Transition from a discrete network to a continuous model in $\mathbb{R}^d$

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The discrete network
Notations

Let Ω a bounded domain of $\mathbb{R}^d$ and $\varepsilon > 0$. We consider a sequence of discrete networks $(N^\varepsilon, E^\varepsilon)$:=(nodes, arcs), an arc $= (x, e)$ with $x \in N^\varepsilon, e \in \mathbb{C}$.
$e = \Theta(\varepsilon) : \exists C > 0 / C\varepsilon \leq |e| \leq \varepsilon$.
Traveling time : $t^\varepsilon(x, e)$ on the arc (x, e).
Mass : $m^\varepsilon(x, e)$ on the arc (x, e).
Congestion : $t^\varepsilon(x, e) = g^\varepsilon(x, e, m^\varepsilon(x, e))$ with $g^\varepsilon \geq 0$, nondecreasing on the last variable.
Transport plan : $\gamma^\varepsilon(x, y)$ mass sent from x to y $(x, y \in N^\varepsilon)$
Paths : σ of the form $(x_0, x_1, \ldots, x_L)$.
$w^\varepsilon(\sigma)$ mass traveling on the path σ.
$\tau^\varepsilon_{m^\varepsilon}(\sigma)$ traveling time of the path σ.
$C^\varepsilon_{x,y} = \{\sigma \text{ free-loop}; \sigma = (x_0, \ldots, x_l) \text{ with } x_0 = x \text{ and } x_L = y\}$.

Equilibrium : the following mass conservation conditions are satisfied

$$\gamma^\varepsilon(x,y) := \sum_{\sigma \in C^\varepsilon_{x,y}} w^\varepsilon(\sigma),\ \forall (x,y) \in N^\varepsilon \times N^\varepsilon$$

and

$$m^\varepsilon(x,e) = \sum_{\sigma \in C^\varepsilon : (x,e) \subset \sigma} w^\varepsilon(\sigma).$$

Definition

A Wardrop equilibrium is the configuration of nonnegative masses $m^\varepsilon : (x,e) \mapsto (m^\varepsilon(x,e))$ and $w^\varepsilon : \sigma \mapsto (w^\varepsilon(\sigma))$ that satisfy the mass conservation conditions and such that $\forall (x,y) \in N^\varepsilon \times N^\varepsilon$ and $\forall \sigma \in C^\varepsilon_{x,y}$, if $w^\varepsilon(\sigma) > 0$ then

$$\tau^\varepsilon_{m^\varepsilon}(\sigma) \leq \tau^\varepsilon_{m^\varepsilon}(\sigma'),\ \forall \sigma' \in C^\varepsilon_{x,y}.$$

Beckmann, McGuire and Winsten in the 50's :
$(w^\varepsilon, m^\varepsilon)$ is an equilibrium if and only if it minimizes

$$\sum_{(x,e)\in E^\varepsilon} G^\varepsilon(x,e,m^\varepsilon(x,e)) \text{ where } G^\varepsilon(x,e,m) := \int_0^m g^\varepsilon(x,e,\alpha)d\alpha$$

subject to nonnegativity constraints and the mass conservation conditions.
The complexity is very important : one variable by path.

Dual formulation :

$$\inf_{t^\varepsilon \in \mathbb{R}_+^{\# E^\varepsilon}} \left(\sum_{(x,e)\in E^\varepsilon} H^\varepsilon(x,e,t^\varepsilon(x,e)) - \sum_{(x,y)\in N^\varepsilon \times N^\varepsilon} \gamma^\varepsilon(x,y)\, T^\varepsilon_{t^\varepsilon}(x,y) \right)$$

where $t^\varepsilon = (t^\varepsilon(x,e))_{(x,e)\in E^\varepsilon}$,

$$H^\varepsilon(x,e,t) := \sup_{m\geq 0} \{mt - G^\varepsilon(x,e,m)\} \ \forall t \in \mathbb{R}_+ \quad \text{(Legendre Transform)}$$

and

$$T^\varepsilon_{t^\varepsilon}(x,y) := \min_{\sigma \in C^\varepsilon_{x,y}} \sum_{(x,e)\subset\sigma} t^\varepsilon(x,e).$$

Transition from discrete to continuous
Some assumptions

- $\gamma^\varepsilon \overset{*}{\rightharpoonup} \gamma$ a finite nonnegative measure on $\overline{\Omega} \times \overline{\Omega}$ ie

$$\lim_{\varepsilon \to 0^+} \varepsilon^{\frac{d}{2}-1} \sum_{(x,y) \in N^\varepsilon \times N^\varepsilon} \gamma^\varepsilon(x,y)\varphi(x,y) = \int_{\overline{\Omega}\times\overline{\Omega}} \varphi d\gamma, \forall \varphi \in C(\overline{\Omega}\times\overline{\Omega}).$$

- $g^\varepsilon(x,e,m) = |e|^{d/2} g\left(x, \frac{e}{|e|}\frac{m}{|e|^{d/2}}\right), \forall \varepsilon > 0, (x,e) \in E^\varepsilon, m \geq 0.$
  Then one has

$$G^\varepsilon(x,e,m) = |e|^d G\left(x, \frac{e}{|e|}, \frac{m}{|e|^{d/2}}\right) \left(G(x,v,m) = \int_0^m g(x,v,\alpha)d\alpha\right)$$

  and

$$H^\varepsilon(x,e,t) = |e|^d H\left(x, \frac{e}{|e|}, \frac{t}{|e|^{d/2}}\right) \text{ where } H(x,v,\cdot) = (G(x,v,\cdot))^\star.$$

  Let $\xi^\varepsilon(x,e) := \frac{t^\varepsilon(x,e)}{|e|^{d/2}}$.

- There exists $p > d$ and constants $0 < \lambda < \Lambda$ such that

$$\lambda(\xi^p - 1) \leq H(x,e,\xi) \leq \Lambda(\xi^p + 1) \; \forall (x,e,\xi) \in \overline{\Omega} \times \mathbb{S}^{d-1} \times \mathbb{R}_+.$$

Transition from discrete to continuous
Some assumptions

- E^ε "weakly" converges in the sense that $\forall \varphi \in C(\overline{\Omega} \times \mathbb{S}^{d-1}, \mathbb{R})$,

$$\lim_{\varepsilon \to 0^+} \sum_{(x,e) \in E^\varepsilon} |e|^d \varphi\left(x, \frac{e}{|e|}\right) = \int_{\Omega \times \mathbb{S}^{d-1}} \varphi(x, v)\, \theta(dx, dv).$$

where

$$\theta(dx, dv) = \sum_{k=1}^{N} c_k(x) \delta_{v_k(x)} dx.$$

c_k smooth, > 0 and v_k ($C^{0,\alpha}$ with $\alpha > d/p$) are the directions. We can decompose every $z \in \mathbb{R}^d$ in the family $\{v_k(x)\}, \forall x \in \Omega$, with nonnegative coefficients : $\exists \lambda_1, \ldots, \lambda_N \in \mathbb{R}_+$ such that $z = \sum \lambda_k v_k(x)$. These λ_i are the weights of z.

Transition from discrete to continuous
Limit functional

Then the continuous limit functional is, $\forall \xi \in L^p_+(\theta)$, :

$$J(\xi) := I_0(\xi) - I_1(\xi) = \int_{\Omega \times \mathbb{S}^{d-1}} H(x, v, \xi(x, v))\theta(dx, dv) - \int_{\overline{\Omega} \times \overline{\Omega}} \overline{c}_\xi d\gamma,$$

where

$$\overline{c}_\xi(x, y)" = " \inf_{\sigma \in C_{x,y}} \inf_{\rho \in \mathcal{P}_\sigma} \int_0^1 \left(\sum_{k=1}^N \xi(\sigma(t), v_k(\sigma(t)))\rho_k(t) \right) dt,$$

with

$$\mathcal{P}_\sigma = \left\{ \rho : t \in [0, 1] \to \rho(t) \in \mathbb{R}^N_+ / \dot{\sigma}(t) = \sum_{k=1}^N v_k(\sigma(t))\rho_k(t) \right\}.$$

Definition

For $\varepsilon > 0$, let $\xi^\varepsilon \in \mathbb{R}_+^{\#E^\varepsilon}$ et $\xi \in L^p_+$, then ξ^ε is said to weakly converge to ξ in L^p ($\xi^\varepsilon \to \xi$) if :
there exists $M > 0$ such that $\forall \varepsilon > 0$, one has :

$$\|\xi^\varepsilon\|_{\varepsilon,p} := \left(\sum_{(x,e)\in E^\varepsilon} |e|^d \xi^\varepsilon(x,e)^p \right)^{1/p} \leq M,$$

and $\forall \varphi \in C(\overline{\Omega} \times \mathbb{S}^{d-1}, \mathbb{R})$, one has

$$\lim_{\varepsilon \to 0^+} \sum_{(x,e)\in E^\varepsilon} |e|^d \varphi\left(x, \frac{e}{|e|}\right) \xi^\varepsilon(x,e) = \int_{\Omega \times \mathbb{S}^{d-1}} \varphi \xi \, d\theta.$$

Definition

For $\varepsilon > 0$, *let* $F^\varepsilon : \mathbb{R}_+^{\#E^\varepsilon} \to \mathbb{R} \cup \{+\infty\}$ *et* $F : L^p_+ \to \mathbb{R} \cup \{+\infty\}$, *then* $(F^\varepsilon)_\varepsilon$ *is said to* Γ*-converge (for the weak* L^p *topology) if and only if the two conditions are satisfied :*
*(*Γ*-liminf inequality)* $\forall \xi \in L^p_+, \xi^\varepsilon \in \mathbb{R}_+^{\#E^\varepsilon}$ *such that* $\xi^\varepsilon \to \xi$, *one has*

$$\liminf_{\varepsilon \to 0^+} F^\varepsilon(\xi^\varepsilon) \geq F(\xi),$$

*(*Γ*-limsup inequality)* $\forall \xi \in L^p_+$, *there exists* $\xi^\varepsilon \in \mathbb{R}_+^{\#E^\varepsilon}$ *such that* $\xi^\varepsilon \to \xi$ *and*

$$\limsup_{\varepsilon \to 0^+} F^\varepsilon(\xi^\varepsilon) \leq F(\xi).$$

Theorem

Under above assumptions, $(J^\varepsilon)_\varepsilon$ Γ-converges (for the weak L^p topology) to J.

We obtain the following result :

Corollary

Under above assumptions, one has :

$$\min_{\xi^\varepsilon \in \mathbb{R}_+^{\#E^\varepsilon}} J^\varepsilon(\xi^\varepsilon) \to \min_{\xi \in L^p_+} J(\xi).$$

Moreover, if for $\varepsilon > 0$, ξ^ε is a solution of the discrete minimization problem, then $\xi^\varepsilon \to \xi$ where ξ is the minimizer of J over L^p_+.

The continuous model
Other minimization problems

One can show the following minimization problems are equivalent

$$\inf_{\xi \in L^p_+} J(\xi) := \int_{\Omega \times \mathbb{S}^{d-1}} H(x, v, \xi(x, v))\theta(dx, dv) - \inf_{\gamma \in \Pi(f_-, f_+)} \int_{\overline{\Omega} \times \overline{\Omega}} \overline{c}_\xi d\gamma,$$

where
$\Pi(f^+, f^-) := \{\gamma \in \mathcal{M}_1(\overline{\Omega} \times \overline{\Omega}) : \gamma \text{ has } f^- \text{ and } f^+ \text{as marginals}\}$,
and

$$\inf_{\sigma \in L^q(\Omega, \mathbb{R}^d)} \left\{ \int_\Omega \mathcal{G}(x, \sigma(x))\, dx;\ -\text{div}\ \sigma = f \right\}, \qquad \text{(P)}$$

where $f = f_+ - f_-$ and

$$\mathcal{G}(x, \sigma) = \inf_{\varrho \in \mathcal{P}^\sigma_x} \sum_{k=1}^N c_k(x) G(x, v_k(x), \varrho_k),$$

with $\mathcal{P}^\sigma_x = \left\{\varrho : \Omega \times \mathbb{S}^{d-1} \to \mathbb{R}_+;\ \sigma = \sum v_k(x)\varrho_k)\right\}$.
The dual problem then is

$$\sup_{u \in W^{1,p}(\Omega)} \left\{ \int_\Omega u\, df - \int_\Omega \mathcal{G}^*(x, \nabla u(x))\, dx \right\}. \qquad \text{(D)}$$

One has $\min(P) = \max(D)$ and the optimal solution σ of (P) (unique by strict convexity) can be characterized as

$$\sigma(x) = \nabla \mathcal{G}^*(x, \nabla u(x)),$$

where u is a solution of (D). This is equivalent to say that u is a weak solution of the Euler-Lagrange equation

$$\begin{cases} -\operatorname{div}\left(\nabla \mathcal{G}^*(x, \nabla u(x))\right) = f & \text{in } \Omega, \\ \nabla \mathcal{G}^*(x, \nabla u(x)) \cdot \nu_\Omega = 0 & \text{on } \partial\Omega. \end{cases}$$

A typical example is $g(x, v_k(x), m) = a_k(x) m^{q-1} + \delta_k$, with $\delta_k > 0$ and the weights a_k smooth and bounded away from zero. By a direct computation, one has

$$\sigma = \sum_{k=1}^{N} \left[b_k(x) (\nabla u \cdot v_k - c_k \delta_k)_+^{p-1} \right] v_k.$$

Open problem : regularity of σ if f Sobolev ?

References

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Thank you for your attention !
Any questions ?