Optimal Transportation With Convex Constraints

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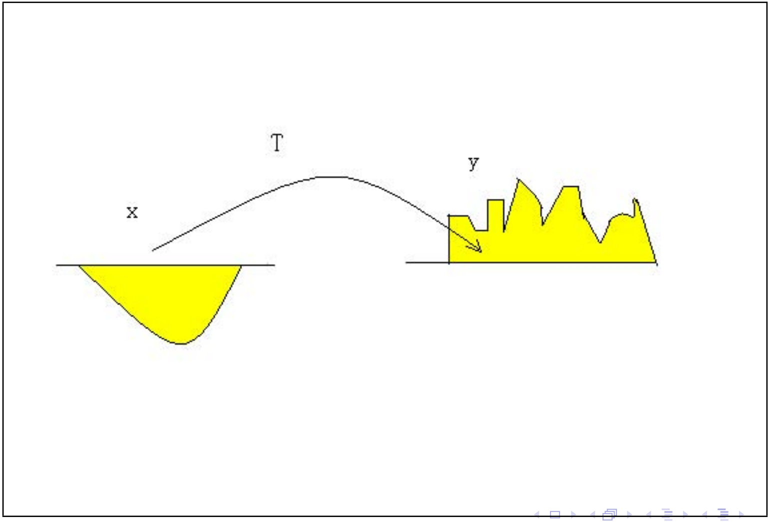
Optimal Transport in the Applied Sciences
Linz December 2014

Outline

1. Optimal transportation
   - Picture of optimal transportation
   - classical problems in Optimal Transportation
   - results on existence of optimal transport maps
   - methods to get existence of optimal transport maps
2. Optimal transportation with convex constraints
   - results on convex constrained optimal transportation problem
   - Difference between CMP and CCOTP
2. Our Main results
   - 2-dimensional cases
   - n-dimensional cases
2. Sketch of the proof
   - Sketch proof in the 2-dimensional cases
   - Sketch proof of the n-dimensional cases
   - main difference in relative results

Picture of optimal transportation



Existence of optimal transport maps

Question

1. Monge problem(MP)

$$\min_{T_\sharp\mu=\nu}\int_X c(x,T(x))d\mu(x), \tag{1}$$

Question

2. Classical Monge problem(CMP)

$$\min_{T_\sharp\mu=\nu}\int_{\mathbb{R}^n} d(x,T(x))d\mu(x), \tag{2}$$

Results on existence of optimal transport maps

- $X = Y = \mathbb{R}^n$,$c(x, y) = |x - y|^p, p \in (0, +\infty)$;$c(x, y) = ||x - y||$,where $|| \cdot ||$ is the general norm.
- $X = Y = (M, d)$ Riemannian manifolds,$c(x, y) = d(x, y)^p, p = 1, 2$
- $X = Y = (M, d_{cc})$ subRiemannian manifolds,$c(x, y) = d_{cc}(x, y)^p, p = 1, 2$(see L. De Pascale and S.Rigot)
- $X = Y = (X, \mu, d_{cc})$ geodesical metric spaces, $c(x, y) = d_{cc}$ (see Y.Brenier,G.Buttazzo,G.Carlier,F.Cavalletti,T.Champion,L.De Pascale,R.McCann,F.Santambrogio etc.)

methods to get existence of optimal transport maps

- Kantorovich dual theory
- variational approximation
- PDEs

Optimal transportation with convex constraints

- convex constraints:the point $y - x$ belongs to a given closed convex set C.
- cost function with convex constraints:

$$c(x,y) = c_{h;C}(x,y) = \begin{cases} h(|x-y|), & \text{if } y - x \in C, \\ +\infty, & \text{otherwise}, \end{cases} \tag{3}$$

- convex constrained optimal transportation problem (CCOTP):

$$\min_{T_\sharp\mu=\nu} \int_{\mathbb{R}^n} c_{h;C}(x, T(x))d\mu(x). \tag{4}$$

- $h(x) = x^2$,i.e.

$$c(x,y) = c_{d^2,C}(x,y) = \begin{cases} |x-y|^2, & \text{if } y-x \in C, \\ +\infty, & \text{otherwise}, \end{cases} \tag{5}$$

  See "Optimal transportation for a quadratic cost with convex constraints and application" (C.Jimenez and F.Santambrogio).

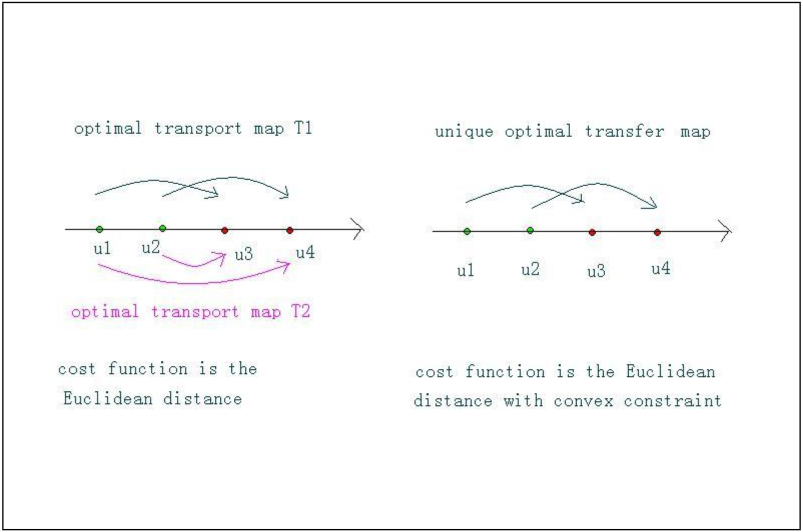
- $h(x)$ is strictly convex See "The optimal mass transportation problem for relativistic costs"(J.Bertrand A.Pratelli and M.Puel)
- $h(x) = x$ the case we considered

Difference Between CMP And CCOTP

Example

Let $u_1 < u_2 < u_3 < u_4 \in \mathbb{R}$ be four points with the following distances: $|u_1 - u_2| = |u_2 - u_3| = |u_3 - u_4| = 1$. Set $\mu = \delta_{u_1} + \delta_{u_2}$ and $\nu = \delta_{u_3} + \delta_{u_4}$.

- Set $c(x, y) = d(x, y)$
  There are two optimal transport maps for the classical Monge problem (2): $Tu_1 = u_3, Tu_2 = u_4$ and $Tu_1 = u_4, Tu_2 = u_3$;
- Set $c(x, y) = d_C(x, T(x))$, where $C = \overline{B(0, r)}, (2 < r < 3)$.
  There is an unique optimal transport map $Tu_1 = u_3, Tu_2 = u_4$.


optimal transport map T1
u1
u2
u3
u4
optimal transport map T2
cost function is the
Euclidean distance
unique optimal transfer map
u1
u2
u3
u4
cost function is the Euclidean
distance with convex constraint

Remark

If the convex constraint C is large enough, for example, if we take $C = \mathbb{R}^n$, then problems with convex constraints are the same as those problems without convex constraints.

Cost functions we considered:

$$c(x,y) = c_d(x,y) = \begin{cases} |x-y|, & \text{if } y-x \in C, \\ +\infty, & \text{otherwise}, \end{cases} \tag{6}$$

Our main results:

- 2-dimensional cases, we get existence and uniqueness of optimal transport map(see Theorem 1)
- n-dimensional cases, we get existence of optimal transport map(see Theorem 2)

Convex Constraints

Theorem

1. Assume that

1. $\mu \ll \mathcal{L}^2$ *and* ν *be probability measures in* $\mathbb{R}^2$,
2. *C is a given closed and convex subset with at most a countable flat parts in* $\mathbb{R}^2$,
3. *for all* $\gamma \in \Pi(\mu, \nu)$, *and* $\gamma - a.e.\ (x, y),\ (x, y'),\ y \neq y'$ *satisfying* $y - x \in C$ *and* $y' - x \in C$, *then* x, y, y' *do not lie in a single line.*
4. *there exists* $\pi \in \Pi(\mu, \nu)$ *such that* $\int_{\mathbb{R}^2\times\mathbb{R}^2} c(x, y) d\pi(x, y) < +\infty$.

Then there exists an optimal transport map for the convex constraint optimal transportation problem (4)with cost function $c_{d,C}(x, y)$.

Strictly Convex Constraints

Remark

Assume that the convex set C in Theorem 1 is strictly convex, then there exists an unique optimal transport map of the convex constraint optimal transportation problem (4).

Geometrical Meaning of Assumption 3 in Theorem 1

Remark

The Assumption 3 in Theorem 1 shows that there are only two cases of transportation of mass located at point x. One is that the mass located at point x is transferred to the only destination y, another is that the mass located at point x is transferred to several possible destinations $y's$. In the second case, starting point x and any two destinations y, y' do not lie in a single line. The following three examples illustrate the idea.

Sufficient condition for the existence and uniqueness of solution of CMP

Generally speaking, the minimizers of (2) are not unique. However we can give a sufficient condition for the existence and uniqueness of solutions of the original Monge's problem with the Euclidean distance cost function in $\mathbb{R}^2$.

Corollary

Let $\mu << \mathcal{L}^2$, ν be probability measures on $\mathbb{R}^2$, assume that

- $\forall \gamma \in \Pi(\mu, \nu)$, *and* $\gamma - a.e.(x, y), (x, y')$ *satisfying* $y \neq y'$*, then* x, y, y' *do not lie in a single line,*
- *there exists* $\pi \in \Pi(\mu, \nu)$ *such that* $\int_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y| d\pi(x, y) < +\infty$,

then the classical Monge problem (2) admits a unique solution.

Convex Constrained problem in R^n

Theorem

[2]. Assume that

- $\mu \ll \mathcal{L}^n$ *and* ν *are Borel probability with compact support in* $\mathbb{R}^n$
- *there is* $\gamma \in \Pi(\mu, \nu)$ *s.t.* $\int_{\mathbb{R}^n \times \mathbb{R}^n} c_1(x, y) d\gamma(x, y) < +\infty$,
- $C \subset \mathbb{R}^n$ *is closed convex subset with at most countable flat parts.*

Then there exists an optimal transport map for the convex constraint optimal transportation problem (4)with cost function $c_{d,C}(x, y)$.

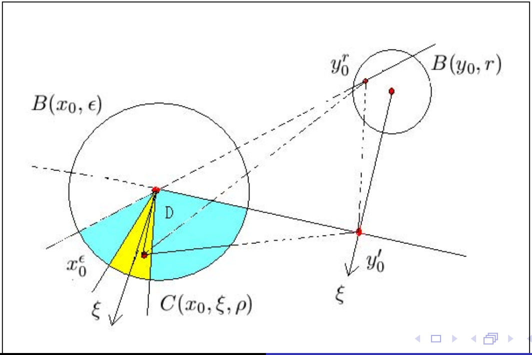
Sketch of the 2-dimensional proof

To get optimal plan γ is induced by a map, we need only prove that if (x, y) and (x, y') are in *support*γ, then $y = y'$.

Proof by contradiction.

- step 1. $c_{d;C}(x, y)$–cyclical monotonicity provides for any two different pairs (x, y) and (x', y') with $y - x \in C$,$y' - x \in C$,$y - x' \in C$ and $y' - x' \in C$, it is true that the segment $[xy]$ and $[x'y']$ do not cross except on the endpoints.

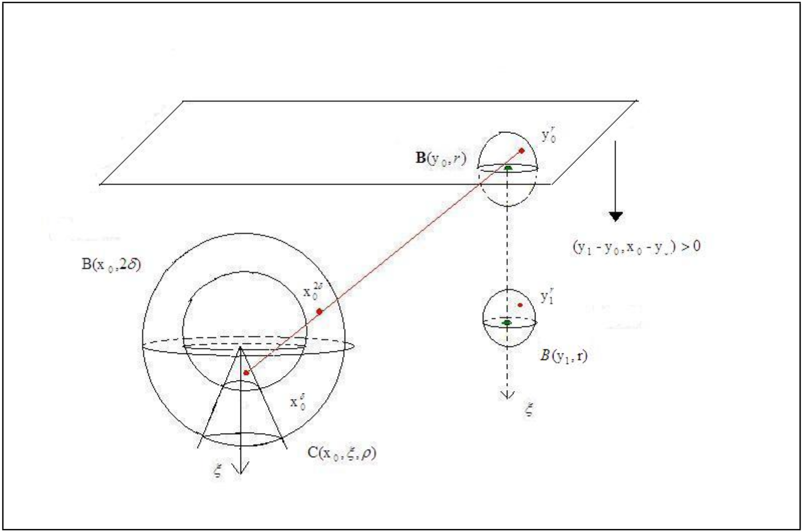
- step 2. One assume that $(x_0, y_0)$and (x_0, y_0') in sup γ. We will show $y_0 = y_1$contradiction.
- 2.1 constructing a perturbation (x_0^ε, y_0^r) of (x_0, y_0)
- 2.2 prove $y^r - x^\varepsilon \in C$,$y' - x^\varepsilon \in C$,$y^r - x' \in C$ and $y' - x' \in C$.
- 2.3 prove segment $[x'y']$ and $[x^\varepsilon y^r]$ cross. It is contradict with that in step 1.



Sketch proof of the n-dimensional cases

The idea inspired from those in ”Monge problem in R^d”(T. Champion and L. De Pascale) and ”Optimal transportation for a quadratic cost with convex constraints and application” (C.Jimenez and F.Santambrogio)

- step 1. Select optimal transport plan by the second variational problem $\min_{\gamma \in Q_1(\mu,\nu)} \int_{\mathbb{R}^n \times \mathbb{R}^n} c_{d^2;C}(x,y) d\gamma(x,y)$.The selected optimal transport plan is non-decreasing (with convex constraints) on the transport set.
- step 2. Variational approximation of the selected optimal transport plans.
- step 3.lower boundedness of the transport set of the selected optimal transport plan.


B(y₀,r)
y₀^r
(y₁ - y₀, x₀ - y.) > 0
B(x₀,2δ)
x₀^{2δ}
y₁^r
B(y₁,r)
x₀^δ
ξ
C(x₀,ξ,ρ)
ξ

difference in our 2-dimensional and n-dimensional papers

- There is only one more assumption in R^2 than those in R^n(see Assumption 3 in Theorem 1)
- If the convex set C is also strictly convex, then the optimal transport map (constrained transport problem)is unique in R^2. But the uniqueness of map(constrained transport problem) in R^n is not clear even the set C is strictly convex.
- The above uniqueness result also give a sufficient condition to get uniqueness of classical Monge problem in R^2. But it's not sufficient to do so in $R^n, n > 2$.

main different proof between the classical Monge problem with and without constraints

Main different proof between the classical Monge problem with and without constraints.

When choosing the perturbation (x_0^δ, y_0^r) of (x_0, y_0) and the perturbation $(x_0^{2\delta}, y_1^r)$ of (x_0, y_1) we must guarantee the perturbations also satisfy the following properties:

- $y_0^r - x_0^{2\delta} \in C$,
- $y_1^r - x_0^\delta \in C$
- $x_0^{2\delta}$ lies in the segment jointing x_0^δ and y_0^r

Thank you!