{-\ell \frac{\pi}{\omega}} \sin \ell \frac{\pi}{\omega} \theta \ f(x) \ dx$
Dirichlet Laplacian in weighted Sobolev spaces on a polygon

**Theorem L.4**

Let $\Omega$ be a plane polygon with corners set $\mathcal{C} \ni c$. Let $\chi_c$ smooth cut-off functions separating the corners.

- Let $m \geq 2$ and $\beta < -1$. Set $\eta = -\beta - 1$. We assume that
  \[ \forall c \in \mathcal{C}, \text{ the line } \text{Re} \lambda = \eta \text{ is disjoint from the Mellin spectrum } \sigma(A_c) \]
  (here this means that $\eta \notin \frac{\pi}{\omega_c} \mathbb{Z}_*$ for all $c \in \mathcal{C}$)

- Let $u \in H^1_0(\Omega)$ such that $\Delta u = f$ in $\Omega$, $f \in K^{m-2}_{\beta+2}(\Omega)$.

- Then

  \begin{equation}
  u = u_{\text{reg}} + \sum_{c \in \mathcal{C}} \chi_c(x) \left\{ \sum_{\ell \in \mathbb{N}_*, \ell \frac{\pi}{\omega_c} < \eta} \gamma_{c,\ell} r_c^{\ell \frac{\pi}{\omega_c}} \sin \left( \ell \frac{\pi}{\omega_c} \theta_c \right) \right\} 
  \end{equation}

  with

  \[ u_{\text{reg}} \in K^{m}_{\beta}(\Omega) \]

**Relations with standard Sobolev spaces**

If $f \in K^{m-2}_{-m+2}(\Omega)$, then $u_{\text{reg}} \in H^m(\Omega)$. But the case $f \in H^{m-2}(\Omega)$ is not straightforward.
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Neumann Laplacian

Let us consider now a solution $H^1(\Omega)$ of the Neumann problem

$$
\begin{cases}
-\Delta u = f & \text{in } \Omega \\
\partial_n u = 0 & \text{on } \partial \Omega.
\end{cases}
$$

with $f \in L^2(\Omega)$. What is different wrt Dirichlet?

1. The Mellin symbol(s)

$$
\mathcal{A}[\lambda] : U \mapsto (\lambda^2 U + U'') : \left\{ U \in H^2(\mathcal{I}), \ U'(0) = U'(\omega) = 0 \right\} \rightarrow L^2(\mathcal{I})
$$

2. The “initial regularity” of $u$, which does not belong to $K_{-1}^1(\Omega)$ in general

Remark: Alternative definitions of the Mellin symbol

$\mathcal{A}[\lambda]$ can also be defined in variational form from $H^1(\mathcal{I})$ into its dual $H^1(\mathcal{I})'$. Or, equivalently, $H^2(\mathcal{I}) \rightarrow L^2(\mathcal{I}) \times \mathbb{R}^2$, with $U \mapsto (\lambda^2 U + U'', U'(0), U'(\omega))$. These choices are more adapted to non-homogeneous Neumann boundary conditions.

Like for Dirichlet, we can write $\mathcal{A}[\lambda] = -\mathcal{P} + \lambda^2$ with the Neumann 1D Laplacian $\mathcal{P}$ on $\mathcal{I} = (0, \omega)$. An orthonormal spectral basis $\phi_\ell$ for $\mathcal{P}$ associated with eigenvalues $\mu_\ell$ is given by

$$
\mu_\ell = \lambda_\ell^2 \quad \text{with} \quad \lambda_\ell = \frac{\ell \pi}{\omega}, \quad \ell \geq 0 \quad \text{and} \quad \phi_\ell(\theta) = \begin{cases} 
\sqrt{\frac{1}{\omega}} & \text{if } \ell = 0 \\
\sqrt{\frac{2}{\omega}} \cos \frac{\ell \pi}{\omega} \theta & \text{if } \ell \geq 1
\end{cases}
$$
Neumann Laplacian on a plane sector

The novelties are

1. The pole at $\lambda = 0$ for $\mathfrak{A}[\lambda]^{-1}$. This pole is **double**:

$$(-\Delta + \lambda^2)^{-1} G = \frac{1}{\lambda^2} \langle \phi_0, G \rangle \phi_0 + \sum_{\ell \in \mathbb{N}^*} \frac{1}{\lambda^2 - \lambda_\ell^2} \langle \phi_\ell, G \rangle \phi_\ell$$

Calculating the residue of $r^\lambda(-\Delta + \lambda^2)^{-1} G[\lambda]$ at $\lambda_0 = 0$ uses that

$$r^\lambda = 1 + \lambda \log r + \frac{1}{2} \lambda^2 \log^2 r + \cdots \quad \lambda \to 0.$$

2. For all positive $\delta$, the variational solution $u$ belongs to $K_{-1+\delta}(\Omega)$. This is due to **Hardy’s inequality** “at infinity”.

We can analyze the solutions on $\Gamma$ in the same way as before. If $u \in K_{-1+\delta}(\Gamma)$, our integration contour in $\mathbb{C}$ is $\text{Re} \lambda = -\delta$, $\text{Re} \lambda = 1$. The resolvent estimates are still valid for the Neumann Mellin symbol as a consequence of Theorem S.2.

**Theorem N.1**

Let $\Gamma$ be a plane sector of opening $\omega \in (0, \pi) \cup (\pi, 2\pi)$ and $u \in K_{-1+\delta}(\Gamma)$ for all $\delta > 0$ with compact support such that

$$\Delta u = f \quad \text{in} \quad \Gamma, \quad f \in L^2(\Gamma) \quad \text{and} \quad \partial_n u = 0 \quad \text{on} \quad \partial \Gamma$$

Then

$$u = u_1 + \gamma_0 + \gamma'_0 \log r + [\text{If } \omega > \pi] (\gamma_1 r^{\frac{\pi}{\omega}} \cos \frac{\pi}{\omega} \theta) \quad \text{with} \quad u_1 \in K^2_{-2}(\Gamma)$$
Hardy’s inequality

For any $f \in C_0^\infty(\mathbb{R}^+)$ and $\gamma \neq 1$:

$$\int_0^\infty r^{\gamma-2} |f(r)|^2 \, dr \leq \frac{4}{(1 - \gamma)^2} \int_0^\infty r^{\gamma} |f'(r)|^2 \, dr.$$  

- When $\gamma < 1$, this inequality still holds for any function $f$ which is zero at 0 "Hardy’s inequality at zero" in the following sense

$$\forall R > 0, \ f' \in L^1(0, R) \quad \text{and} \quad f(r) = \int_0^r f'(s) \, ds$$

- When $\gamma > 1$, this inequality still holds for any function $f$ which is zero at infinity "Hardy’s inequality at infinity" in the following sense

$$\forall R > 0, \ f' \in L^1(R, \infty) \quad \text{and} \quad f(r) = \int_r^\infty f'(s) \, ds$$
Neumann Laplacian in weighted Sobolev spaces on a polygon

**Theorem N.2**

Let \( \Omega \) be a plane polygon with corners set \( \mathcal{C} \ni \mathcal{C} \). Let \( \chi_\mathcal{C} \) smooth cut-off functions separating the corners.

- Let \( m \geq 2 \) and \( \beta < -1 \). Set \( \eta = -\beta - 1 \). We assume that
  \[
  \forall \mathcal{C} \in \mathcal{C}, \text{ the line } \Re \lambda = \eta \text{ is disjoint from the Mellin spectrum } \sigma(\mathcal{A}_\mathcal{C})
  \]
  (here this means that \( \eta \not\in \frac{\pi}{\omega_\mathcal{C}} \mathbb{Z}_* \) for all \( \mathcal{C} \in \mathcal{C} \))
- Let \( u \in H^1(\Omega) \) s. t. \( \Delta u = f \) in \( \Omega \), \( f \in K_{\beta+2}^{m-2}(\Omega) \) and \( \partial_n u = 0 \) on \( \partial\Omega \)
- Then

\[
(N.2) \quad u = u_{\text{reg}} + \sum_{\mathcal{C} \in \mathcal{C}} \chi_\mathcal{C}(x) \left\{ \gamma_{\mathcal{C},0} + \sum_{\ell \in \mathbb{N}_*, \ell \frac{\pi}{\omega_\mathcal{C}} < \eta} \gamma_{\mathcal{C},\ell} r_\mathcal{C}\ell \frac{\pi}{\omega_\mathcal{C}} \cos \left( \ell \frac{\pi}{\omega_\mathcal{C}} \theta_\mathcal{C} \right) \right\}
\]

with

\[
u_{\text{reg}} \in K_{\beta}^{m}(\Omega)
\]

**Corollary**

If \( f \in L^2(\Omega) \) and \( \Omega \) is *convex* then \( u \in H^2(\Omega) \). Same consequence for Dirichlet.
Dirichlet-Neumann Laplacian on a plane sector

Let \( \partial_0 \Gamma \) and \( \partial_\omega \Gamma \) be the two sides of the sector \( \Gamma \) of opening \( \omega \). We consider the Laplace equation on \( \Gamma \) with the mixed boundary conditions

\[
    u = 0 \text{ on } \partial_0 \Gamma \text{ and } \partial_n u = 0 \text{ on } \partial_\omega \Gamma
\]

- The Mellin symbol \( \mathfrak{A}[\lambda] \) incorporates these boundary conditions:

\[
    \mathfrak{A}[\lambda] : U \mapsto (\lambda^2 U + U'') : \left\{ U \in H^2(\mathcal{I}), \; U(0) = 0, \; U'(\omega) = 0 \right\} \rightarrow L^2(\mathcal{I})
\]

- The spectrum of the corresponding operator \( \mathfrak{P} \) is positive (as for Dirichlet). An orthonormal spectral basis \( \phi_{\ell} \) for \( \mathfrak{P} \) associated with eigenvalues \( \mu_{\ell} \) is given by

\[
    \mu_{\ell} = \lambda_{\ell}^2 \quad \text{with} \quad \lambda_{\ell} = \left( \ell - \frac{1}{2} \right) \frac{\pi}{\omega}, \quad \text{and} \quad \phi_{\ell}(\theta) = \sqrt{\frac{2}{\omega}} \sin \lambda_{\ell} \theta, \quad \ell \geq 1
\]

- Any \( u \in H^1(\Gamma) \) such that \( u = 0 \) on \( \partial_0 \Gamma \), belongs to \( K_{-1}^1(\Gamma) \).

Theorem M.1

\( \Gamma \) plane sector of opening \( \omega \not\in \{ \frac{\pi}{2}, \frac{3\pi}{2} \} \). Let \( u \in H^1(\Gamma) \) with compact support s. t.

\[
    \Delta u = f \text{ in } \Gamma, \; f \in L^2(\Gamma) \text{ and } u = 0 \text{ on } \partial_0 \Gamma, \; \partial_n u = 0 \text{ on } \partial_\omega \Gamma
\]

Then

\[
    (M.1) \quad u = u_{1} + \left[ \begin{array}{l}
    \text{If } \omega > \frac{\pi}{2}, \\
    \text{If } \omega > \frac{3\pi}{2}
    \end{array} \right] \left( \gamma_{1} r^{\lambda_{1}} \sin \lambda_{1} \theta \right) + \left( \gamma_{2} r^{\lambda_{2}} \sin \lambda_{2} \theta \right)
\]

with

\[
    u_{\text{reg}} \in K_{-2}^2(\Gamma)
\]
### Theorem M.2

Let $\Omega$ be a plane polygon with corners set $C \ni c$. Let $\chi_c$ smooth cut-off functions separating the corners. Let $\{\lambda_c, \ell, \ell \geq 1\}$ be the positive part of the spectrum $\sigma(A_c)$.

- We assume that
  $$\forall c \in C, \text{ the line } \text{Re } \lambda = 1 \text{ is disjoint from the Mellin spectrum } \sigma(A_c)$$

- Let $u \in H^1(\Omega)$ such that
  $$\Delta u = f \text{ in } \Omega, \ f \in L^2(\Omega) \quad \text{and} \quad u = 0 \text{ on } \partial_{\text{Dir}} \Omega, \ \partial_n u = 0 \text{ on } \partial_{\text{Neu}} \Omega$$

- Then
  $$u = u_{\text{reg}} + \sum_{c \in C} \chi_c(x) \left\{ \sum_{\ell \in \mathbb{N}^*, \lambda_c, \ell < 1} \gamma_{c, \ell} \cdot r_c^{\lambda_c, \ell} \cdot \phi_{c, \ell}(\theta_c) \right\}$$

  with
  $$u_{\text{reg}} \in H^2(\Omega)$$

### Remark

“Corner” has the extended sense of any transition point between Dir. and Neu. bc’s. The opening $\pi$ is thus admissible, and associated with the singularity $r^{1/2} \sin^{1/2} \theta$. 
A last slide on the 2D Laplacian

**Terminology**

In expansions \((L.2), (L.4), (N.2), (M.2)\)

- the functions \(u_{\text{reg}}\) represents a *regular part*,
- the sum of terms in \(r^{\lambda c, \ell}_c\) is the singular part,
- each term is a *singular function* or *singularity*.

Two more things.

1. A comment on the condition

   \[ \forall \mathbf{c} \in \mathcal{C}, \text{the line } \Re \lambda = 1 \text{ is disjoint from the Mellin spectrum } \sigma(\mathcal{A}_c) \]

   - This condition is necessary if one wants a regular part in \(K^2_{-2}(\Omega)\) (in this case one has to consider constants functions at corners as singularities).
   - If one is satisfied with a regular part in \(H^2(\Omega)\), the constant functions are no more “singular”, and the condition can be relaxed to the new condition of “Injectivity modulo polynomials” that we will see later on.

2. These various bc’s can be combined with *transmission conditions* for which

   \[ a(u, v) = \sum_{k=1}^{K} \int_{\Omega_k} a_k \nabla u \cdot \nabla v \, dx, \quad a_k > 0, \quad \Omega_k \text{ polygonal partition of } \Omega \]

   - The collection of corners is the union of the corners of all the \(\Omega_k\).
   - Model pbs are posed on sectors (possibly \(\mathbb{R}^2\)) with partition by sectors of same vertex.
Outline

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2. Planned – Typical examples
3. Standard regularity in smooth domains for coercive forms
4. 2D polygon – Corner localization
5. 2D sector: Δ Dirichlet
   - Changes of variables
   - Mellin transform
   - Mellin symbol
   - Conclusions
6. 2D – Δ Dirichlet in weighted spaces
7. 2D – Δ, miscellaneous in weighted spaces
8. 2D, general
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   - 3D corners
   - Regular cones in $\mathbb{R}^n$ and unweighted spaces
   - Injectivity modulo polynomials
The result generalizes Theorem L.4 and proved in a very similar way (next slides)

**Theorem G.1**

Let $\Omega$ be a plane polygon with corners set $\mathcal{C} \ni \mathbf{c}$, with $\chi_c$ smooth cut-off functions separating the corners.

Let $a$ be a $V$-coercive form homogeneous with constant coefficients ($a^{\alpha\beta} \in \mathbb{C}$ are 0 if $|\alpha| + |\beta| < 2$). Assume that $X(\Omega) = H^1(\Omega)^d$. Recall that $A$ is the op. associated with $a$.

- Let $m \geq 2$ and $\beta < -1$. Set $\eta = -\beta - 1$. We assume that
  \[ \forall c \in \mathcal{C}, \text{ the line } \Re \lambda = \eta \text{ is disjoint from the Mellin spectrum } \sigma(A_c) \]

- Let $u \in V$ s. t. $Au = f$, $f \in K^{m-2}_{\beta+2}(\Omega)^d$. Zero natural bc’s are included!

- Then
  \[ u = u_{\text{reg}} + \sum_{c \in \mathcal{C}} \chi_c(x) \left\{ \sum_{\lambda \in \sigma(A_c)}, \Re \lambda \in [0, \eta] \Phi_{c, \lambda} \right\} \]

with

\[ u_{\text{reg}} \in K^m_{\beta}(\Omega)^d \]

Here $\Phi_c$ are singular functions associated with the corner $\mathbf{c}$

\[ \Phi_{c, \lambda}(x) = \sum_{\eta=1}^{Q} r_{c}^{\lambda} \log q_{c} \phi_{c, \lambda, \eta}(\theta_{c}) \]
Mellin symbol, in general

The problem \( Au = f \) with \( f \in L^2(\Omega)^d \) (or \( K^0(\Omega)^d \) with \( \beta' < 1 \)) is equivalent to a pde

\[ Lu = f \quad \text{with } L \text{ homogeneous of deg. 2 with constant coeffs} \]

At each corner \( c \), the Euler change of variables \( x \mapsto (t, \theta) \) transforms \( Lu = f \) into

\[ LC((\theta; \partial_t, \partial_\theta))\hat{u} = e^{2t} \hat{f} \]

In the full Dirichlet case \( V = H^1_0(\Omega)^d \), the Mellin symbol \( A_c[\lambda] \) is defined on \( H^1_0(\mathcal{I}_c)^d \) as

\[ U \mapsto LC(\theta; \lambda, \partial_\theta) \]

A suitable modification has to be performed for more general bc’s.

**Theorem**

The corner Mellin symbol \( A \) associated with a coercive form (homogeneous with constant coefficients) has a meromorphic resolvent \( \lambda \mapsto A[\lambda]^{-1} \) that satisfies the uniform parameter estimates

\[ \forall \lambda \in \Lambda, \quad \forall G \in H^{m-2}(\mathcal{I})^d, \quad \|A[\lambda]^{-1}G\|_{H^m(\mathcal{I};\lambda)} \leq C_{\lambda} \|G\|_{H^{m-2}(\mathcal{I};\lambda)} \]

on any set \( \Lambda \) of the form \( \Lambda = \{ \lambda \in \mathbb{C}, \ \text{Re } \lambda \in [\eta_0, \eta_1] \} \setminus \bigcup_{\lambda_0 \in \sigma(A)} B(\lambda_0, \delta) \)

The proof is based on elliptic estimates (Th S.2) and the Analytic Fredholm Theorem.
Residues, in general

The general resolvent estimates in the previous lemma allows to justify the residue formula related with inverse Mellin transform (after localization at each corner $c$):

$$u_{\eta} - u = \sum_{\lambda_0 \in \sigma(\mathfrak{A}_c)} \text{Res} \ r^\lambda \mathfrak{A}_c[\lambda]^{-1} G_c[\lambda] = \sum_{\lambda \in \sigma(\mathfrak{A}_c)} \Phi_{c,\lambda}$$

$L \Phi_{c,\lambda} = 0$ in $\Gamma_c$

and $\Phi_{c,\lambda}$ satisfies zero bc's on $\partial \Gamma_c$.

**Proof.** We have, for a suitable closed contour $\gamma$ around $\lambda$

$$\Phi_{c,\lambda} = \frac{1}{2i\pi} \int_{\gamma} r^\lambda \mathfrak{A}_c[\lambda]^{-1} G_c[\lambda] \, d\lambda$$

Therefore

$$L \Phi_{c,\lambda} = \frac{1}{2i\pi} \int_{\gamma} r^{-2} L_c(\theta; r\partial_r, \partial_\theta) \left( r^\lambda \mathfrak{A}_c[\lambda]^{-1} G_c[\lambda] \right) \, d\lambda$$

$$= \frac{1}{2i\pi} \int_{\gamma} r^{-2} r^\lambda L_c(\theta; \lambda, \partial_\theta) \mathfrak{A}_c[\lambda]^{-1} G_c[\lambda] \, d\lambda$$

$$= \frac{1}{2i\pi} \int_{\gamma} r^{-2} r^\lambda \mathfrak{A}_c[\lambda] \mathfrak{A}_c[\lambda]^{-1} G_c[\lambda] \, d\lambda = 0$$
Smooth coefficients

In the case of smooth coefficients, and with possible lower order terms, the results are similar if we

- Define the corner Mellin symbol $A_c$ after freezing the coeffs of $a$ at $c$ according to

$$a_c^\alpha\beta(x) = \begin{cases} a^\alpha\beta(c) & \text{if } |\alpha| + |\beta| = 2 \\ 0 & \text{if } |\alpha| + |\beta| \leq 1 \end{cases}$$

- Modify the singular functions $\Phi_c$ by completing them by their *shadows* that have higher degrees in $r_c$: Replace $\Phi_c$ by $\Psi_c$

$$\Psi_c = \Phi_c + \sum_{p \geq 1} \sum_{q=1}^{Q} r_c^{p+\lambda} \log^q r_c \psi_{c,\lambda,p,q}(\theta_c)$$

With this definition of the corner Mellin symbol $A_c$, the regularity theorem is the same in homog./non-homog. and constant/variable cases.

**Theorem G.2**

Let $\Omega$ be a plane polygon with corners set $C \ni c$. Let $a$ be a $V$-coercive form. Assume that $X(\Omega) = H^1(\Omega)^d$.

- Let $m \geq 2$ and $\beta < -1$. Set $\eta = -\beta - 1$. Assume

$$\forall c \in C, \text{ the strip } \text{Re} \lambda = [0, \eta] \text{ is disjoint from the Mellin spectrum } \sigma(A_c)$$

- Let $u \in V$ such that $Au = f$ with $f \in K_{\beta+2}^{m-2}(\Omega)^d$.

- Then $u \in K_{\beta}^m(\Omega)^d$
And Maxwell in all that?

1. **Theorem S.3** is still valid when Ω is a *convex polygon*.
   - The reason for this is the global $H^2(Ω)$ regularity of the solutions of Dirichlet or Neumann Laplacians with right hand sides $f \in L^2(Ω)$.
   - This means that the variational spaces $X_N(Ω)$ or $X_T(Ω)$ for electric or magnetic cases are subspaces of $H^1(Ω)^2$.
   - As a consequence **Theorem G.1** applies.
   - The spectrum of the Mellin symbols $\mathcal{A}_c$ for electric or magnetic cases can be made explicit: $\sigma(\mathcal{A}_c) = \left( \frac{π \omega_c}{\pi} \mathbb{Z}_* - 1 \right) \cup \left( \frac{π \omega_c}{\pi} \mathbb{Z}_* + 1 \right)$.

2. If the polygon Ω has non-convex corners, $H_N(Ω)$ is a strict, closed, subspace of $X_N(Ω)$ and $H_T(Ω)$ is a strict, closed, subspace of $X_T(Ω)$.
   - **Theorem G.1** does not apply as is.
   - But, by the same method of proof, we can prove a similar theorem in the Maxwell case: *The integration contour has to be enlarged* and (G.1) replaced with
   \[
   (G.1^*) \quad u = u_{\text{reg}} + \sum_{c \in C} \chi_c(x) \sum_{\lambda \in \sigma(\mathcal{A}_c), \ \text{Re}\ \lambda \in (-1, \eta)} \Phi_{c,\lambda}
   \]
   with, still, $\sigma(\mathcal{A}_c) = \left( \frac{π \omega_c}{\pi} \mathbb{Z}_* - 1 \right) \cup \left( \frac{π \omega_c}{\pi} \mathbb{Z}_* + 1 \right)$. 

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### 3D cones

#### Definition

1. A subset $\Gamma$ of $\mathbb{R}^3$ is called a cone if $\forall x \in \Gamma$, $\forall \rho > 0$, $\rho x \in \Gamma$.

2. Denote for $x \neq 0$:

   $$ r = |x| \quad \text{and} \quad \hat{x} = \frac{x}{r} $$

3. Define the *section* $\hat{\Gamma}$ of a cone $\Gamma$

   $$ \hat{\Gamma} = \Gamma \cap S^2 $$

#### Definition

Let $\Gamma$ be a cone in $\mathbb{R}^3$ such that $\hat{\Gamma}$ is a connected open subset of $S^2$.

1. $\Gamma$ is called a *regular cone* if $\hat{\Gamma}$ is a smooth submanifold with boundary of $S^2$.

2. $\Gamma$ is called a *wedge* if after a possible rotation, $\Gamma = \Gamma' \times \mathbb{R}$ with a sector $\Gamma'$.

3. $\Gamma$ is called a *polyhedral cone* if $\partial \Gamma$ is a finite union of plane faces. Then the boundary of $\hat{\Gamma}$ is a finite union of great-circle arcs.

4. $\Gamma$ is called a *cone with polygonal section* if $\hat{\Gamma}$ is a polygonal subset of $S^2$.

Note that 2 is a particular case of 3, which is a particular case of 4.

Here we understand that a wedge has an edge, and that a cone has a vertex.
3D domains with corners

**Definition**

Let $\Omega$ be an open bounded and connected subset of $\mathbb{R}^3$. If for any $x \in \partial \Omega$, $\Omega$ is locally diffeomorphic

1. to $\mathbb{R}^2 \times \mathbb{R}_+$ or to a regular cone, $\Omega$ is called a *domain with conical points*.
2. to $\mathbb{R}^2 \times \mathbb{R}_+$ or to a wedge, $\Omega$ is called a *domain with edges*.
3. to $\mathbb{R}^2 \times \mathbb{R}_+$, to a wedge or to a polyhedral cone, $\Omega$ is called a *polyhedral domain*.
4. to $\mathbb{R}^2 \times \mathbb{R}_+$, to a wedge or to a cone with polygonal section, $\Omega$ is called a *domain with corners*.

In fact, (3) (and 4) are more general than the *"manifold with corners"* for which the cones have 3 plane faces.

We are going to

1. Give hints on model problems (homogeneous with constant coefficients) in regular cones,
2. Tackle the problem of unweighted Sobolev spaces.
3. Chat on polyhedral cones.
3D domains with corners: Examples

1. Domains with conical points

2. Domains with edges
3D corners

3D domains with corners: Examples

3. A polyhedral domain (Fichera corner) — on the left

4. A domain with corners (non polyhedral) — on the right
Regular cones in $\mathbb{R}^n$

The variable $\hat{x} \in \hat{\Gamma} \subset \mathbb{S}^{n-1}$ plays the same role as the angle $\theta \in (0, \omega) = \mathcal{I}$:

$$\Delta \quad \text{becomes} \quad e^{-2t}(\partial_t^2 + (n-2)\partial_t - \mathcal{L}_{n-1})$$

with $\mathcal{L}_{n-1}$ the (positive) Laplace-Beltrami operator on $\mathbb{S}^{n-1}$. Let $(\mu_\ell, \phi_\ell)$ be its Dirichlet eigenpairs on $\hat{\Gamma}$. The Mellin symbol of the Dirichlet Laplacian on $\Gamma$ is

$$\mathfrak{A}[\lambda] : U \mapsto (\lambda^2 + (n-2)\lambda - \mathcal{L}_{n-1}) \quad (H^2 \cap H^1_0)(\hat{\Gamma}) \rightarrow L^2(\hat{\Gamma})$$

The spectrum $\sigma(\mathfrak{A}) = \{\lambda_\ell, \ell \in \mathbb{Z}_*\}$ with

$$\lambda_{\pm \ell} = 1 - \frac{n}{2} \pm \sqrt{\mu_\ell + \left(\frac{n}{2} - 1\right)^2}, \quad \ell \in \mathbb{N}_*$$

The equivalence $u \in K^m_\beta(\Gamma) \iff e^{-\eta t} \hat{u} \in H^m(\mathbb{R} \times \hat{\Gamma})$ holds with $\eta = -\beta - \frac{n}{2}$.

**Theorem L.5**

Let $\Gamma$ be a regular cone in $\mathbb{R}^n$. Let $m \geq 2$ and $\beta < -1$. Set $\eta = -\beta - \frac{n}{2}$. Assume

The line $\text{Re } \lambda = \eta$ is disjoint from the Mellin spectrum $\sigma(\mathfrak{A})$

- Let $u \in H^1_0(\Gamma)$ with compact support such that $\Delta u \in K^{m-2}_{\beta+2}(\Gamma)$

- Then

$$u = u_\eta + \sum_{\ell \in \mathbb{Z}_*, 1 - \frac{n}{2} < \lambda_\ell < \eta} \gamma_\ell r^{\lambda_\ell} \phi_\ell(\hat{x}) \quad \text{with} \quad u_\eta \in K^m_\beta(\Gamma)$$

"The line $\text{Re } \lambda = \eta$ is disjoint from the Mellin spectrum $\sigma(\mathfrak{A})"
Unweighted spaces in 3D (odd $n$D) and Taylor expansions

The question of a right hand side $f$ posed in a standard unweighted Sobolev space $H^{m-2}(\Omega)$ is more or less still pending.

1. If $f \in L^2(\Omega) = K_0^0(\Omega)$, OK with $\beta = -2$ and $m = 2$. Regular part in $K_{-2}^2(\Omega) \subset H^2(\Omega)$

2. If $f \in H^1(\Omega)$, then $f \in K_{-1}^1(\Omega)$ because in dimension $n = 3$, the two spaces coincide! OK with $\beta = -3$ and $m = 3$. Regular part in $K_{-3}^3(\Omega) \subset H^3(\Omega)$

3. If $f \in H^2(\Omega)$, then $f \in C^{\frac{1}{2}}(\overline{\Omega})$ and, moreover

$$f - \sum_{c \in \mathcal{C}} \chi_c(x) f(c) \in K_{-2}^2(\Omega)$$

4. General case in $n$D, $n$ odd: If $f \in H^k(\Omega)$, $f \in C^{k - \frac{n}{2}}(\overline{\Omega})$ and

$$[Taylor] \quad f - \sum_{c \in \mathcal{C}} \chi_c(x) \sum_{|\alpha| < k - \frac{n}{2}} \frac{(x - c)^\alpha}{\alpha!} \partial_x^\alpha f(c) \in K_{-k}^k(\Omega)$$

Proofs use Hardy’s inequality

The curse of even dimensions

If $n = 2$, or more generally if $n$ is even, we are in limit cases of Sobolev embedding theorem for integer regularity exponents. [Taylor] can be extended to odd dimensions with $|\alpha| < k - \frac{n}{2}$ but not to even dimensions.
Theorem G.3

For odd $n \geq 3$, let $\Omega \subset \mathbb{R}^n$ be a domain with set of conical points $\mathcal{C} \ni c$ [here for simplicity we assume that local diffeomorphisms at corners are affine functions]. Let $a$ be a $V$-coercive form homogeneous with constant coefficients. Assume that $X(\Omega) = H^1(\Omega)^d$. Recall that $A$ is the op. associated with $a$.

- Let $m \geq 2$. Set $\eta = m - \frac{n}{2}$. We assume that
  \[ \forall c \in \mathcal{C}, \text{ the line } \Re \lambda = \eta \text{ is disjoint from the Mellin spectrum } \sigma(A_c) \]

- Let $u \in V$ such that $Au = f$, $f \in H^{m-2}(\Omega)^d$.

- Then

\[
(G.3) \quad u = u_{\text{flat}} + \sum_{c \in \mathcal{C}} \chi_c(x) \left\{ \sum_{\lambda \in \sigma(A_c) \cup \mathbb{N}_2, \Re \lambda \in [1-\frac{n}{2}, \eta)} \psi_{c,\lambda} \right\}
\]

with \[ u_{\text{flat}} \in K_{-m}^m(\Omega)^d \]

Here $\psi_c$ are singular functions (or polynomials) associated with the corner $c$

\[
\psi_{c,\lambda}(x) = \sum_{q=1}^{Q} r_\lambda^q r_c \log^q r_c \psi_{c,\lambda,p}(\hat{x}_c)
\]
General residues in 3D (odd \(n\)D) cones

Difference between Th. G.3 and Th. G.1: The regularity of the rhs and the set of poles:

\[
\lambda \in \sigma(\mathcal{A}_c) \cup \{2, 3, \ldots\} \quad \text{with} \quad \Re \lambda \in [1 - \frac{n}{2}, \eta)
\]

The proof is still based on the residue formula issued from inverse Mellin transform:

\[
u_\eta - u = \sum_{\lambda_0 \in \sigma(\mathcal{A}_c) \cup \mathbb{N}_2} \text{Res} \ r^\lambda \mathcal{A}_c[\lambda]^{-1} G_c[\lambda] = \sum_{\lambda \in \sigma(\mathcal{A}_c) \cup \mathbb{N}_2} \Psi_{c, \lambda}
\]

Now \(G_c[\lambda]\) has poles on integers \(\lambda \geq 2\) corresponding to the Taylor expansion of \(f\) at \(c\)

\[
\text{Res} r^\lambda G[\lambda] = r^2 \sum_{|\alpha| = k-2} \frac{(x-c)^\alpha}{\alpha!} \partial_x^\alpha f(c)
\]

### Lemma

\[
\begin{cases}
L \Psi_{c, \lambda} = 0 & \text{if } \lambda \in \sigma(\mathcal{A}_c) \setminus \mathbb{N}_2 \\
L \Psi_{c, \lambda} = \sum_{|\alpha| = \lambda-2} \frac{(x-c)^\alpha}{\alpha!} \partial_x^\alpha f(c) & \text{if } \lambda \in \mathbb{N}_2
\end{cases}
\]

and \(\Psi_{c, \lambda}\) satisfies zero boundary conditions on \(\partial \Gamma_c\)
Injectivity modulo polynomials

The notion of singular function is relative. A function is considered as singular if it does not belong to some space of regular function. With a space like $K_{-m}^m$, any nonzero polynomial of degree $< m - \frac{n}{2}$ is singular, but with an unweighted Sobolev space $H^m$, the same polynomial is regular.

Definition

Let $\Gamma$ be a regular cone in $\mathbb{R}^n$, with $\hat{\Gamma}$ its section. Let $\lambda \in \mathbb{C}$.

1. Let $S^\lambda$ be the space of quasi-homogeneous $d$-component functions

   $$S^\lambda(\Gamma) = \left\{ \psi = r^\lambda \sum_{q=0}^{Q} \log^q r \psi_q(\hat{x}), \quad \psi_q \in \mathcal{C}^\infty(\hat{\Gamma})^d \right\}$$

   We mention zero boundary conditions by an index: $S^\lambda_{bc}(\Gamma)$, e.g. $S^\lambda_0(\Gamma)$ for Dirichlet, and $S^\lambda_N(\Gamma)$ for tangential bc’s (if $n = d$).

2. Let $P^\lambda$ be the space of homogeneous polynomials in Cartesian var. $x$, of deg. $\lambda$

3. The operator $L$ with boundary conditions bc is said to be injective modulo polynomials if

   $$\psi \in S^\lambda_{bc}(\Gamma) \quad \text{and} \quad L\psi \in P^{\lambda-2} \implies \psi \in P^\lambda.$$

   This condition can be written as

   $$L : S^\lambda_{bc}(\Gamma)/P^\lambda_{bc}(\Gamma) \rightarrow S^{\lambda-2}(\Gamma)/P^{\lambda-2} \quad \text{injective}$$
Theorem G.4  The dimension is any integer \( n \geq 2 \).

Let \( \Omega \subset \mathbb{R}^n \) be a domain with set of conical points \( C \ni c \) [here for simplicity we assume that local diffeomorphisms at corners are affine functions]. Let \( a \) be a \( V \)-coercive form homogeneous with constant coefficients. Assume that \( X(\Omega) = H^1(\Omega)^d \). Recall that \( A \) is the operator, \( L \) is pde system, and \( \text{bc} \) the boundary conditions associated with \( a \).

- Let \( m \geq 2 \). Set \( \eta = m - \frac{n}{2} \). We assume that
  \[
  (\forall c \in C), \ (\forall \lambda, \ Re \lambda = \eta), \ L \ \text{is injective mod. polynomials on } S_{bc}^\lambda(\Gamma_c)
  \]

- Let \( u \in V \) such that \( Au = f \), \( f \in H^{m-2}(\Omega) \).

- Then
  \[
  (G.4) \quad u = u_{\text{reg}} + \sum_{c \in C} \chi_c(x) \sum_{\lambda \in \sigma^*(a,c), \ Re \lambda \in [1 - \frac{n}{2}, \eta]} \Psi_{c,\lambda}
  \]
  with
  \[
  u_{\text{reg}} \in H^m(\Omega)
  \]

and \( \Psi_c \) singular functions associated with the corner \( c \) (similar as in Th.G.3)

Here \( \sigma^*(a,c) \) is the set of \( \lambda \)'s such that \( L \) is not injective mod. polynomials on \( S_{bc}^\lambda(\Gamma_c) \).
Comments on the optimal theorem and conclusion

1. The regular part is in $H^m$ and is not “flat” in general: It may have a nonzero Taylor expansion at corners.

2. When $\lambda \notin \mathbb{N}$, the injectivity modulo polynomials is equivalent to $(\lambda \notin \sigma(A_c))$.

3. Importantly, we have overcome the curse of even dimensions! But at the expense of introducing modified parameter-dependent norms $H^m(\hat{\Gamma}_c; \lambda; \star)$ in order to relax $H^m(\hat{\Gamma}_c; \lambda)$ in a suitable way around positive integers.

4. **2D cracks**: The condition of injectivity modulo polynomials is satisfied for homogeneous Dirichlet or Neumann conditions for $\Delta$ at $\lambda = 1$ (and any positive integer) when $\omega = 2\pi$.

5. **Half-planes**: The condition of injectivity modulo polynomials is satisfied for homogeneous Dirichlet or Neumann for $\Delta$ at any $\lambda$ when $\omega = \pi$ ... This means that our theory is coherent with the standard regularity results at regular points of the boundary.

Regularity theorems on edge or polyhedral domains are in the same spirit: At each singular point is associated a condition of injectivity modulo polynomials (IMP) and regularity is obtained if the IMP condition is satisfied in certain complex strips.

Expansions in regular and singular terms are much more involved. There is a competition between corners and edges. Both contribute to expansions, and they combine with each other in non-unique and non-canonical ways.