Transition from a discrete network to a continuous model in $\mathbb{R}^d$

Roméo Hatchi

Ceremade - Université Paris-Dauphine

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Let $\Omega$ a bounded domain of $\mathbb{R}^d$ and $\varepsilon > 0$. We consider a sequence of discrete networks $(N^\varepsilon, E^\varepsilon) := (\text{nodes}, \text{arcs})$, an arc $= (x, e)$ with $x \in N^\varepsilon$, $e \in \mathbb{C}$.

$e = \Theta(\varepsilon) : \exists C > 0/C\varepsilon \leq |e| \leq \varepsilon$.

**Traveling time**: $t^\varepsilon(x, e)$ on the arc $(x, e)$.

**Mass**: $m^\varepsilon(x, e)$ on the arc $(x, e)$.

**Congestion**: $t^\varepsilon(x, e) = g^\varepsilon(x, e, m^\varepsilon(x, e))$ with $g^\varepsilon \geq 0$, nondecreasing on the last variable.

**Transport plan**: $\gamma^\varepsilon(x, y)$ mass sent from $x$ to $y$ ($x, y \in N^\varepsilon$).

**Paths**: $\sigma$ of the form $(x_0, x_1, \ldots, x_L)$.

$w^\varepsilon(\sigma)$ mass traveling on the path $\sigma$.

$\tau_{m^\varepsilon}(\sigma)$ traveling time of the path $\sigma$.

$C^\varepsilon_{x, y} = \{\sigma \text{ free-loop}; \sigma = (x_0, \ldots, x_l) \text{ with } x_0 = x \text{ and } x_L = y\}$. 

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The discrete network
Mass conservation and Wardrop equilibrium

**Equilibrium**: the following mass conservation conditions are satisfied

\[ \gamma^\epsilon(x, y) := \sum_{\sigma \in C^\epsilon_{x,y}} w^\epsilon(\sigma), \quad \forall (x, y) \in N^\epsilon \times N^\epsilon \]

and

\[ m^\epsilon(x, e) = \sum_{\sigma \in C^\epsilon:(x,e) \subset \sigma} w^\epsilon(\sigma). \]

**Definition**

A Wardrop equilibrium is the configuration of nonnegative masses \( m^\epsilon : (x, e) \mapsto (m^\epsilon(x, e)) \) and \( w^\epsilon : \sigma \mapsto (w^\epsilon(\sigma)) \) that satisfy the mass conservation conditions and such that \( \forall (x, y) \in N^\epsilon \times N^\epsilon \) and \( \forall \sigma \in C^\epsilon_{x,y}, \text{ if } w^\epsilon(\sigma) > 0 \) then

\[ \tau^\epsilon_{m^\epsilon}(\sigma) \leq \tau^\epsilon_{m^\epsilon}(\sigma'), \quad \forall \sigma' \in C^\epsilon_{x,y}. \]
Beckmann, McGuire and Winsten in the 50’s:

$$(w^\varepsilon, m^\varepsilon)$$ is an equilibrium if and only if it minimizes

$$\sum_{(x, e) \in E^\varepsilon} G^\varepsilon(x, e, m^\varepsilon(x, e))$$ where

$$G^\varepsilon(x, e, m) := \int_0^m g^\varepsilon(x, e, \alpha) d\alpha$$

subject to nonnegativity constraints and the mass conservation conditions.
The complexity is very important: one variable by path.
Dual formulation:

$$\inf_{t^\varepsilon \in \mathbb{R}^+} \left( \sum_{(x,e) \in E^\varepsilon} H^\varepsilon(x, e, t^\varepsilon(x, e)) - \sum_{(x,y) \in N^\varepsilon \times N^\varepsilon} \gamma^\varepsilon(x, y) T_{t^\varepsilon}(x, y) \right)$$

where $t^\varepsilon = (t^\varepsilon(x, e))(x,e) \in E^\varepsilon$,

$$H^\varepsilon(x, e, t) := \sup_{m \geq 0} \{ mt - G^\varepsilon(x, e, m) \} \ \forall t \in \mathbb{R}_+ \ \text{(Legendre Transform)}$$

and

$$T_{t^\varepsilon}(x, y) := \min_{\sigma \in C_{x,y}^\varepsilon} \sum_{(x,e) \subset \sigma} t^\varepsilon(x, e).$$
Transition from discrete to continuous

Some assumptions

- $\gamma^\varepsilon \rightharpoonup \gamma$ a finite nonnegative measure on $\overline{\Omega} \times \overline{\Omega}$, i.e.

\[
\lim_{\varepsilon \to 0^+} \varepsilon^{\frac{d}{2} - 1} \sum_{(x, y) \in N^\varepsilon \times N^\varepsilon} \gamma^\varepsilon(x, y) \varphi(x, y) = \int_{\overline{\Omega} \times \overline{\Omega}} \varphi \, d\gamma, \forall \varphi \in C(\overline{\Omega} \times \overline{\Omega}).
\]

- $g^\varepsilon(x, e, m) = |e|^{d/2} g\left(x, \frac{e}{|e|}, \frac{m}{|e|^{d/2}}\right)$, $\forall \varepsilon > 0$, $(x, e) \in E^\varepsilon$, $m \geq 0$.

Then one has

\[
G^\varepsilon(x, e, m) = |e|^d G\left(x, \frac{e}{|e|}, \frac{m}{|e|^{d/2}}\right) \left( G(x, v, m) = \int_0^m g(x, v, \alpha) \, d\alpha \right)
\]

and

\[
H^\varepsilon(x, e, t) = |e|^d H\left(x, \frac{e}{|e|}, \frac{t}{|e|^{d/2}}\right) \text{ where } H(x, v, \cdot) = (G(x, v, \cdot))^*.
\]

Let $\xi^\varepsilon(x, e) := \frac{t^\varepsilon(x, e)}{|e|^{d/2}}$.

- There exists $p > d$ and constants $0 < \lambda < \Lambda$ such that

\[
\lambda(\xi^p - 1) \leq H(x, e, \xi) \leq \Lambda(\xi^p + 1) \forall (x, e, \xi) \in \overline{\Omega} \times S^{d-1} \times \mathbb{R}_+.
\]
Transition from discrete to continuous
Some assumptions

- \( E^\varepsilon \) "weakly" converges in the sense that
  \[ \forall \varphi \in C(\Omega \times S^{d-1}, \mathbb{R}), \]
  \[ \lim_{\varepsilon \to 0^+} \sum_{(x, e) \in E^\varepsilon} |e|^d \varphi \left( x, \frac{e}{|e|} \right) = \int_{\Omega \times S^{d-1}} \varphi(x, v) \theta(dx, dv). \]

where

\[ \theta(dx, dv) = \sum_{k=1}^{N} c_k(x) \delta_{v_k(x)} dx. \]

\( c_k \) smooth, \( > 0 \) and \( v_k \) (\( C^{0,\alpha} \) with \( \alpha > d/p \)) are the directions. We can decompose every \( z \in \mathbb{R}^d \) in the family \( \{v_k(x)\}, \forall x \in \Omega \), with nonnegative coefficients:

\[ \exists \lambda_1, \ldots, \lambda_N \in \mathbb{R}_+ \text{ such that } z = \sum \lambda_k v_k(x). \]

These \( \lambda_i \) are the weights of \( z \).
Then the continuous limit functional is, \( \forall \xi \in L^p_+(\theta) \), :

\[
J(\xi) := I_0(\xi) - I_1(\xi) = \int_{\Omega \times \mathbb{S}^{d-1}} H(x, v, \xi(x, v))\theta(dx, dv) - \int_{\Omega \times \Omega} c_\xi d\gamma,
\]

where

\[
c_\xi(x, y) = \inf_{\sigma \in C_{x,y}} \inf_{\rho \in \mathcal{P}_\sigma} \int_0^1 \left( \sum_{k=1}^N \xi(\sigma(t), v_k(\sigma(t)))\rho_k(t) \right) dt,
\]

with

\[
\mathcal{P}_\sigma = \left\{ \rho : t \in [0, 1] \rightarrow \rho(t) \in \mathbb{R}_+^N/\dot{\sigma}(t) = \sum_{k=1}^N v_k(\sigma(t))\rho_k(t) \right\}.
\]
Definition

For $\varepsilon > 0$, let $\xi^\varepsilon \in \mathbb{R}^{\#E^\varepsilon}_+$ and $\xi \in L^p_+$, then $\xi^\varepsilon$ is said to weakly converge to $\xi$ in $L^p$ ($\xi^\varepsilon \rightharpoonup \xi$) if:

there exists $M > 0$ such that $\forall \varepsilon > 0$, one has:

$$\|\xi^\varepsilon\|_{\varepsilon, p} := \left( \sum_{(x,e) \in E^\varepsilon} |e|^d \xi^\varepsilon(x, e)^p \right)^{1/p} \leq M,$$

and $\forall \varphi \in C(\overline{\Omega} \times S^{d-1}, \mathbb{R})$, one has

$$\lim_{\varepsilon \to 0^+} \sum_{(x,e) \in E^\varepsilon} |e|^d \varphi \left( x, \frac{e}{|e|} \right) \xi^\varepsilon(x, e) = \int_{\Omega \times S^{d-1}} \varphi \xi \, d\theta.$$
Transition from discrete to continuous
The $\Gamma$-convergence

**Definition**

For $\varepsilon > 0$, let $F^\varepsilon : \mathbb{R}_+^{#E^\varepsilon} \to \mathbb{R} \cup \{+\infty\}$ and $F : L^p_+ \to \mathbb{R} \cup \{+\infty\}$, then $(F^\varepsilon)_\varepsilon$ is said to $\Gamma$-converge (for the weak $L^p$ topology) if and only if the two conditions are satisfied:

**($\Gamma$-liminf inequality)** $\forall \xi \in L^p_+, \xi^\varepsilon \in \mathbb{R}_+^{#E^\varepsilon}$ such that $\xi^\varepsilon \to \xi$, one has

$$\liminf_{\varepsilon \to 0^+} F^\varepsilon(\xi^\varepsilon) \geq F(\xi),$$

**($\Gamma$-limsup inequality)** $\forall \xi \in L^p_+$, there exists $\xi^\varepsilon \in \mathbb{R}_+^{#E^\varepsilon}$ such that $\xi^\varepsilon \to \xi$ and

$$\limsup_{\varepsilon \to 0^+} F^\varepsilon(\xi^\varepsilon) \leq F(\xi).$$
Theorem

Under above assumptions, $(J^\varepsilon)_\varepsilon \Gamma$-converges (for the weak $L^p$ topology) to $J$.

We obtain the following result :

Corollary

Under above assumptions, one has :

$$\min_{\xi^\varepsilon \in \mathbb{R}^E_+} J^\varepsilon(\xi^\varepsilon) \to \min_{\xi \in L^p_+} J(\xi).$$

Moreover, if for $\varepsilon > 0$, $\xi^\varepsilon$ is a solution of the discrete minimization problem, then $\xi^\varepsilon \to \xi$ where $\xi$ is the minimizer of $J$ over $L^p_+$. 
One can show the following minimization problems are equivalent

\[
\inf_{\xi \in L^p_+} J(\xi) := \int_{\Omega \times S^{d-1}} H(x, v, \xi(x, v)) \theta(dx, dv) - \inf_{\gamma \in \Pi(f^-, f^+)} \int_{\overline{\Omega} \times \overline{\Omega}} c_\xi d\gamma,
\]

where

\[
\Pi(f^+, f^-) := \{ \gamma \in M_1(\overline{\Omega} \times \overline{\Omega}) : \gamma \text{ has } f^- \text{ and } f^+ \text{ as marginals}\},
\]

and

\[
\inf_{\sigma \in L^q(\Omega, \mathbb{R}^d)} \left\{ \int_{\Omega} G(x, \sigma(x)) \, dx ; -\text{div } \sigma = f \right\}, \tag{P}
\]

where \( f = f^+ - f^- \) and

\[
G(x, \sigma) = \inf_{\varrho \in \mathcal{P}_x^\sigma} \sum_{k=1}^N c_k(x) G(x, v_k(x), \varrho_k),
\]

with \( \mathcal{P}_x^\sigma = \{ \varrho : \Omega \times S^{d-1} \to \mathbb{R}_+ ; \sigma = \sum v_k(x) \varrho_k \} \).

The dual problem then is

\[
\sup_{u \in W^{1,p}(\Omega)} \left\{ \int_{\Omega} u \, df - \int_{\Omega} G^*(x, \nabla u(x)) \, dx \right\}. \tag{D}
\]
The continuous model
An elliptic PDE

One has \( \min(P) = \max(D) \) and the optimal solution \( \sigma \) of \( P \) (unique by strict convexity) can be characterized as

\[
\sigma(x) = \nabla G^*(x, \nabla u(x)),
\]

where \( u \) is a solution of \( D \). This is equivalent to say that \( u \) is a weak solution of the Euler-Lagrange equation

\[
\begin{cases}
- \text{div} (\nabla G^*(x, \nabla u(x))) = f & \text{in } \Omega, \\
\nabla G^*(x, \nabla u(x)) \cdot \nu_\Omega = 0 & \text{on } \partial \Omega.
\end{cases}
\]

A typical example is \( g(x, v_k(x), m) = a_k(x)m^{q-1} + \delta_k \), with \( \delta_k > 0 \) and the weights \( a_k \) smooth and bounded away from zero. By a direct computation, one has

\[
\sigma = \sum_{k=1}^{N} \left[ b_k(x)(\nabla u \cdot v_k - c_k \delta_k)^{p-1} \right] v_k.
\]

Open problem : regularity of \( \sigma \) if \( f \) Sobolev?
• Some first results on traffic congestion: "Optimal transportation with traffic congestion and Wardrop equilibrium", Carlier, Jimenez, Santambrogio
• Isotropic networks: "Congested traffic dynamics, weak flows and very degenerate elliptic equations", Brasco, Carlier, Santambrogio
• Cartesian model in $\mathbb{R}^2$: "From discrete to continuous Wardrop equilibria", Baillon, Carlier
• Regularity of solutions: "Congested traffic equilibria and degenerate anisotropic PDEs", Brasco, Carlier
• $\mathbb{R}^d$ case: "Wardrop equilibrium: rigorous study of continuous limit of discrete networks models", in progress, Hatchi
Thank you for your attention!
Any questions?