NTRU Cryptosystem: Recent Developments

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Outline of the talk

1- Introduction
- Background: Why study NTRU?

2- NTRU Cryptosystem: Review

3- Recent Developments on NTRU Security
- NTRU variant provably as secure as worst-case lattice problems
- Tools: Discrete Gaussians, Fourier analysis, Ring-LWE

4- Recent Developments on NTRU Applications
- Fully-Homomorphic Encryption (FHE) from NTRU
- Cryptographic Multilinear Maps from NTRU

5- Concluding Remarks

The NTRU Cryptosystem

NTRUEncrypt: A public-key encryption scheme.

- 1996: Proposed by Hoffstein, Pipher & Silverman.
- 1997: Lattice attacks by Coppersmith & Shamir.
- 1998: Revised by Hoffstein et al.

In the last 15 years:

- Several minor improvements to the lattice attacks.
- Attacks for isolated sets of parameters.
- **But the design has proved very robust.**

In the last 3 years (this talk):

- Variants with a provable security foundation
- Variants with new functionality

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- Standardized: IEEE P1363.
- Commercialized: Security Innovation.
- Super-fast (comparison to 1024-bit RSA, based on an NTRU brochure):
  - Encryption ~ 10 times faster
  - Decryption ~ 100 times faster
  - Asymptotically: $\widetilde{O}(\lambda)$ versus $\widetilde{O}(\lambda^6)$, for security 2^λ
- Interesting security features:
  - No integer factoring nor discrete logs
  - Seems to resist practical attacks
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Polynomial Rings

Take $\phi \in \mathbb{Z}[x]$ monic of degree n.

$$R^{\phi} := \Big[\mathbb{Z}[x]/(\phi), +, \times\Big].$$

Interesting ϕ's:

- $\phi = x^n - 1 \rightarrow R^-$, $\phi = x^n + 1 \rightarrow R^+$.
- For n a power of 2, the ring R^+ is isomorphic to the ring of integers of $K = \mathbb{Q}[e^{i\pi/n}]$:

$$K \simeq \mathbb{Q}[x]/(x^n + 1)$$
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$\Rightarrow$ Rich algebraic structure (great for design and proofs).

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- R_q^+ is isomorphic to $\mathcal{O}_K/(q)$.

The key to decryption correctness

If $f \in R^\phi$ is known to have coefficients in $(-q/2, q/2)$, then

$f \bmod q$ uniquely determines f.

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NTRU Cryptosystem: Key Generation

Parameters: n prime, $q \approx n$ a power of 2, p small, $\phi = x^n - 1$.
(e.g. $(n, q, p) = (503, 256, 3)$).

- Secret key sk: $f, g \in R^-$ sampled indep. from distrib. χ_σ with:
  - f is invertible mod q and mod p
  - The coeffs of f and g are small
    - $\mathrm{Supp}(\chi_\sigma) = \{-1, 0, 1\}^n$.
- Public key pk: $h = g/f \bmod q$.

Security intuition

Given $h \in R_q^-$, finding $g, f \in R^-$ small s.t. $h = g/f \ [q]$ is hard.

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NTRU Cryptosystem: Encryption and Decryption

- sk: $f, g \in R^-$ small with f invertible mod q and mod p
- pk: $h = g/f \bmod q$

Encryption of $M \in R$ with coeffs in $\{0, \ldots, p-1\}$:

- Sample $s, e \in R_q^-$ from distrib. χ_ρ, χ_β resp. with small coeffs – $\mathrm{Supp}(\chi_\rho) = \{-1, 0, 1\}^n, \mathrm{Supp}(\chi_\beta) = \{0\}$).
- Send $C := p(hs + e) + M \bmod q$

Decryption of $C \in R_q^-$:

- $f \times C = p(gs + fe) + fM \bmod q$
- Smallness $\Rightarrow$ equality holds over R^-
- $(f \times C \bmod q) \bmod p = fM \bmod p$
- Multiply by the inverse of f mod p

Security intuition

The mask $p(hs + e)$ hides the plaintext M in the ciphertext C.

Lattices Background: Approx-SVP

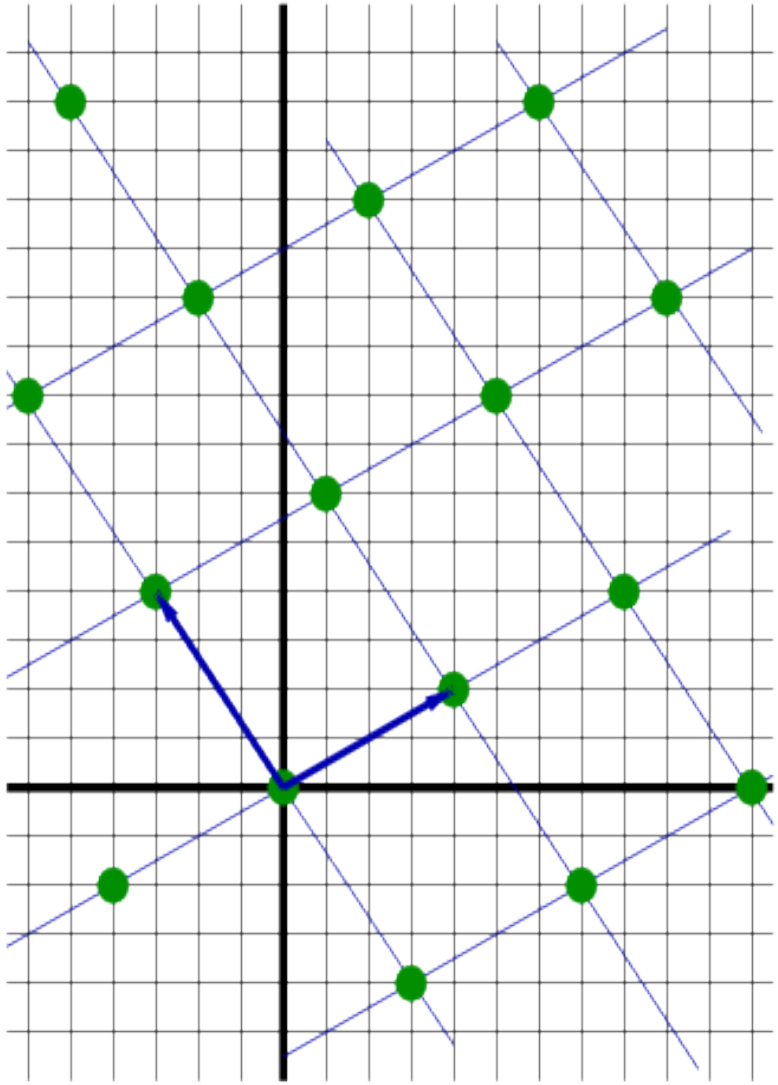
Lattice $\equiv \{\sum_{i \le n} x_i \mathbf{b}_i : x_i \in \mathbb{Z}\}$,
for some lin. independent $\mathbf{b}_i$'s.

Minimum: $\lambda(L) = \min(\|\mathbf{b}\| : \mathbf{b} \in L \setminus \mathbf{0})$

γ-SVP

Find $\mathbf{b} \in L$ with: $0 < \|\mathbf{b}\| \le \gamma \cdot \lambda(L)$.

- No known sub-exp. algorithm for $\gamma = Poly(n)$.
- Not even quantumly.
- Seems harder than Int-Fac and DLog.



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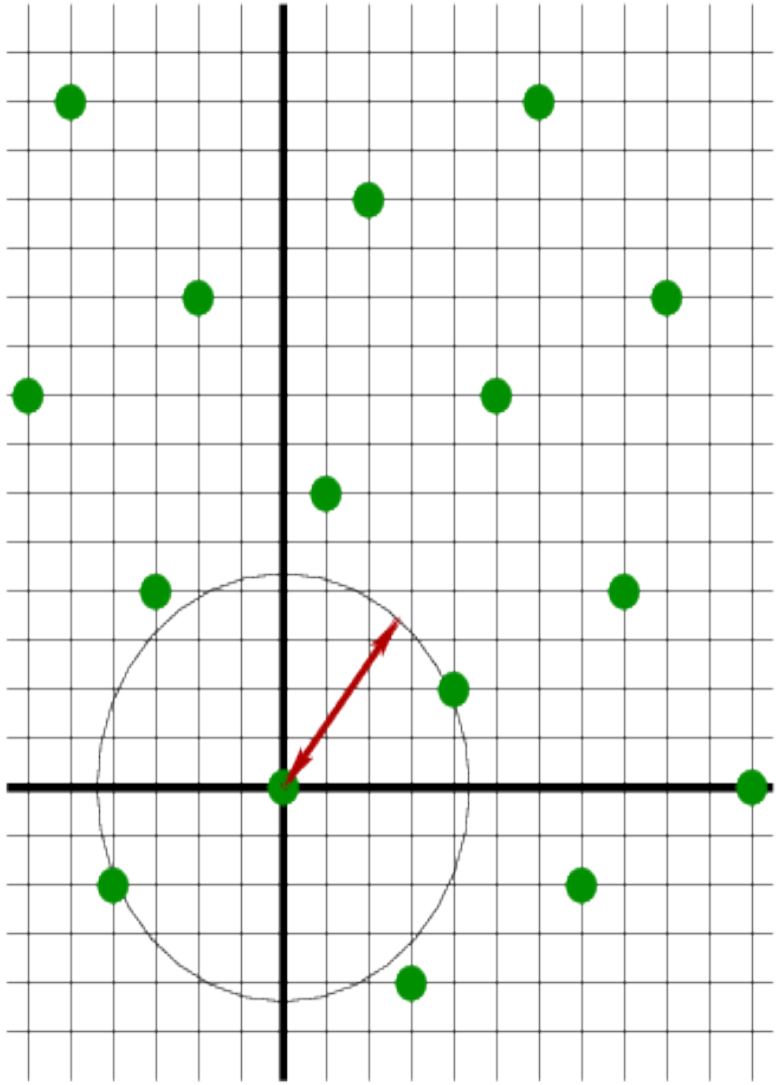

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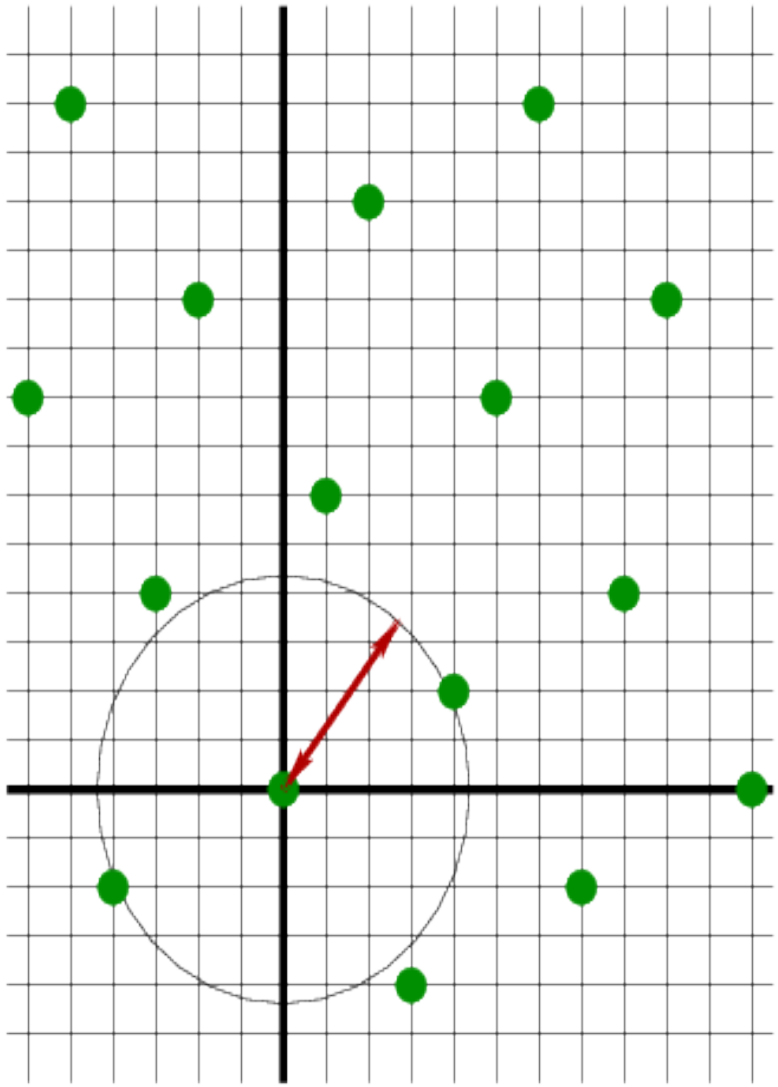

Lattice Background: Approx-Ideal-SVP

- $I \subseteq R^\phi$ is an ideal if:

$$\forall a, b \in I, \forall r \in R^\phi : \quad a + b \cdot r \in I.$$

- We identify polynomials to vectors via their coefficients:

$$\begin{array}{ccc} R^\phi & \rightarrow & \mathbb{Z}^n \\ \sum_{i<n} f_i x^i & \mapsto & (f_0, \ldots, f_{n-1})^t \end{array}$$

- An ideal I is mapped to an integer lattice.

$\mathcal{P}oly(n)$-Ideal-SVP: $\mathcal{P}oly(n)$-SVP restricted to ideal lattices.

No known computational advantage for this family of inputs.

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Security of NTRU: Lattice Attacks

Coppersmith-Shamir Lattice attack: Given $h = g/f \in R_q$, small secret key $(f, g) \in R^2$) satisfies:

$$f \cdot h - g = 0 \bmod q.$$

Set of all solutions $(f', f'h + qR) \in R^2$ to above is a $2n$-dim. $\mathbb{Z}$-lattice L_{NTRU} with row basis

$$\begin{bmatrix} I_n & \mathrm{rot}(h) \\ 0 & qI_n \end{bmatrix}, \tag{1}$$

called the NTRU lattice.

Attack: $\|(f, g)\| = \lambda_1(L_{\mathrm{NTRU}}) \rightarrow$ Run γ-SVP on L_{NTRU}, hope to get small multiple of (f, g).

Catch: $(q, 0) \in L_{\mathrm{NTRU}}$, so need $\gamma \leq q/\|(f, g)\| = O(\mathcal{P}oly(n))$.

- Recall: Best known alg. for $\mathcal{P}oly(n)$-SVP take time $2^{\Omega(n)}$!

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NTRU variant provably as secure as worst-case lattice problems

Motivation:

- NTRU lattices have a special algebraic structure.
- What if an efficient approx-SVP algorithm could be tailored to NTRU lattices?
- Could there exist a non-negligible fraction of "weak" NTRU lattices?

Theorem[Stehlé, Steinfeld 2011]

There is a choice of parameters for NTRU Cryptosystem so that:

- Encryption/decryption of λ bits still cost $\widetilde{O}(\lambda)$,
- Any polynomial-time IND attack leads to a polynomial-time quantum algorithm for $\mathcal{P}oly(n)$-Ideal-SVP that works for all inputs.

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Indistinguishability (IND) Security Definition

Modern security definition for public-key encryption (against passive eavesdropping)

For $b \in \{0, 1\}$, two phase game G_b with adversary A:

- Phase 1:
  - A is given public key pk from key generation algorithm $\mathrm{KG}(n)$.
  - A outputs two challenge messages m_0, m_1.
- Phase 2:
  - A is given challenge ciphertext $c_b = \mathrm{Enc}(pk, m_b)$.
  - A outputs an estimate b' for bit b.

Indistinguishability (IND) Security

For all $\mathcal{P}oly(n)$-time A,

$$Adv(A) \stackrel{\text{def}}{=} |\Pr_{G_1}[b' = 1] - \Pr_{G_0}[b' = 1]| = n^{-\omega(1)} = \mathrm{neg}(n).$$

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Security of NTRU: Computational/Statistical Problems

Essentially two ways to break the IND security of NTRU:

- Crack the **public key**:

NTRU Decision Key Cracking Problem $\mathtt{DNKC}_{n,q,\phi,\chi_\sigma}$

Given (n, q, ϕ) and h, distinguish

- NTRU key distribution $D_0 = \{h = g/f \in R_q : f, g \hookleftarrow \chi_\sigma\}$.
- Uniform key distribution $D_1 = \{h \hookleftarrow U(R_q^*)\}$.

- Crack the **ciphertext** for a uniform key:

NTRU Decision Ciphertext Cracking Problem $\mathtt{DNCC}_{n,q,\phi,\chi_\rho,\chi_\beta}$

Given (n, q, ϕ), h sampled from $U(R_q^*)$, and c, distinguish

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IND Security of NTRU: Sufficient Condition

Proposition (Adapted from [StSt11])

If DNKC and DNCC are both hard, then NTRUcryptosystem achieves semantic (IND) security.

Proof by contradiction – three 'games' with adversary A:

- IND_b – pk: $h = g/f$, ciph: $c_b = p \cdot (hs + e) + m_b$, $p_b = \Pr_{\text{IND}_b}[A(h, c_b) = 1]$.
- IND'_b – pk: $h \leftarrow\!\!\shortmid\; U(R_q^*)$, ciph: $c_b = p \cdot (hs + e) + m_b$, $p'_b = \Pr_{\text{IND'}_b}[A(h, c_b) = 1]$.
  - $|p'_b - p_b| = \text{non-neg}(n) \rightarrow A$ breaks DNKC.
- IND''_b – pk: $h \leftarrow\!\!\shortmid\; U(R_q^*)$, ciph: $c_b = p \cdot U(R_q) + m_b$, $p'' = \Pr_{\text{IND''}_b}[A(h, c_b) = 1]$.
  - $|p''_b - p'_b| = \text{non-neg}(n) \rightarrow A$ breaks DNCC.

Else, A can distinguish IND''_0 from IND''_1: contradiction – $p \cdot U(R_q)$ term perfectly hides m_b!

How to make both DNKCand DNCCproblems hard?

StSt11 strategy to prove hardness of DNKC and DNCC problems:

- Choose χ_σ for f, g to make DNKC statistically hard.
  - $f, g \hookleftarrow \chi_\sigma \to h = g/f$ almost uniformly distributed on R_q^*.
  - Must work in statistical region: $|\mathrm{Supp}(\chi_\sigma)| > |R_q^*| \to \sigma > \sqrt{q}$.
  - Use a (modified) discrete Gaussian distribution χ_σ.
  - Proof ingredients: Fourier analysis, duality of lattices, algebraic structure of R_q.
- Choose $\chi_\rho = \chi_\beta$ for s, e to make DNCC computationally hard.
  - Change rings: $R_q^- = \mathbb{Z}_q[x]/(x^n - 1) \to R_q^+ = \mathbb{Z}_q[x]/(x^n + 1)$, $n = 2^k$.
  - $h \hookleftarrow U(R_q^*), s, e \hookleftarrow \chi_\beta \to (h, c = hs + e)$ computationaly indist. from $U(R_q^* \times R_q)$, if $\approx q/\beta$-Ideal-SVP is hard.
  - Must work in computational region: $|\mathrm{Supp}(\chi_\beta)| < |R_q| \to \beta < \sqrt{q}$.
  - Use a rounded Gaussian distribution χ_β.

The modified scheme

Parameters: n, q a power of 2, $R = R^-$.

Key generation:

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  - Coeffs of f and g in $\{-1, 0, 1\}$
- pk: $h = g/f \bmod q$.

Encryption of $M \in R$ with coeffs in $\{0, 1\}$:

- $C := phs + M \bmod q$, with coeffs of s in $\{-1, 0, 1\}$.

Decryption of $C \in R_q$:

- $f \times C \bmod q \;=\; pgs + fM \qquad$ (over R)
- $(f \times C \bmod q) \bmod p \;=\; fM \bmod p$.
- Multiply by the inverse of f mod p.

The modified scheme

Parameters: n a power of 2, q prime, $R = R^+$.

Key generation:

- sk: $f, g \in R$ with:
  - f invertible mod q and p.
  - Coeffs of f and g of magnitude $\approx \sqrt{q}$
- pk: $h = g/f \bmod q$.

Encryption of $M \in R$ with coeffs in $\{0, 1\}$:

- $C := p(hs + e) + M \bmod q$, with coeffs of s, e of magnitude $\approx \beta$.

Decryption of $C \in R_q$:

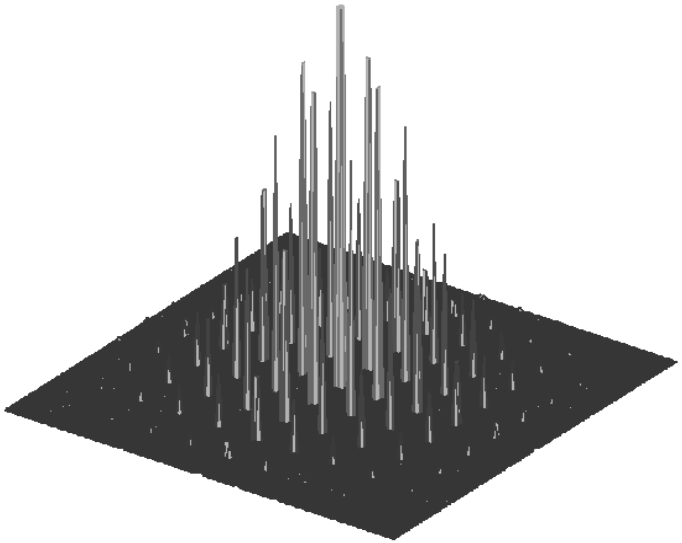
- $f \times C \bmod q \;=\; p(gs + fe) + fM \qquad$ (over R)
- $(f \times C \bmod q) \bmod p \;=\; fM \bmod p$.
- Multiply by the inverse of $f \bmod p$.

The distribution $\chi_\sigma = D_\sigma^\times$ of f and g

1. Sample f from the discrete Gaussian $D_{\mathbb{Z}^n,\sigma}$ (using [GePeVa'08]):

$$\forall x \in \mathbb{Z}^n : \quad D_{\mathbb{Z}^n,\sigma}[x] \sim \exp\left(-\pi \frac{\|x\|^2}{\sigma^2}\right).$$

2. If f is not invertible in R_q, restart.



- Discrete Gaussian with odd support.
- If $f \hookleftarrow D_\sigma^\times$, then $\|f\| \leq \sigma\sqrt{n}$, with overwhelming prob.
- Here, we need $\sigma \geq \sqrt{q}$.

- We also want f invertible mod p: handled by tweaking $D_\sigma^\times$.

Making $h = g/f$ statistically close to uniform

Main technical contribution of StSt11

If $\sigma \geq n \cdot q^{\frac{1}{2}+\varepsilon}$ with $\varepsilon > 0$, then:

$$\Delta\left[\frac{D_\sigma^\times}{D_\sigma^\times} \bmod q\ ,\ U(R_q^\times)\right] \ \leq\ q^{-\Omega(\varepsilon \cdot n)},$$

where $\Delta(D_1, D_2) = \frac{1}{2}\sum_t |D_1(t) - D_2(t)|$ is the stat. distance.

- We don't get uniformity in R_q but only in $R_q^\times$.
- Proof based on smoothing phenomenon of Gaussians modulo lattices (MR04).

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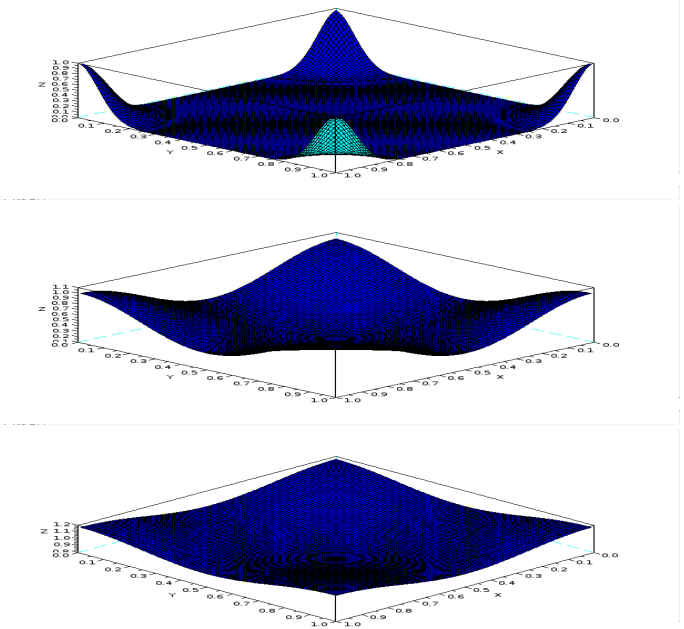
Gaussians Modulo Lattices: Smoothing Phenomenon

Take a continuous Gaussian density function $\nu_\sigma(x)$ on $\mathbb{R}^n$, width parameter σ:

$$\nu_\sigma(x) = \sigma^{-n} \cdot \rho_\sigma(x), \rho(x) \stackrel{\text{def}}{=} e^{-\pi\|x\|^2/\sigma^2},$$

and reduce it modulo a lattice $L \subseteq \mathbb{R}^n$:

$$\nu'_\sigma(x) \stackrel{\text{def}}{=} (\nu_\sigma \bmod L)(x) = \sum_{v \in L} \nu_\sigma(x+v).$$



Two regions for ν'_σ depending on σ:

- Wavy region: σ small
- Smooth region: σ large

As σ increases, ν'_σ approaches uniformity – Micciancio-Regev (2004) studied this smoothing phenomenon.

Quantifying Smoothing Phenomenon: Fourier Analysis

ν'_σ naturally extends to an L-periodic function on $\mathbb{R}^n$:

$$\nu'_\sigma(x) \stackrel{\text{def}}{=} \sum_{v \in L} \nu_\sigma(x+v) \to \nu'_\sigma(x+t) = \nu'_\sigma(x) \ \forall \ t \in L.$$

L-Periodic functions have a Fourier series decomposition:

$$\nu'_\sigma(x) = \det \hat{L} \cdot \sum_{w \in \hat{L}} c_{\sigma,w} e^{2\pi i \langle x,w \rangle},$$

- $\hat{L}$ is the dual lattice: $\{w \in \mathbb{R}^n : \forall v \in L, \langle w, v \rangle \in \mathbb{Z}\}$.
- w'th Fourier component $\psi_w(x) = e^{2\pi i \langle x,w \rangle}$
  - $w = 0$ – uniform component, $w \in \hat{L} \setminus \{0\}$ – non-uniform wavy component, wave period $= 1/\|w\|$ in direction of w
- w'th Fourier coefficient $c_{\sigma,w}$ is Fourier transform of ν_σ evaluated at w: $c_{\sigma,w} = \int_{\mathbb{R}^n} \nu_\sigma(x) e^{2\pi i \langle x,w \rangle} dx = \rho_{1/\sigma}(w)$.

Quantifying Smoothing Phenomenon: Fourier Analysis

$$\nu'_\sigma(x) = \det \hat{L} \cdot \sum_{w \in \hat{L}} \rho_{1/\sigma}(w) e^{2\pi i \langle x, w \rangle},$$

Stat. dist. $\Delta \stackrel{\text{def}}{=} \int_{P(L)} |\nu'_\sigma(x) - \det L^{-1}| dx$ of ν'_σ to uniform:

$$\Delta \leq S_\sigma(L) \stackrel{\text{def}}{=} \sum_{w \in \hat{L} \setminus 0} \rho_{1/\sigma}(w).$$

ε-Smoothing Parameter $\eta_\varepsilon(L)$ of lattice L

$\eta_\varepsilon(L) \stackrel{\text{def}}{=}$ Smallest σ such that $S_\sigma(L) \leq \varepsilon$.

Smooth region: $1/\sigma < \lambda_1(\hat{L})$ – terms in $S_\sigma(L)$ in 'tail' of $\rho_{1/\sigma}$.

Theorem [MR04]

Fix $\varepsilon > 0$. We have:

$$\eta_\varepsilon(L) \leq \sqrt{n \ln(2n(1 + 1/\varepsilon))} \cdot \lambda(\hat{L})^{-1}.$$

[GPV08]: Extends to discrete Gaussians: $\nu_\sigma \to D_{\mathbb{Z}^n,\sigma}$ for $L \subseteq \mathbb{Z}^n$.

Proving uniformity of $h = g/f$ (1)

Outline of proof

- Goal: $\Delta = \frac{1}{2}\sum_{h \in R_q^*} |\Pr_{f,g}[g/f = h] - |R_q^*|^{-1}| \leq \varepsilon$.
- Sufficient term-wise condition:
  $|\Pr_{f,g}[g/f = h] - |R_q^*|^{-1}| < |R_q^*|^{-1} \cdot \varepsilon$.
- Since $g/f = h$ equivalent to $fh - g = 0$, suffices to show

$$|\Pr_{f,g}[fh - g = 0] - |R_q^*|^{-1}| < |R_q^*|^{-1} \cdot \varepsilon.$$

- **Observation:** $\Pr_{f,g}[fh - g = 0]$ is prob. that (f, g) falls in NTRU lattice

$$L_h \stackrel{\text{def}}{=} \{(f, g) \in R^2 : fh - g = 0 \bmod q\}.$$

- $\rightarrow$ suffices to show distrib. $\chi_\sigma = (D^*_{\mathbb{Z}^n,\sigma})^2$ of (f, g) reduced modulo lattice L_h is close to uniform on $\mathbb{Z}^{2n}/L_h$.
- $\rightarrow$ Can almost directly apply smoothing!

Proving uniformity of $h = g/f$ (2)

Q: How to deal with non-lattice support of $\chi_\sigma = (D^*_{\mathbb{Z}^n,\sigma})^2$?

A: Decompose it in terms of lattices:

- $\mathbb{Z}^{2n} \cap (R_q^*)^2 = \mathbb{Z}^{2n} \setminus \bigcup_{S \subseteq \{1,\ldots,n\}} I_S$
  - I_S denotes the ideal of R_q generated by $\prod_{i \in S} \phi_i(x)$
  - $\phi_1, \ldots, \phi_n$ denote the irreducible factors of $\phi = x^n + 1 \bmod q$
- $L_h^* \cap (R_q^*)^2 = L_h \setminus \bigcup_{S \subseteq \{1,\ldots,n\}} L_h(I_S)$
  - $L_h(I_S) \stackrel{\text{def}}{=} L_h \cap (I_S \times I_S)$.

Apply inclusion-exclusion to reduce termwise to lattice smoothing:

$$\Pr_{f,g}[(f,g) \in L_h] = \frac{\sum_{S \subseteq \{1,\ldots,n\}} (-1)^{|S|} \cdot D_{\mathbb{Z}^{2n},\sigma}(L_h(I_S))}{\left(\sum_{S \subseteq \{1,\ldots,n\}} (-1)^{|S|} \cdot D_{\mathbb{Z}^n,\sigma}(I_S)\right)^2}.$$

Proving uniformity of $h = g/f$ (3)

Apply smoothing-parameter method – need lower bound on minimum of dual $\lambda(\widehat{L_h(I_S)})$.
Dual of $L_h(I_S)$ has a simple algebraic description!
Counting argument based on algebraic structure of $R_q \rightarrow$ probabilistic bound on minimum.

Smoothing paramter of gen. NTRU lattices $L_h(I_S)$ [StSt11]

Fix $\varepsilon, \varepsilon' > 0$. Let $n \geq 8$ be a power of 2 such that $\phi = x^n + 1$ splits into n linear factors modulo a prime $q \geq 5$ and let $S \subseteq [n]$ with $|S| \leq \varepsilon' n$. Then for all except a fraction $\leq 2^{8n} q^{-2\varepsilon n}$ of $h \in (R_q^*)$, we have

$$\eta_\varepsilon(L_h(I_S)) \leq \sqrt{n \ln(4n(1+1/\varepsilon))/\pi} \cdot q^{\frac{1}{2}+\varepsilon'/2+\varepsilon}$$

Hardness of DNCC: The R-LWE Problem

- The error distribution ν_β:
  - n-dimensional Gaussian of standard deviation $\beta \ll q$,
  - rounded to $\mathbb{Z}^n$,
  - looked at as an element of R^+.

  $\Rightarrow$ Small element of R^+.

- The R-LWE distribution D_β:
  - Sample $a \hookleftarrow U(R_q^+)$, $s \hookleftarrow \nu_\beta$, $e \hookleftarrow \nu_\beta$,
  - Return $(a, as + e) \in R_q^+ \times R_q^+$.

R-LWE (simplified)

Distinguish between D_β and $U(R_q^+ \times R_q^+)$.

$\rightarrow$ If $\chi_\beta = \chi_\rho = \nu_\beta$, DNCC coincides with R-LWE!

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Computational hardness of R-LWE

Hardness of R-LWE [LyPeRe'10]

$\mathcal{P}oly(n)$-Ideal-SVP reduces to R-LWE in quantum polynomial-time.

- Security under R-LWE implies security under Ideal-SVP.
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Homomorphic Encryption

Homomorphic encryption with respect to function f – Given:

- Ciphertexts for messages $m_1, \ldots, m_t$ (encrypted under pk_A):
  - $c_1 = \text{Enc}_{pk_A}(m_1), \ldots, c_t = \text{Enc}_{pk_A}(m_t)$
- a function $f(m_1, \ldots, m_t)$,
- and Alice's public key pk_A,

Bob can compute: $c = \text{Enc}_{pk_A}(m)$ for $m = f(m_1, \ldots, m_t)$.

Primary Application: Private 'cloud computing':

- **Setup:** Alice generates (sk_A, pk_A), keeps sk_A private, stores $c_i = \text{Enc}_{pk_A}(m_i)$ on cloud server (Bob).
- **Query:** Alice issues to Bob search query f: $f(m_1, \ldots, m_t) = \{m_i : \text{Title}(m_i) = \text{'Bank Statement'}\}$.
- **Response:** Bob uses f and $c_1, \ldots, c_t$ to compute $c = \text{Enc}_{pk_A}(f(m_1, \ldots, m_t))$, and returns c to Alice.
- **Decryption:** Alice uses sk_A to decrypts c and obtain $m = f(m_1, \ldots, m_t)$.

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Homomorphic Encryption: Realization

History:

- Concept proposed in the 1970s [RAD78]
- 1970s – 2009: Very limited classes of functions f allowed.
- 2009: First plausible scheme supporting arbitrary functions f (Fully Homomorphic Encryption - FHE)
  - Based on hardness of approx-Ideal-SVP variants
  - Inefficient – Huge parameters
- 2010-2013: Significant improvements: Most efficient schemes based on R-LWE (BV11,BGV12,B12,LTV12,GHS13)

One of the more efficient schemes (LTV12): NTRU variant

- Ciphertext same as standard NTRU encryption
- Hardness Assumptions:
  - DNKC problem in the computational region: $\sigma < q^{1/2}$.
  - 'circular' variant of R-LWE problem = DNCC problem.

Homomorphic Encryption: NTRU Variant (LTV12)

Observation: NTRU has natural 'semi homomorphic' properties

Given ciphertexts $c_i = h \cdot s_i + pe_i + m_i \in R_q$, $i \in \{1, 2\}$:

- **Addition** (ciphertexts over R_q, messages over R_p):
  - $c = c_1 + c_2 = h \cdot (s_1 + s_2) + p(e_1 + e_2) + (m_1 + m_2)$.
  - c decrypts with $sk = f$ to message $m = m_1 + m_2 \in R_p$
  - 'Semi-homomorphic' catch: Need $\|p(gs + fe) + m\|_\infty < q/2$), with $s = s_1 + s_2$, $e = e_1 + e_2$.
- **Multiplication** (ciphertexts over R_q, messages over R_p):
  - $c = c_1 \cdot c_2 =$ $h^2 s_1 s_2 + h(s_1 e_2' + s_2 e_1') + p(e_1 e_2 + e_1 m_2 + e_2 m_1) + m_1 m_2$.
  - c decrypts with $sk = f^2$ to message $m = m_1 \cdot m_2 \in R_p$.
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Multiplication is the bottleneck: Even 1 mult requires $(\sigma \cdot \beta \mathcal{P}oly(n))^2 \leq q/2$.

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Homomorphic Encryption: NTRU Variant (LTV12)

For depth d homomorphic multiplications, 'doubly exponential norm blowup' difficulty:

- Need to decrypt with $sk = f^{2^d}$ – need $(\sigma\beta\mathcal{P}oly(n))^{2^d} < q/2$.
- For hardness of DNKC, allows d only up to $O(\log n)$!

Improvement to support larger d (based on BV11): Relinearization

- Apply relinearization procedure to $c = c_1 \cdot c_2$ after each homomorphic multiplication
- Produce $\hat{c}$ encrypting same message $m_1 \cdot m_2 \in R_p$ as c, but decryptable with the f, not f^2 $\rightarrow$ avoid the exponential degree blowup for sk.

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Homomorphic Encryption: NTRU Variant (LTV12)

Idea of Relinearization:

- Modify key generation – Alice publishes $\approx \log q$ additional ring elements ζ_τ : 'pseudo-encryptions' of f^2:

$$\zeta_\tau = h \cdot s_\tau + p e_\tau + 2^\tau f^2 \in R_q \text{ for } \tau = 0, \ldots, \lfloor \log q \rfloor,$$

- Relinearization procedure – split c into its binary reprsentation $\sum_\tau c_\tau 2^\tau$ and compute

$$\hat{c} = \sum_\tau c_\tau \cdot \zeta_\tau = h \cdot (\sum_\tau c_\tau s_\tau) + p \cdot (\sum_\tau c_\tau e_\tau) + f^2 \cdot (\sum_\tau c_\tau 2^\tau).$$

Second Improvement (based on BV11): Modulus Reduction.

- Scales down the ciphertext $\hat{c}$ from R_q to $R_{q'}$ with $q' < q$
- Noise in $\hat{c}'$ also scaled down by the ratio q'/q.

Overall: Can support f of mult. depth $d = O(n^\varepsilon)$.

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Relinearization $\rightarrow$ security now relies on new variant of DNKC:

NTRU Decision Circular Key Cracking Problem $\mathtt{DNCKC}_{n,q,\phi,\chi_\sigma,\chi_\beta,\ell}$

Given (n, q, ϕ) and $(h, \{\zeta_\tau\}_\tau)$, distinguish

- NTRU circular key distribution $D_0 = \{(h = g/f \in R_q, \zeta_\tau = h \cdot s_\tau + pe_\tau + 2^\tau f^2 \in R_q : f, g \hookleftarrow \chi_\sigma, s_\tau, e_\tau \hookleftarrow \chi_\beta\}$.
- Uniform key distribution $D_1 = U(R_q^*) \times U(R_q^\ell)$.

Q: Hardness??

Cryptographic Multilinear Maps

Example Motivation: Non-interactive Key exchange
Classical Diffie-Hellman Non-Interactive 2-party Key Exchange (1976)

- Publish a cyclic group G (generator g, order q) where Discrete Log (DL) problem is hard.
- Alice chooses random $x_1 \in \mathbb{Z}_q$, publishes $y_1 = g^{x_1}$.
- Bob chooses random $x_2 \in \mathbb{Z}_q$, publishes $y_2 = g^{x_2}$.
- Both Alice and Bob compute agreed secret key $K = g^{x_1 x_2} = y_1^{x_2} = y_2^{x_1}$.

Cryptographic Multilinear Maps

Q: How to generalize Diffie-Hellman to $N > 2$ parties?

A[J00,BS02]: Use a group where DL is hard and there is an efficient $(N-1)$-linear map $e : G_1 \times \cdots \times G_{N-1} \to G_T$:

$$e(g^{x_1}, g^{x_2}, \ldots, g^{x_{N-1}}) = e(g,g)^{x_1 \cdots x_{N-1}} \forall x_1, \ldots, x_{N-1} \in \mathbb{Z}_q.$$

N-party Non-Interactive Key Exchange

- Publish a cyclic group G (generator g, order q) where Discrete Log (DL) problem is hard, with an efficient $(N-1)$-linear map e.
- For $i \in \{1, \ldots, N\}$, party P_i chooses random $x_i \in \mathbb{Z}_q$, publishes $y_i = g^{x_i}$.
- All parties can compute agreed secret key $K = e(g, \ldots, g)^{x_1 \cdots x_N} = e(y_2, y_3, \ldots, y_N)^{x_1}$.

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k-linear Maps: History

- 2000: Bilinear ($k = 2$) via Weil pairings on algebraic curves, applications:
  - 2000: 3-party non-interactive key agreement [J00]
  - 2000-2001: Identity-Based Encryption (IBE) [SK00,BF01]
  - 2001: Short signatures [BS01]
  - 2000-2013: **lots** of others
- 2002: Applications for k-linear maps [BS02]
  - $(k + 1)$-party non-interactive key agreement
  - Efficient Broadcast Encryption
  - and others...
- 2012: First plausible realization for $k > 2$, via ideal lattices [GGH12], applications:
  - 2012-2013: Functional Encryption for arbitrary functions
  - 2013: Program obfuscation notions for arbitrary functions
  - and others...

k-linear Maps: GGH Realization

[GGH12] realization: not quite a k-linear map, but essentially the same

Technically, a k-graded encoding scheme:

- Replace groups $\mathbb{Z}_q, G$ by
  - Rings R_p, R_q.
- Replace $x \to g^x$ by
  - $x \to \mathrm{Enc}_1(x; \rho)$ – randomized 'level 1 encoding' of x.
- Replace $e(g^{x_1}, \ldots, g^{x_k}) = e(g, \ldots, g)^{x_1 \cdots x_k}$ by
  - Homomorphic up to 'level k':
    $\mathrm{Enc}_1(x_1; \rho_1) \cdots \mathrm{Enc}_1(x_k; \rho_k) = \mathrm{Enc}_k(x_1 \cdots x_k; \rho)$ for some ρ.
  - Randomness-independent extraction at level k –
    $\mathrm{Ext} : R_q \to \{0,1\}^\ell$: $\mathrm{Ext}(\mathrm{Enc}_k(x; \rho)) = r(x) \in \{0,1\}^n$ is independent of randomness ρ, and uniformly random for $x \hookleftarrow U(R_p)$.

GGH k-graded encoded scheme (LSS13 simplified - GGHLite)

Public Params: Choose a secret 'small' $p \in R$. Publish (h_1, h_2, e_k), with:

- $h_1 = pg_1/f, h_2 = pg_2/f \in R_q$, with 'small' $f, g_1, g_2 \hookleftarrow \chi_\sigma$ with $\sigma < q^{1/k}$.
- $e_k = uf^k/p \in R_q$, for u of norm $\|u\| = \mathcal{P}oly(n) \cdot q^{1/2}$.

Level 1 Encoding of $m \in R_p$:

$c = \mathrm{Enc}_1(m) = h_1 s_1 + h_2 s_2 + m \in R_q$, with s_i 'small'.

- Note: $\mathrm{Enc}_1(m) = pg'/f + m$ for a small g'
- By homomorphic property, $\mathrm{Enc}_1(m_1) \cdots \mathrm{Enc}_1(m_k) = pg'/f^k + m = \mathrm{Enc}_1(m)$, with $m = m_1 \cdots m_k \bmod p$.

Level k Representative Extraction: $\mathrm{Ext}(e_k, c) = MSB_\ell(e_k \cdot c)$.

- Note: $c = \mathrm{Enc}_1(m; \rho) = pg'/f^k + m \to \mathrm{Ext}(e_k, c) = MSB_\ell(ug' + uf^k/p \cdot m) = MSB_\ell(uf^k/p \cdot m)$ since ug' small.

GGH k-graded encoded scheme (LSS13 simplified - GGHLite)

Scheme security depends on hardness of a new variant of DNKC problem:

k-graded NTRU Discrete-Log Problem $\mathtt{DNDL}_{n,q,\phi,\chi_\sigma,\chi_\beta,\ell}$

Given

- $(n, q, h_1 = pg_1/f, h_2 = pg_2/f, e_k = uf^k/p)$
- $c = h_1 s_1 + h_2 s_2 + m \in R_q$,

find m' with $\|m'\|$ 'small' (less than q) such that $m' = m \bmod p$.

Note:

- Without knowing e_k, similar to standard DNCC problem.
- With e_k, may be easier (e.g. checking c is encoding of m is easy!).

Conclusions

Intersting recent developments in both security analysis and applications of NTRU.

Important open problems:

- Hardness of NTRU key cracking problem in the computational region $\sigma < q^{1/2}$ (applications: efficient parameters, FHE, multilinear maps)
  - Starting point: statistical properties of $h = g/f \in R_q$ in this region?
- Hardness of circular NTRU ciphertext cracking problem (application: FHE)
- Hardness of k-graded NTRU discrete-log problem (application: multilinear maps)