The connections between the Erdős distance problem and the restriction problem for spheres in the finite field setting

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Outline of Today's Talk

1. The Erdős distance problem in Euclidean space
2. The Erdős distance problem in the finite field setting
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7. The results on the Erdős distance problem

The Erdős Distance Problem in $\mathbb{R}^d$

In 1946, Erdős addressed a question that "how many distinct distances can be determined by a finite subset of $\mathbb{R}^d$?"

Definition

Given a finite set $E \subset \mathbb{R}^d, d \geq 2$, the distance set is defined by

$$\Delta(E) = \{|x - y| : x, y \in E\},$$

Taking the set E as a piece of the integer lattice, Erdős conjectured the following.

Conjecture

For every $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that $|\Delta(E)| \geq c_\varepsilon |E|^{\frac{2}{d}-\varepsilon}$.

In dimension two, the conjecture has recently been solved by Guth and Katz but it remains open in higher dimensions $d \geq 3$.

The Erdős Distance Problem in Finite Fields

In 2004, Bourgain, Katz, and Tao initially addressed the finite field version of the Erdős distance problem. In 2007, Iosevich and Rudnev developed the problem and obtained concrete results on the problem.

- $\mathbb{F}_q$: a finite field with q elements ($Char\mathbb{F}_q > 2$).
- $\mathbb{F}_q^d$: a d-dimensional vector space over $\mathbb{F}_q$.

Definition

For $E \subset \mathbb{F}_q^d$, the distance set $\Delta(E)$ is defined by

$$\Delta(E) = \{||x - y|| \in \mathbb{F}_q : x, y \in E\},$$

where $\|\alpha\| = \alpha_1^2 + \cdots + \alpha_d^2$ for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{F}_q^d$.

Problem

Express the minimal cardinality of $\Delta(E)$ in terms of $|E|$.

New Question

The Erdős distance conjecture in Euclidean space can not be true in the finite field case. For example, suppose that $-1 \in \mathbb{F}_q$ is a square number (say that $i^2 = -1$ for some $i \in \mathbb{F}_q$). Let $E \subset \mathbb{F}_q^2$ be defined by $E = \{(t, it) \in \mathbb{F}_q^2 : t \in \mathbb{F}_q\}$.

- $|E| = q = q^{d/2}$ but $|\Delta(E)| = |\{0\}| = 1.$

Iosevich and Rudnev asks for the minimal cardinality of $E \subset \mathbb{F}_q^d$ such that $|\Delta(E)| \gg |\mathbb{F}_q| = q$.

Question

Let $E \subset \mathbb{F}_q^d$ with $|E| \sim q^{\alpha}$. Find the smallest $\alpha > 0$ such that $|\Delta(E)| \gg q$.

Conjecture and Known Results

In 2007, Iosevich and Rudnev conjectured the following.

Conjecture

Let $E \subset \mathbb{F}_q^d, d \geq 2$. If $|E| \geq Cq^{\frac{d}{2}}$ with $C > 0$ a sufficiently large constant, then $|\Delta(E)| \geq cq$ for some $0 < c \leq 1$.

In addition, Iosevich and Rudnev obtained the following result.

Theorem

Let $E \subset \mathbb{F}_q^d, d \geq 2$. If $|E| \geq Cq^{\frac{d+1}{2}}$ with $C > 0$ a sufficiently large constant, then $|\Delta(E)| \geq cq$ for some $0 < c \leq 1$.

In 2011, Hart, Iosevich, Koh, and Rudnev found examples to show that the conjecture can not be true if the dimensions $d \geq 3$ are odd. In fact, they showed that the exponent $(d+1)/2$ gives a sharp result for odd dimensional case.

New Conjecture and Main Point of the Talk

Although the conjecture of Iosevich and Rudnev is not true in odd dimensions, it has been believed that the conjecture is true for even dimensional case.

Conjecture

Let $E \subset \mathbb{F}_q^d$. If $d \geq 2$ is even and $|E| \geq Cq^{\frac{d}{2}}$, then $|\Delta(E)| \geq cq$.

In case of even dimensions $d \geq 2$, we discuss about how to improve the sharp exponent $(d+1)/2$ in odd dimensions. The key idea is to use the results on the restriction problem for spheres in finite fields. In particular, if $d = 2$, then we are able to improve the exponent $(d+1)/2 = 3/2$ to $4/3$.

Theorem

If $E \subset \mathbb{F}_q^2$ and $|E| \geq Cq^{\frac{4}{3}}$, then $|\Delta(E)| \geq cq$.

Basic Approach to Deducing the Size of Distance Set

Let $E \subset \mathbb{F}_q^d$. For each $t \in \mathbb{F}_q$, consider a counting function

$$\nu(t) = |\{(x,y) \in E \times E : \|x-y\| = t\}| = \sum_{x,y\in E} S_t(x-y).$$

$$= \sum_{x,y\in E} \sum_{m\in\mathbb{F}_q^d} \psi(m\cdot(x-y))\widehat{S}_t(m) = q^{2d} \sum_{m\in\mathbb{F}_q^d} |\widehat{E}(m)|^2 \widehat{S}_t(m),$$

where $S_t := \{x \in \mathbb{F}_q^d : x_1^2 + \cdots + x_d^2 = t\}$. Applying the Cauchy-Schwarz inequality, we see that if $|E| \geq Cq^{\frac{d}{2}}$, then

$$|E|^4 \sim (|E|^2 - \nu(0))^2 = \left(\sum_{t\neq 0} \nu(t)\right)^2 \leq |\Delta(E)| \sum_{t\in\mathbb{F}_q^*} \nu^2(t).$$

Formula for the Lower Bound of the Distance Set

We have seen that if $|E| \geq Cq^{\frac{d}{2}}$, then $|\Delta(E)| \gg \frac{|E|^4}{\sum_{t\in\mathbb{F}_q^*} \nu^2(t)}$.
Applying the discrete Fourier analysis, one can derive the upper bound of $\sum_{t\in\mathbb{F}_q^*} \nu^2(t)$. In conclusion, we have the following formula.

Lemma

Let $E \subset \mathbb{F}_q^d, d \geq 2$. Suppose that $|E| \geq Cq^{\frac{d}{2}}$ with C sufficiently large. Then

$$|\Delta(E)| \gg \min\left\{q, \frac{q}{\mathbb{M}_E(q)}\right\},$$

where

$$\mathbb{M}_E(q) := \frac{q^{3d+1}}{|E|^4} \sum_{t\in\mathbb{F}_q^*} \left(\sum_{m\in S_t} |\widehat{E}(m)|^2 \right)^2 \leq \frac{q^{2d+1}}{|E|^3} \max_{t\in\mathbb{F}_q^*} \sum_{m\in S_t} |\widehat{E}(m)|^2.$$

Restriction Problem for Spheres S_t in Finite Fields

In 2004, Mockenhaupt and Tao first posed the restriction problem for various algebraic varieties in finite fields. In 2008, Iosevich and Koh addressed nontrivial results on the spherical restriction problems in finite fields.

- $(\mathbb{F}_q^d, dm)$: d-dimensional vector space over $\mathbb{F}_q$ with the counting measure dm.
- $(\mathbb{F}_q^d, dx)$: the dual space of $(\mathbb{F}_q^d, dm)$ where we endow the dual space with the normalized counting measure dx.
- For the sphere $S_t \subset (\mathbb{F}_q^d, dx)$, we endow S_t with the normalized surface measure $d\sigma$ which can be defined by the relation

$$\int g(x)\, d\sigma(x) = \frac{1}{|S_t|} \sum_{x \in S_t} g(x), \quad g : (\mathbb{F}_q^d, dx) \to \mathbb{C}.$$

Restriction Problem for Spheres S_t in Finite Fields

The restriction problem for the sphere S_t is as follows.

Problem (Restriction Problem)

Determine $1 \leq p, r \leq \infty$ *such that* $\|\widehat{f}\|_{L^r(S_t, d\sigma)} \leq C\|f\|_{L^p(\mathbb{F}_q^d, dm)}$, *where* $C > 0$ *is independent of* q *and* $f : (\mathbb{F}_q^d, dm) \to \mathbb{C}$.

Definition

We write $R(p \to r) \ll 1$ to indicate that the restriction estimate above holds for $1 \leq p, r \leq \infty$.

By duality, the restriction problem is same as the following extension problem.

Problem (Extension Problem)

Find $1 \leq p, r \leq \infty$ *such that* $\|(g d\sigma)^{\vee}\|_{L^{p'}(\mathbb{F}_q^d, dm)} \ll \|g\|_{L^{r'}(S_t, d\sigma)}$ *where* $p' = p/(p-1)$.

$L^p - L^2$ Restriction Estimates Imply the Erdős Distance Results

Combining the Formula for distance sets with the restriction estimate, we can deduce the following result.

Lemma

If $R(p \to 2) \ll 1$ then we have

$$|\Delta(E)| \gg \min\left\{q, \ \frac{|E|^{3-\frac{2}{p}}}{q^{d-1}}\right\}.$$

The following corollary follows immediately from the above lemma.

Corollary

Suppose that $R(p \to 2) \ll 1$. If $E \subset \mathbb{F}_q^d$ with $|E| \geq Cq^{\frac{dp}{3p-2}}$, then $|\Delta(E)| \gg q$.

Conjecture for $R(p \to 2) \ll 1$ and Known Results

We introduce the conjecture of the $L^p - L^2$ restriction estimates for spheres.

Conjecture

If $d \geq 3$ is odd, then

$$R(p \to 2) \ll 1 \iff 1 \leq p \leq \frac{2d+2}{d+3}.$$

On the other hand, if $d \geq 2$, then

$$R(p \to 2) \ll 1 \iff 1 \leq p \leq \frac{2d+4}{d+4}.$$

The best known results are due to Iosevich and Koh.

Theorem

If $d \geq 3$ is odd or $d = 2$, then the above conjecture is true. In addition, if $d \geq 4$ is even and $1 \leq p \leq \frac{2d+2}{d+3}$, then $R(p \to 2) \ll 1$.

Erdős Distance Results from Restriction Estimates

Using the connection between distance sets and the restriction phenomena, we can deduce following result.

Theorem

Let $E \subset \mathbb{F}_q^d$.

- *If* $d \geq 3$ *and* $|E| \geq Cq^{\frac{d+1}{2}}$, *then* $|\Delta(E)| \gg q$.
- *If* $d = 2$ *and* $|E| \geq Cq^{\frac{4}{3}}$, *then* $|\Delta(E)| \gg q$.

Remark

1. *The result for dimension two is the best possible one which can be deduced by the restriction results. However, it is far from the conjectured result on the Erdős distance problem.*
2. *If one could establish the restriction conjecture for spheres in even dimensions, then one can deduce that* $|E| \geq Cq^{\frac{d}{2}+\frac{d}{2d+2}} \Rightarrow |\Delta(E)| \gg q$.