Ramanujan Graphs, Ramanujan Complexes and Zeta Functions

Emerging Applications of Finite Fields

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Ramanujan's conjectures on the τ function

The Ramanujan τ-function

$$\Delta(z) = \sum_{n\geq 1} \tau(n) q^n = q \prod_{n\geq 1} (1-q^n)^{24}, \qquad \text{where} \;\; q = e^{2\pi i z},$$

is a weight 12 cusp form for $SL_2(\mathbb{Z})$.

In 1916 Ramanujan conjectured the following properties on $\tau(n)$:

- $\tau(mn) = \tau(m)\tau(n)$ for $(m,n) = 1$;
- for each prime p, $\tau(p^{n+1}) - \tau(p)\tau(p^n) + p^{11}\tau(p^{n-1}) = 0$ for all $n \geq 1$;
- $|\tau(p)| \leq 2p^{11/2}$ for each prime p.

The first two statements can be rephrased as the associated L-series having an Euler product:

$$L(\Delta, s) = \sum_{n \geq 1} \tau(n) n^{-s} = \prod_{p \text{ prime}} \frac{1}{1 - \tau(p)p^{-s} + p^{11-2s}}, \quad \Re(s) > 11,$$

$\Leftrightarrow$ Δ is a common eigenfunction of T_p with eigenvalue $\tau(p)$. Proved by Mordell in 1917 for Δ, by Hecke in 1937 for all modular forms.

The third statement $\Leftrightarrow$ in the factorization

$$1 - \tau(p)p^{-s} + p^{11-2s} = (1 - \alpha(p)p^{-s})(1 - \beta(p)p^{-s})$$

we have

$$|\alpha(p)| = |\beta(p)| = p^{11/2}.$$

This is called Ramanujan conj., proved by Deligne for Δ and cusp forms of wt ≥ 3, Eichler-Shimura (wt 2), Deligne-Serre (wt 1).

Generalized Ramanujan conjecture

The L-function attached to an auto. cuspidal rep'n π of GL_n over a global field K has the form

$$L(\pi, s) \approx \prod_{\pi \text{ unram. at } v} \frac{1}{1 + a_1(v)Nv^{-s} + \cdots + a_n(v)Nv^{-ns}}.$$

They are equal up to finitely many places where π is ramified.

Suppose that the central character of π is unitary.

π satisfies the Ramanujan conjecture
"$\Leftrightarrow$" at each unram. v all roots of

$$1 + a_1(v)u + \cdots + a_n(v)u^n$$

have the same absolute value 1.

For K a function field (= finite extension of $\mathbb{F}_q(t)$):

- Ramanujan conj. for GL_n over K is proved by Drinfeld for $n = 2$ and Lafforgue for $n \geq 3$.
- Laumon-Rapoport-Stuhler (1993) proved R. conj. for auto. rep'ns of (the multiplicative group of) a division algebra H over K which are Steinberg at a place where H is unram.

For K is a number field, there is also a statement for the Ramanujan condition at the archimedean places; when $n = 2$, this is the Selberg eigenvalue conj.

The Ramanujan conjecture over number fields is proved for holomorphic cusp. repn's for GL_2 over $K = \mathbb{Q}$ and K totally real (Brylinski-Labesse-Blasius).

Luo-Rudnick-Sarnak and Blomer-Brumley gave subconvexity bounds for $n = 2, 3, 4$ and K any number field.

Ramanujan graphs

- X: d-regular connected undirected graph on n vertices
- Its eigenvalues satisfy

$$d = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n \geq -d.$$

- Trivial eigenvalues are $\pm d$, the rest are nontrivial eigenvalues.
- X is a *Ramanujan graph*

  $\Leftrightarrow$ its nontrivial eigenvalues λ satisfy

$$|\lambda| \leq 2\sqrt{d-1}$$

  $\Leftrightarrow$ for each nontrivial eigenvalue λ, all roots of $1-\lambda u+(d-1)u^2$ have the same absolute value $(d-1)^{-1/2}$.

Spectral theory of regular graphs

- $[-2\sqrt{d-1},\ 2\sqrt{d-1}]$ is the spectrum of the d-regular tree, the universal cover of X.
- $\{X_j\}$: a family of undirected d-regular graphs with $|X_j| \to \infty$.

  Alon-Boppana :

$$\liminf_{j\to\infty} \max_{\lambda \text{ of } X_j} \lambda \geq 2\sqrt{d-1}.$$

  Li, Serre : if the length of the shortest odd cycle in X_j tends to ∞ as $j \to \infty$, or if X_j contains few odd cycles, then

$$\limsup_{j\to\infty} \min_{\lambda \text{ of } X_j} \lambda \leq -2\sqrt{d-1}.$$

- A Ramanujan graph is spectrally optimal; excellent communication network.

Examples of Ramanujan graphs

Lipton-Tarjan separator theorem : For a fixed d, there are only finitely many planar Ramanujan d-regular graphs.

$Cay(PSL_2(\mathbb{Z}/5\mathbb{Z}), S) = C_{60}$

Other examples: C_{80} and C_{84}.

Ihara zeta function of a graph

The Selberg zeta function, defined in 1956, counts geodesic cycles in a compact Riemann surface obtained as

$$\Gamma\backslash\mathfrak{H} = \Gamma\backslash SL_2(\mathbb{R})/SO_2(\mathbb{R}),$$

where Γ is a torsion-free discrete cocompact subgroup of $SL_2(\mathbb{R})$.

Extending Selberg zeta function to a nonarchimedean local field F with q elements in its residue field, Ihara in 1966 considered the zeta function for

$$\Gamma\backslash \mathrm{PGL}_2(F)/\mathrm{PGL}_2(\mathcal{O}_F),$$

where Γ is a torsion-free discrete cocompact subgroup of $\mathrm{PGL}_2(F)$.

Serre pointed out that Ihara's definition of zeta function works for finite graphs.

- X : connected undirected finite graph
- A cycle (i.e. closed walk) has a starting point and an orientation.
- Interested in geodesic tailless cycles.

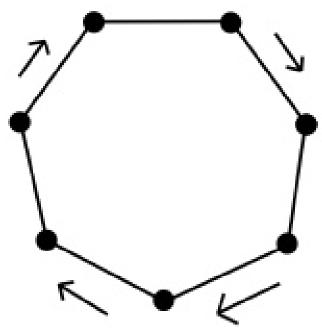

Figure 1: without tail

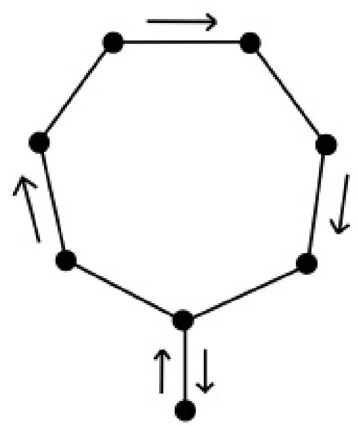

Figure 2: with tail

- Two cycles are *equivalent* if one is obtained from the other by shifting the starting point.

- A cycle is *primitive* if it is not obtained by repeating a cycle (of shorter length) more than once.
- $[C]$: the equivalence class of C.

The Ihara zeta function of X counts the number $N_n(X)$ of geodesic tailless cycles of length n:

$$Z(X;u) = \exp\left(\sum_{n \geq 1} \frac{N_n(X)}{n} u^n \right) = \prod_{[C]} \frac{1}{1 - u^{l(C)}},$$

where $[C]$ runs through all equiv. classes of primitive geodesic and tailless cycles C, and $l(C)$ is the length of C.

Properties of the zeta function of a regular graph

Ihara (1966): *Let X be a finite d-regular graph on n vertices. Then its zeta function $Z(X,u)$ is a rational function of the form*

$$Z(X;u) = \frac{(1-u^2)^{\chi(X)}}{\det(I - Au + (d-1)u^2 I)} \ ,$$

where $\chi(X) = n - nd/2 = -n(d-2)/2$ is the Euler characteristic of X and A is the adjacency matrix of X.

Note that

$$\det(I - Au + (d-1)u^2) = \prod_{1 \le i \le n} (1 - \lambda_i u + (d-1)u^2).$$

RH and Ramanujan graphs

- $Z(X,u)$ satisfies RH if the nontrivial poles of $Z(X,u)$ (arising from the nontrivial λ) all have the same absolute value $(d-1)^{-1/2}$

  $\Leftrightarrow$ all nontrivial eigenvalues λ satisfy the bound
$$|\lambda| \leq 2\sqrt{d-1}.$$

- $Z(X,u)$ satisfies RH if and only if X is a Ramanujan graph.

Zeta functions of varieties over finite fields

V: smooth irred. proj. variety of dim. d defined over $\mathbb{F}_q$
The zeta function of V counts $N_n(V) = \#V(\mathbb{F}_{q^n})$:

$$Z(V,u) \; = \; \exp(\sum_{n \geq 1} \frac{N_n(V)}{n} u^n) = \prod_{v \text{ closed pts}} \frac{1}{(1 - u^{\deg v})}.$$

Grothendieck proved

$$Z(V,u) = \frac{P_1(u) P_3(u) \cdots P_{2d-1}(u)}{P_0(u) P_2(u) \cdots P_{2d}(u)},$$

where $P_i(u) \in \mathbb{Z}[u]$.
RH : the roots of $P_i(u)$ have absolute value $q^{-i/2}$.
Proved by Hasse and Weil for curves and Deligne in general.

Explicit constructions of Ramanujan graphs

Construction by Lubotzky-Phillips-Sarnak, and independently by Margulis.

Fix an odd prime p, valency $p+1$.

The $(p+1)$-regular tree $= PGL_2(\mathbb{Q}_p)/PGL_2(\mathbb{Z}_p) = Cay(\Lambda, S_p)$.

Let H be the Hamiltonian quaternion algebra over $\mathbb{Q}$, ramified only at 2 and ∞. Let $D = H^\times$/center. The cosets can be represented by a group Λ from $D(\mathbb{Z})$ so that the tree can be expressed as the Cayley graph $Cay(\Lambda, S_p)$ with $S_p = \{x \in \Lambda : N(x) = p\}$.

Such S_p is symmetric of size $|S_p| = p + 1$.

By taking quotients mod odd primes $q \neq p$, one gets a family of finite $(p+1)$-regular graphs

$$Cay(\Lambda \mod q,\ S_p \mod q) = Cay(\Lambda(q)\backslash\Lambda,\ S_p \mod q).$$

Lubotzky-Phillips-Sarnak: *For* $p \geq 5$, $q > p^8$, *the graphs*

- $Cay(PGL_2(\mathbb{F}_q),\ S_p \mod q)$ *if* p *is not a square mod* q, *and*
- $Cay(PSL_2(\mathbb{F}_q),\ S_p \mod q)$ *if* p *is a square mod* q

are $(p+1)$*-regular Ramanujan graphs.*

Ramanujan: Regard the vertices of the graph as

$$\begin{aligned}\Lambda(q)\backslash\Lambda &= \Lambda(q)\backslash PGL_2(\mathbb{Q}_p)/PGL_2(\mathbb{Z}_p)\\ &= D(\mathbb{Q})\backslash D(\mathbb{A}_{\mathbb{Q}})/D(\mathbb{R})D(\mathbb{Z}_p)K_q,\end{aligned}$$

where K_q is a congruence subgroup of the max'l compact subgroup outside ∞ and p.

Adjacency operator = Hecke operator at p

The nonconstant functions on graphs are automorphic forms on D, which by JL correspond to classical wt 2 cusp forms.

Eigenvalue bound follows from the Ramanujan conjecture established by Eichler-Shimura.

Can replace H by other definite quaternion algebras over $\mathbb{Q}$; or do this over function fields to get $(q+1)$-regular Ramanujan graphs (for q a prime power) using the Ramanujan conjecture established by Drinfeld.

Ramanujan graphs for bi-regular bipartite graphs

- Li-Solé: The covering radius for the spectrum of the (c, d)-biregular bipartite tree is $\sqrt{c-1} + \sqrt{d-1}$.
- A finite (c, d)-biregular bigraph has trivial eigenvalues $\pm\sqrt{cd}$.
- Feng-Li : the analogue of Alon-Boppana theorem holds for bi-regular bigraphs:

  Let $\{X_m\}$ be a family of finite connected (c, d)-biregular bigraphs with $|X_m| \to \infty$ as $m \to \infty$. Then

  $$\liminf \lambda_2(X_m) \geq \sqrt{c-1} + \sqrt{d-1}.$$

- A bi-regular bigraph is called *Ramanujan* if its nontrivial eigenvalues in absolute value are bounded by the covering radius of its universal cover. This is also the definition of an irregular Ramanujan graph in general.

Infinite family of Ramanujan biregular bigraphs

- The explicit construction by Margulis, Lubotzky-Phillips-Sarnak, Morgenstern using number theory gives an infinite family of Ramanujan graphs for $d = q + 1$, where q is a prime power.
- Question: Is there an infinite family of Ramanujan d-regular graphs for any $d \geq 3$?
- Friedman: A random large d-regular graph X is very close to being Ramanujan, i.e., given any $\varepsilon > 0$, the probability of $\lambda_2(X) < 2\sqrt{d-1} + \varepsilon$ goes to 1 as $|X| \to \infty$.
- Adam Marcus, Daniel Spielman and Nikhil Srivastava (2013): There exists an infinite family of Ramanujan (c, d)-biregular bigraphs. When $c = d$, this answers the question in the affirmative.

The existence proof by Marcus-Spielman-Srivastava

- Strategy: To show that any connected Ramanujan bigraph has a 2-fold unramified cover which is also Ramanujan.

  Hence starting with any bipartite Ramanujan graph, there is an infinite tower of Ramanujan graphs.

  Since a complete (c, d)-regular bigraph is Ramanujan, we obain an infinite tower of (c, d)-regular Ramanujan bigraphs.

- To get a 2-fold unramified cover Y of X, take two copies of X, line up the vertices, and reconnect some edges.

- The adjacency matrix of Y is given by $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, where A is the adjacency matrix of X, and $B = B(Y)$ is the matrix with uv entry equal to ± 1 if $\overline{uv}$ is an edge, and 0 otherwise.

- Conjecture (Bilu and Linial): *For any finite connected graph X, there is a 2-fold unramified cover Y such that all eigenvalues of B are bounded by the covering radius r of the universal cover of X, i.e., lie in the interval $[-r, r]$.*

- Marcus-Spielman-Srivastava proved the existence of Y such that eigenvalues of B lie in $(-\infty, r]$.

  If, in addition, X is bipartite, then all eigenvalues of B lie in $[-r, r]$. Hence Y is Ramanujan if X is.

Open Question. Find an algorithm to pick such Y, i.e., make the construction explicit.

The Bruhat-Tits building of PGL_n

- F: local field with q elements in its residue field, ring of integers $\mathcal{O}_F$, eg. $F = \mathbb{Q}_p$ or $F = \mathbb{F}_q((t))$
- $\mathcal{B}_{n,F} = PGL_n(F)/PGL_n(\mathcal{O}_F)$: Bruhat-Tits building attached to $PGL_n(F)$.

  It is a contractible $(n-1)$-dim'l simplicial complex.
- Types of vertices parametrized by $\mathbb{Z}/n\mathbb{Z}$. Adjacent vertices have different types.
- According to type differences, the neighbors of a vertex are partitioned into $n-1$ sets.
- For $1 \le i \le n-1$, the type-difference-i neighbors are described by $A_{n,i}$. They generate the Hecke algebra of $PGL_n(F)$.

Spectral theory for finite quotients of $\mathcal{B}_{n,F}$

- $A_{n,1}, \ldots, A_{n,n-1}$ can be simultaneously diagonalized, spectra $\Omega_{n,i}$ known.
- Li: Analog of Alon-Boppana holds for finite quotients of $\mathcal{B}_{n,F}$.
- A finite quotient X of $\mathcal{B}_{n,F}$ is called a *Ramanujan complex* $\Leftrightarrow$ all nontrivial eigenvalues of $A_{n,i}$ on X lie in $\Omega_{n,i}$ $\forall$ i $\Leftrightarrow$ for each $(n-1)$-tuple of simultaneous nontrivial eigenvalues $(\lambda_1, \ldots, \lambda_{n-1})$, all roots of

$$\sum_{i=0}^{n} (-1)^i q^{i(i-1)/2} \lambda_i u^i$$
$$= 1 - \lambda_1 u + q\lambda_2 u^2 - \cdots + (-1)^n q^{n(n-1)/2} u^n$$

have the same absolute value $q^{-(n-1)/2}$.

Explicit constructions of Ramanujan complexes

Li: *For $n \geq 3$ and $F \cong \mathbb{F}_q((t))$, there exists an explicitly constructed infinite family of Ramanujan complexes arising as finite quotients of $\mathcal{B}_{n,F}$.*

The construction is similar to LPS, but over a function field K with $F =$ the completion of K at a place v.

In order to obtain finite complexes, one considers quotients

$$\Gamma\backslash\mathcal{B}_{n,F} = \Gamma\backslash PGL_n(F)/PGL_n(\mathcal{O}_F)$$

by suitable (torsion-free) discrete cocompact subgroups Γ of $PGL_n(F)$ arising from the multiplicative groups of division algebras over K of dimension n^2 and unram. at v. Then one uses JL correspondence to see that the eigenvalues of $A_{n,i}$ on auto. forms on division algebras are also eigenvalues of auto. forms on $GL_n(F)$, and then apply Lafforgue to get the desired Ramanujan bound.

Catch: JL correspondence is established only for prime n.

My construction used the Ramanujan conjecture established by Laumon-Rapoport-Stuhler.

Assuming JL, Lubotzky-Samuels-Vishne and Sarveniazi independently gave explicit constructions, the one by Sarveniazi is similar to LPS.

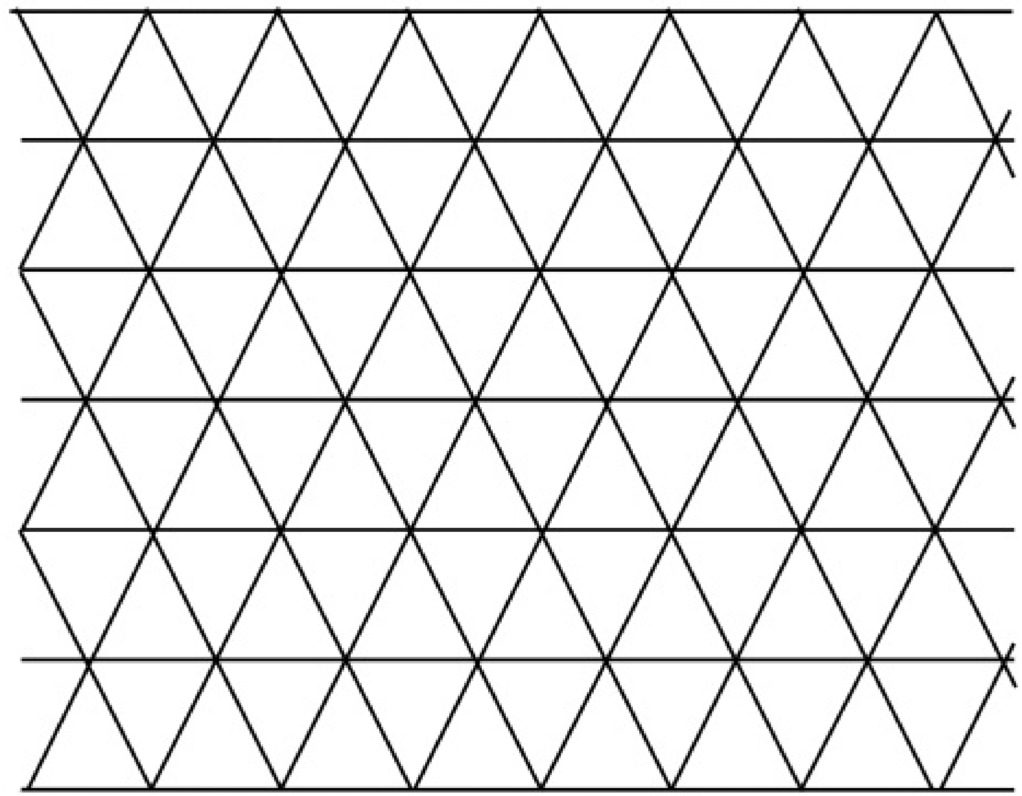

Figure 3: an apartment of $\mathcal{B}_{3,F}$

Zeta functions of finite quotients of $\mathcal{B}_{3,F}$

Let $X_\Gamma = \Gamma\backslash\mathcal{B}_{3,F}$ be a quotient of $\mathcal{B}_{3,F}$ by a discrete torsion-free cocompact subgroup Γ of $PGL_3(F)$. The zeta function of X_Γ counts the number N_n of tailless geodesic cycles of length n contained in the 1-skeleton of X_Γ, defined as

$$Z(X_\Gamma, u) = \exp(\sum_{n\geq 1} \frac{N_n(X_\Gamma)u^n}{n}) = \prod_{[C]} \frac{1}{1 - u^{l_A(C)}},$$

where $[C]$ runs through the equiv. classes of primitive tailless geodesic cycles in the 1-skeleton of X_Γ, and $l_A(C)$ is the algebraic length of the cycle C.

RH and Ramanujan complexes

Kang-Li: $Z(X_\Gamma, u)$ *is a rational function given by*

$$Z(X_\Gamma, u) = \frac{(1-u^3)^{\chi(X_\Gamma)}}{\det(I - A_{3,1}u + qA_{3,2}u^2 - q^3u^3I)\det(I + L_B u)},$$

where $\chi(X_\Gamma) = \#V - \#E + \#C$ *is the Euler characteristic of* X_Γ*, and* L_B *is the adjacency operator on directed chambers.*

Kang-Li-Wang:

X_Γ *is a Ramanujan complex*

$\Leftrightarrow$ *the nontrivial zeros of* $\det(I - A_{3,1}u + qA_{3,2}u^2 - q^3u^3I)$ *have the same absolute value* q^{-1}

$\Leftrightarrow$ *the nontrivial zeros of* $\det(I - L_B u)$ *have absolute values* 1, $q^{-1/2}$ *and* $q^{-1/4}$

$\Leftrightarrow$ $Z(X_\Gamma, u)$ satisfies RH.