Abstracts SCHOOL 2

L. De Pascale (U. Pisa):
"Multimarginal optimal transport problem. Basic theory, open problems and applications"
First I will introduce the multimarginal optimal transportation problems in general then I will discuss some specific costs which are relevant in the applications. I will present the basic theory and discuss the main difficulty in advancing the theory.

J.-A. Carrillo (Imperial College):
"Gradient Flows: Qualitative Properties & Numerical Schemes "
We will review some of the basic properties of different gradient flows in the space of measures obtained using optimal transport tools. We will also concentrate in how to develop numerical schemes based on optimal transport maps and variational schemes.

J.-D. Benamou (INRIA Rocquencourt):
“Computational Optimal Transport”
I will present three numerical methodologies for solving classical Monge and Monge-Kantorovich (MK) optimal transport problems as well as several extensions or related problems. Whenever possible, we will experiment with toy implementations either in Matlab, Python or FreeFem++. I will also discuss a few applications and present numerical results. Comparing these numerical methods in term of cost/accuracy performance is difficult because they are generally applied to different classes of data and regularity of the solutions. We will distinguish three classes of source/target data: Discrete to Discrete (D2D) that is transporting atomic measures to atomic measures (weighted sums of Diracs) - Continuous to Discrete (C2D) that is sending a density function to an atomic measure - Continuous to Continuous (C2C) that is transporting a density function to a density function. The problem of the discretisation of continuous data of course also raises questions, and can be understood as a OT problem.

The first session will be devoted to the Monge-Kantorovich problem. In this approach one does not directly compute the transport map but the transport plan and possibly the Kantorovich potentials.
Using D2D data, it can be solved with state of the art Linear programming method or more “efficiently" using the assignment structure of the MK problem with a $O(N^3)$ cubic cost “Auction” algorithm, N is the size of the data. For most applications both the computational and the storage cost are strong limitations. Note that just storing the cost matrix and transport plan implies $O(N^2)$ operations/memory spaces. A line of attack to improve the computational speed is to regularise the problem by adding the an entropy penalisation with a given temperature parameter. This old idea, going back to Schroedinger, yield a strictly convex problems which can be reformulate in terms of a projection in the sense of the Kullback-Leibler Divergence on an intersection of convex sets of transport plans.
In the MK case, one can easily apply the iterative Bregman or Dykstra procedures which consist in cyclically projecting on each convex set. The projections are explicitly computed. The computational cost of each iterate is quadratic $O(N^2)$.
The convergence of the iterate is observed to be fast and depends on the temperature parameter.
Ones get a good approximation of the OT distance between source and target but the transport plan is a diffuse representation of the exact transport map. There is a recent and promising surge of research along these lines in connection with image processing (cf. G. Peyre talk at the WS).

The second session will address the Computational Fluids Dynamic formulation of OT and the Augmented Lagrangian numerical method ALG2.
The CFD formulation uses an artificial time dimension and a change of variable to transform the non-linear volume preserving constraint into a linear divergence constraint. The resulting problem is amenable to the Augmented Lagrangian approach advocated by Fortin and Glowinski.
The derived numerical algorithm are iterative and can deal with non-smooth convex cost functions, in the classical MK problem the indicatrix of a convex set. The CFD approach treats C2C data but densities

maybe be discontinuous and have non convex support.
On can consider that each iteration has a linear cost in the size of the discretisation which now is $O(N_t \times N)$ where N_t is the number of time steps. ALG2 is however a first order method and the iterates may converge slowly. Recent works have linked this approach to splitting methods and proximal operators. The method has been extended to other OT problems and we also recently discovered that the ALG2 algorithm can be used for the original Monge cost, for static congestion and also for some classes of Mean Field games.

The last session will present methods based on the Monge-Ampere (MA) equation. Since Brenier pioneering work on the quadratic cost MK problem, OT is also understood as a variational formulation for the MA equation and has been a pivotal tool in the recent Caffarelli regularity theory. We will review the different notion of weak solutions. The historic construction of MA solutions by Pogorelov correspond to Brenier solutions for D2C data. The dual (reverse transport) of Pogorelov solutions are MA solutions in the sense of Aleksandrov. Pogorelov solutions are conveniently computed by maximising the dual of the MK problem which simplifies for C2D data. Each iterates relies on the construction of Power Diagrams, a generalisation of the Voronoi diagrams used in computational geometry. The cost of this construction is quadratic $O(N^2)$. The classical maximisation method is a simple first order ascent method but a Newton solver can be used close enough to the solution. The worst computational cost for this approach is therefore again $O(N^3)$ cubic but in most instances one can hope for $O(N^2)$.
The discretisation of the MA operator using Power Diagram is also used in a recent numerical algorithm for the minimisation of density functionals penalized with a Wasserstein distance. This configuration typically appears in Wasserstein Gradient flows interpretation of non linear diffusion equations (cf Merigot talk at the WS).

Finally we will consider the slightly more regular viscosity solution setting for the MA equation. This corresponds to C2C data with the additional assumption that the target density is Lipschitz, does not vanish, and has convex support. In this framework, monotone and consistent and therefore convergent discretisation of the MA operator may be constructed. This is closely linked to the convexity constraint on the solution.
A Newton solver can be used and work fine. There are observed to converge in very few iterations.
Each iterate is the resolution of a linear Elliptic equation. We therefore empirically obtain a linear $O(N)$ optimal method.
Note that the MA approach requires the least storage as we only discretize a potential on a physical grid.

Abstracts WORKSHOP 4

Mathias Beiglböck (Vienna):
“Optimal Transport and Skorokhod Embedding”
The Skorokhod embedding problem is to represent a given probability as the distribution of Brownian motion at a chosen stopping time. Over the last 50 years this has become one of the important classical problems in probability theory and a number of authors have constructed solutions with particular optimality properties. These constructions employ a variety of techniques ranging from excursion theory to potential and PDE theory and have been used in many different branches of pure and applied probability.
We develop a new approach to Skorokhod embedding based on ideas and concepts from optimal mass transport. In analogy to the celebrated article of Gangbo and McCann on the geometry of optimal transport, we establish a geometric characterization of Skorokhod embeddings with desired optimality properties. This leads to a systematic method to construct optimal embeddings. It allows us, for the first time, to derive all known optimal Skorokhod embeddings as special cases of one unified construction and leads to a variety of new embeddings. While previous constructions typically used particular properties of Brownian motion, our approach applies to all sufficiently regular Markov processes.

L. Brasco (Université de Marseille):
"A widely degenerate elliptic equation arising in congested optimal transport."
We present some gradient estimates for local solutions of a widely degenerate quasilinear elliptic equation. We also discuss applications of these estimates to the so-called Beckmann's problem , the latter being an Optimal Transport problem with congestion effects. The results presented in this talk have been obtained in collaboration with Pierre Bousquet (Toulouse), Guillaume Carlier (Paris Dauphine) and Vesa Julin (Jyväskylä).

Y. Brenier (CNRS, Ecole Polytechnique):
"Reconstruction of the early universe via optimal transport"
About 10 years ago, Uriel Frisch (Observatoire de Nice) addressed the inverse problem in Cosmology of reconstructing the distribution of mass of the early universe from the presently observed data. He pointed out that the problem could be reduced to the Monge optimal transport problem with a quadratic cost function, provided the Zeldovich approximation of the full Einstein equations was used. Using this idea, he could design an efficient numerical method (MAK=Monge-Ampere-Kantorovich) to solve this important inverse problem and we got involved in this effort.
Surprisingly enough, we more recently discovered that the Zeldovich approximation could itself be interpreted in terms of optimal transport, by substituting the Monge-Ampere equation for the Poisson equation used by Zeldovich in its approximation of the Einstein equations. With this idea, it has been later possible to design a (less efficient) numerical method that takes into account inelastic collisions in the gravitational model, which was not possible in the original MAK method.

G. Buttazzo (U. di Pisa):
"Optimal regions for congested transport"
We consider a given region Ω where the traffic flows according to two regimes: in a region C we have a low congestion, where in the remaining part $\Omega\setminus C$ the congestion is higher. The two congestion functions H_1 and H_2 are given, but the region C has to be determined in an optimal way in order to minimize the total transportation cost. Various penalization terms on C are considered, together with some numerical computations. In particular, if a density of resources to be spent on C is considered, this give raise to some interesting free boundary problems.

Guido De Philippis (University of Zurich):
“BV estimates in optimal transportation and applications”
I will study BV regularity for solutions of variational problems in Optimal Transportation and present some applications. In particular I will focus on BV bounds for the Wasserstein projection on the set of measure

with density bounded by a prescribed $BV_{\rm loc}$ function f. I will also show how to recover BV estimates for solutions of some non-linear parabolic PDE by means of optimal transportation techniques. This is a joint work with A. Meszaros, F. Santambrogio and B. Velichkov.

C. Jimenez (UBO, Brest):
"Growing Sand Piles on a table with side-walls"
We prove existence and uniqueness of solutions for a system of PDEs which describes the growth of a sandpile in a silos with flat bottom under the action of a vertical measure source. The tools we use are a discrete approximation of the source and the duality theory for optimal transportation problems.

I. Kim (UCLA):
“Evolution equations with congestion”
We will discuss several situations where we have a L^∞-constraint on the evolution equation. The link between free boundary problems and the evolution equation will be discussed, using approximations both by gradient flow approach in Wasserstein space and also by porous medium equation.

R. J. McCann (U. Toronto):
"Academic wages, singularities, phase transitions and pyramid schemes"
In this lecture we introduce a mathematical model which couples the education and labor markets, in which steady-steady competitive equilibria turn out to be characterized as the solutions to an infinite-dimensional linear program and its dual.
In joint work with Erlinger, Shi, Siow and Wolthoff, we use ideas from optimal transport to analyze this program, and discover the formation of a pyramid-like structure with the potential to produce a phase transition separating singular from non-singular wage gradients.
Wages are determined by supply and demand. In a steady-state economy, individuals will choose a profession, such as worker, manager, or teacher, depending on their skills and market conditions. But these skills are determined in part by the education market. Some individuals participate in the education market twice, eventually marketing as teachers the skills they acquired as students. When the heterogeneity amongst student skills is large, so that it can be modeled as a continuum, this feedback mechanism has the potential to produce larger and larger wages for the few most highly skilled individuals at the top of the market. We analyze this phenomena using the aforementioned model.
We show that a competitive equilibrium exists, and it displays a phase transition from bounded to unbounded wage gradients, depending on whether or not the impact of each teacher increases or decreases as we pass through successive generations of their students.
We specify criteria under which this equilibrium will be unique, and under which the educational matching will be positive assortative. The latter turns out to depend on convexity of the equilibrium wages as a function of ability, suitably parameterized.

Daniel Matthes (TU Muenchen), Joint work with Horst Osberger (TU Muenchen):
"Discretizing Nonlinear Diffusion the Lagrangian Way"
One of the ground-breaking observations from the theory of optimal transport is that various nonlinear diffusion equations can be written as gradient flows with respect to the L2-Wasserstein metric. In this context, diffusion is interpreted as the motion of a particle density along a (gradient) vector field that sensitively depends on the density itself.
We use that interpretation to define spatio-temporal "Lagrangian" discretizations of these diffusion equations. That is, instead of calculating the change in density at given points, we trace the trajectories of "mass particles". The resulting schemes inherit various nice features of the original diffusion equations, like conservation of mass, preservation of positivity, energy dissipation and convexity.
Our main result concerns the rigorous analysis of the discrete-to-continuous-limit for a particular Lagrangian discretization for certain fourth order equations (like thin film and QDD) in one spatial dimension. The key estimates are obtained from the dissipation of an auxiliary discrete Lyapunov functional.
Quentin Mérigot (Ceremade, Paris Dauphine)

“Discretization of functionals involving the Monge-Ampère operator”

Gradient flows in the Wasserstein space have become a powerful tool in the analysis of diffusion equations, following the seminal work of Jordan, Kinderlehrer and Otto (JKO). The numerical applications of this formulation have been limited by the difficulty to compute the Wasserstein distance in dimension two or more. One step of the JKO scheme is equivalent to a variational problem on the space of convex functions, which involves the Monge-Ampère operator. Convexity constraints are notably difficult to handle numerically, but in our setting the internal energy plays the role of a barrier for these constraints. This enables us to introduce a consistent discretization, which inherits convexity properties of the continuous variational problem. We show the effectiveness of our approach on nonlinear diffusion and crowd-motion models. Joint work with G. Carlier, J.D Benamou and É. Oudet.

Brendan Pass (Edmonton)

“Wasserstein barycenters over Riemannian manifolds”

I will describe joint work in progress with Y.-H. Kim (UBC). We study barycenters in the space of probability measures on a Riemannian manifold M, equipped with the Wasserstein metric. Under reasonable conditions, we prove absolute continuity of the barycenter (which is itself a measure on M). As applications, we prove Jensen type inequalities for various displacement convex functionals, as well as a Riemannian version of Vitale's random Brunn-Minkowski inequality.

Gabriel Peyré, CNRS and Ceremade, Univ. Paris-Dauphine, FRANCE

“A Review of Wasserstein Barycenter Algorithms, with a New One”

The computation of Wasserstein barycenters is the cornerstone that allows to unleash the power of optimal transport (OT) methods to applications such a imaging sciences and machine learning. Indeed, it enables to extend traditional problems in these fields, such as texture synthesis, color normalization, clustering and principal component analysis from Euclidean settings to the OT's world. This is crucial to advance the front of research in these areas, because OT takes into account some underlying ground metric, which exploit correlations and structures between the considered variables (color and edges locations/directions for images, bag of features for machine learning). The computation of OT barycenters is however a challenging task when one considers large-scale setups where thousands of histograms of giga-samples need to be averaged. A recent breakthrough in this area is the idea of extending the entropic regularization of OT (which dates back to early works of Schrodinger, and was recently revitalized by Marco Cuturi in machine learning) to the setting of more advance variational problems such as Wasserstein barycenters. Relying on this idea, I will show how several algorithms can be derived from either the primal or the dual formulation, and how this relates to classical convex optimization methods exploiting the geometry of the Kullback-Leibler divergence. I will show several promising results, and argue that this new computational paradigm has the potential to deeply impact the machine learning and imaging communities. This is a joint work with Marco Curturi, Luca Nenna, Jean-David Benamou and Guillaume Carlier.

Aldo Pratelli (Erlangen)

“The mass transport problem with relativistic cost functions”

In the last years, there has been some interest on the mass transport problem with relativistic or highly relativistic cost functions: this comes as a natural generalization of the case of the relativistic heat cost, introduced by Brenier and then studied also by others. Basically, a relativistic cost function is a cost function which is bounded and strictly convex in a strictly convex subset of \R^N, and +\infty outside. The cost is highly relativistic if the function approaches its values on the boundary of the convex set with an infinite slope (as it happens for Brenier's relativistic heat cost and in most of the examples). We will describe the general situation and we will concentrate on some main tasks, namely the "Chain lemma", the continuity of the total cost relative to the time, and the existence of a Kantorovich potential, hence of an optimal transport map. Joint work with J. Bertrand and M. Puel.

Walter Schachermayer (Vienna)
"Optimal transport and stochastic portfolio theory"
In a recent paper S. Pal and T. Wong established a remarkable connection between the the following two notions. On the one hand functionally generated portfolios as introduced by R. Fernholz some fifteen years ago. On the other hand a multiplicative version of the concept of cyclical monotonicity. We shall try to discuss and motivate the underlying ideas.

Juliette Venel (Valenciennes)
"Differential inclusions and applications"
My talk will deal with differential inclusions. Such evolution problems appear when the state-variable is submitted to some constraints and therefore has to stay in an admissible set.
Especially we will be interested in the study of sweeping processes and of second order differential inclusions involving proximal normal cones. We propose to detail some results about these different problems by pointing out the geometrical assumptions regarding the admissible sets.
Furthermore we will consider some applications in crowd motion modelling and in granular media.

Marie Therese Wolfram (RICAM), joint with J.A. Carrillo and H. Ranetbauer
"On numerical simulations of nonlinear convection-aggregation equations by evolving diffeomorphisms
In this talk we present a numerical algorithm for solving nonlinear convection-aggregation equations, which is based on the Lagrangian coordinate representation of the original problem. These equations can be considered as gradient ows with respect to the quadratic transportation distance for a free energy functional. Here the evolution of the density is represented by evolving diffeomorphisms, mapping the unknown density to the constant distribution of mass on the domain. The proposed numerical algorithm is based on the variational formulation and automatically adapts the mesh to the shape of the mass distribution. Hence it allows to track the formulation of singularities in a natural manner. A feature that we illustrate with various numerical experiments.

Qinglan Xia, UC Davis
"Some applications of ramified optimal transportation in economics"
In this talk, I will talk about two applications of ramified (branching) optimal transportation to economics. The first one is about the exchange value embedded in some ramified transport systems. The second one is on the ramified optimal allocation problem. Both of them are joint works with Shaofeng Xu.