Ultraquadrics and its application to the reparametrization of rational complex surfaces

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1. Introduction: $\mathbb{K}$-Algebraic Optimality Problem (Parametric Version)

- **Rational parametric representations of algebraic varieties (in particular, of curves and surfaces) are a useful tool in many applied fields, such as CAGD.**
  - **J. R. Sendra, F. Winkler, and S. Pérez-Díaz. Rational algebraic curves: A computer algebra approach volume 22 of *Algorithms and Computation in Mathematics* Springer, Berlin, 2008.**
  - **J. Schicho. Rational parametrization of surfaces *J. Symbolic Comput.*, 26(1):1–29, 1998.**
- **But a rational variety might be described through many different (although related) parametrizations:**
  - $\mathcal{P}_1(s,t) = (t, t^2, s)$
  - $\mathcal{P}_2(s,t) = (t^2, t^4, s)$
  - $\mathcal{P}_3(s,t) = \left(\mathtt{i}t, -t^2, \frac{1}{s}\right)$
  - $\mathcal{P}_4(s,t) = \left(\frac{100\mathtt{i}t}{t-s}, -\frac{10000t^2}{(t-s)^2}, s\right)$

  are parametrizations of the same cylindric surface $F(x,y,z) = y - x^2 = 0$

1. Introduction: $\mathbb{K}$-Algebraic Optimality Problem (Parametric Version)

- **Given:**
  - $\mathbb{K} \subseteq \mathbb{L} \subseteq \mathbb{F}$ where $\mathbb{K}$ is a computable field of characteristic zero (ground field), $\mathbb{L} = \mathbb{K}(\alpha)$ an algebraic extension of $\mathbb{K}$, and $\mathbb{F}$ is the algebraic closure of $\mathbb{K}$,
  - a unirational map $\varphi : \mathbb{F}^m \to \mathbb{F}^n$ where $\varphi = (\varphi_1, \ldots, \varphi_n)$ and

$$\varphi_i(T_1, \ldots, T_m) = \frac{h_i(T_1, \ldots, T_m)}{g_i(T_1, \ldots, T_m)} \in \mathbb{L}(T_1, \ldots, T_m)$$

  - $\mathcal{V}$ the Zariski closure of $\varphi(\mathbb{F}^m)$.
- **Decide:** whether $\mathcal{V}$ can be parametrized over $\mathbb{K}$.
- **Find:** (in the affirmative case) a $\mathbb{K}$-rational parametrization of $\mathcal{V}$.

1. Introduction: $\mathbb{K}$-Algebraic Optimality Problem (Parametric Version)

- **For curves the reparametrization problem can be approached by means of hypercircles**
  - **C. Andradas, T. Recio, and J. R. Sendra. Base field restriction techniques for parametric curves. In *Proceedings of the 1999 International Symposium on Symbolic and Algebraic Computation*, pages 17–22, 1999.**
  - **T. Recio and J. R. Sendra. Real reparametrizations of real curves. *J. Symbolic Comput.*, 23(2-3):241–254, 1997.**
  - **T. Recio, J. R. Sendra, L. F. Tabera, and C. Villarino. Generalizing circles over algebraic extensions. *Mathematics of Computation*, 79(270):1067–1089, 2010.**
  - **L. F. Tabera. Two Tools in Algebraic Geometry: Construction of Configurations in Tropical Geometry and Hypercircles for the Simplification of Parametric Curves. PhD thesis, Universidad de Cantabria, Université de Rennes I, 2007.**
  - **C. Villarino. Algoritmos de optimalidad algebraica y de cuasi-polinomialidad para curvas racionales. PhD thesis, Universidad de Alcalá, 2007.**
- **For higher dimensional algebraic varieties, generalizing the ideas for curves, a theoretical frame is established (by means of ultraquadrics):**
  - **C. Andradas, T. Recio, J. R. Sendra, and L. F. Tabera. On the simplification of the coefficients of a parametrization. *J. Symbolic Comput.*, 44(2):192–210, 2009.**

1. Introduction: $\mathbb{K}$-Algebraic Optimality Problem (Parametric Version)

- Let $\mathbb{K} \subseteq \mathbb{R}$ be a computable field; (for instance, $\mathbb{K}$ may by $\mathbb{Q}$, $\mathbb{Q}(\sqrt{2})$, etc).
- Given a rational parametrization over $\mathbb{K}(\mathtt{i})$ of a ruled surface $\mathcal{S}$, say

$$\mathcal{P}(T_1,T_2) = (\varphi_1(T_1) + T_2\psi_1(T_1), \varphi_2(T_1) + T_2\psi_2(T_1), \varphi_3(T_1) + T_2\psi_3(T_1))$$

  where $\varphi_i(T_1), \psi_i(T_1) \in \mathbb{K}(\mathtt{i})(T_1)$

- Given a rational swung surface parametrized as

$$\mathcal{P}(T_1,T_2) = (\varphi_1(T_1)\psi_1(T_2), \varphi_1(T_1)\psi_2(T_2), \varphi_2(T_1))$$

  where $\varphi_i(T_1), \psi_i(T_2) \in \mathbb{K}(\mathtt{i})(T_i)$

- Determine whether $\mathcal{S}$ can be reparametrized over $\mathbb{K}$, and in the affirmative case,
- Compute a parametrization of $\mathcal{S}$ with coefficients in $\mathbb{K}$.

1. Introduction: $\mathbb{K}$-Algebraic Optimality Problem (Parametric Version)

- **For rational ruled surfaces the reparametrization problem has been solved in**
  - **C. Andradas, T. Recio, J. R. Sendra, L. F. Tabera, and C. Villarino.**
    Proper real reparametrization of rational ruled surfaces.
    In *Computer Aided Geometric Design*, 28(2), pages 102–113 , 2011.
- **For rational swung surfaces the reparametrization problem has been solved in**
  - **C. Andradas, T. Recio, J. R. Sendra, L. F. Tabera, and C. Villarino.**
    Reparametrizing Swung Surfaces over the reals.
    Submitted, 2013.

2. Preliminaires: $\mathtt{i}$-Hypercircles

- Let $u(t)$ be a unit of $\mathbb{K}(\mathtt{i})(t)$; i.e. $u(t) = (at+b)/(ct+d) \in \mathbb{K}(\mathtt{i})(t)$ such that $ad - bc \neq 0$.

- We expand $u(t)$ as
$$u(t) = \phi_0(t) + \mathtt{i}\phi_1(t),$$
where $\phi_i(t) \in \mathbb{K}(t)$, for $i = 0, 1$.

The $\mathtt{i}$-hypercircle $\mathcal{U}$ generated by $u(t)$ is the rational curve in $\mathbb{F}^2$ parametrized by
$$\phi(t) = (\phi_0(t), \phi_1(t)).$$

- Let $u : \mathbb{Q}(\mathtt{i})(t) \to \mathbb{Q}(\mathtt{i})(t)$ be the authomorphism $u(t) = \dfrac{1}{t+\mathtt{i}} = \dfrac{t-\mathtt{i}}{t^2+1}$.

- Then the $\mathtt{i}$-hypercircle $\mathcal{U}$ defined by u is the rational curve parametrized by:
$$\phi(t) = \left(\frac{t}{t^2+1}, -\frac{1}{t^2+1}\right).$$

- The $\mathtt{i}$-hypercircle is the real circle: $x^2 + y^2 + y = 0$.

2. Preliminaires: $\mathtt{i}$-Weil(descent) Parametric Variety of a Curve

- **Let $\Phi(t) = (\xi_1(t), \ldots, \xi_n(t)) \in \mathbb{K}(\mathtt{i})(t)^n$ be a proper parametrization of a curve $\mathcal{C}$.**
- **We consider, then, the formal substitution $\Phi(t_0 + \mathtt{i}t_1)$, and we express $\xi_i(t_0 + \mathtt{i}t_1)$ as**

$$\xi_i(t_0 + \mathtt{i}t_1) = \frac{\psi_{i0}(t_0, t_1) + \mathtt{i}\psi_{i1}(t_0, t_1)}{\delta(t_0, t_1)}, \quad i = 1, \ldots, n,$$

where $\psi_{ij}, \delta \in \mathbb{K}[t_0, t_1]$ and $\gcd(\psi_{10}, \psi_{11}, \ldots, \psi_{n1}, \delta) = 1$.

- **Let Y be the algebraic variety in $\mathbb{F}^2$ defined by the polynomials $\{\psi_{i1}(t_0, t_1)\}_{i=1,\ldots,n}$ (i.e. the imaginary parts of the numerators of $\xi_i(t_0 + \mathtt{i}t_1)$).**
- **Let Δ be the algebraic variety in $\mathbb{F}^2$ defined by $\delta(t_0, t_1)$.**

Then, the $\mathtt{i}$-Weil (descent) parametric variety of $\mathcal{C}$ via $\Phi(t)$ **is defined as the Zariski closure of $Y \setminus \Delta$. We denote it by** $\mathrm{Weil1D}(\Phi)$.

2. Preliminaires: $\mathtt{i}$-Weil(descent) Parametric Variety of a Curve

The following theorems shows the role that $\mathrm{Weil}\mathbf{1D}(\Phi)$ plays in the reparametrization problem. In this context we need to recall that the $\mathbb{K}$-definability of a curve (or surface) means that the ideal of the curve (or surface) can be generated by polynomials over $\mathbb{K}$.

Theorem 1 *It holds that*

1. *The curve $\mathcal{C}$ is $\mathbb{K}$-definable iff $\mathrm{Weil}\mathbf{1D}(\Phi)$ contains a 1-dimensional component.*
2. *The curve $\mathcal{C}$ can be parametrized over $\mathbb{K}$ iff the 1-dimensional component of $\mathrm{Weil}\mathbf{1D}(\Phi)$ is an $\mathtt{i}$-hypercircle. In this case, if $(\chi_1(t), \chi_2(t))$ is a proper $\mathbb{K}$-parametrization of the $\mathtt{i}$-hypercircle, then $\Phi(\chi_1(t) + \mathtt{i}\chi_2(t))$ is a $\mathbb{K}$-parametrization of $\mathcal{C}$*

Theorem 2 $\mathrm{Weil}\mathbf{1D}(\Phi)$ *contains a component of dimension* 1 *if and only if*

$$\gcd(\psi_{11}, \ldots, \psi_{n1}) \neq 1.$$

If so, $\mathrm{Weil}\mathbf{1D}(\Phi)$ *is defined by the* $\gcd(\psi_{11}, \ldots, \psi_{n1})$.

Algorithm 1 of Real Reparametrization of Space Curves

Input: A parametrization $\xi = (\xi_1, \dots, \xi_n)$ of a spatial curve with coefficients in $\mathbb{K}(\mathtt{i})$.
Output: "$\mathcal{C}$ is not a real curve", else a linear fraction $u(t)$ such that $\xi \circ u$ is real.

1. Find $\xi^*(t) = (\xi_1^*, \dots, \xi_n^*)$, a proper parametrization of $\mathcal{C}$.
2. Write $\xi_i^*(t_0 + \mathtt{i}t_1) = \frac{\psi_{i0}(t_0,t_1)+\mathtt{i}\psi_{i1}(t_0,t_1)}{\delta(t_0,t_1)}, \quad i = 1, \dots, n$, with $\psi_{i1}, \delta \in \mathbb{K}[t_0, t_1]$.
3. Compute $\psi(t_0, t_1) = \gcd(\psi_{11}, \dots, \psi_{n1})$.
4. If $\psi = 1$, return: "$\mathcal{C}$ is not a real curve"
5. If ψ is a linear polynomial
   (a) Compute a linear real (over $\mathbb{K}$) parametrization $\chi = (\chi_1(t), \chi_2(t))$ of the line defined by ψ.
   (b) Return $t \mapsto \chi_1(t) + \mathtt{i}\chi_2(t)$

Algorithm 1 of Real Reparametrization of Space Curves

6. Check whether ψ is a real circle.
7. If ψ is not a real circle, return: "$\mathcal{C}$ is not a real curve"
8. Compute a real (over a real field extension of $\mathbb{K}$ of degree at most 2) parametrization $\chi = (\chi_1, \chi_2)$ of the real circle ψ.
9. Return $t \mapsto \chi_1(t) + \mathtt{i}\chi_2(t)$

We consider the space curve given by

$$\Phi(t) = \left(\frac{3\,t^2+1+2\,\mathtt{i}t^3}{1+4\,t^2}, -\frac{t^2\,(-1+2\,\mathtt{i}t)}{1+4\,t^2}, -\frac{3\,t^2+1-2\,t^4+5\,\mathtt{i}t^3+\mathtt{i}t}{t\,(1+4\,t^2)}\right).$$

$\mathrm{Weil}\mathbf{1D}(\Phi)$ contains the curve defined by $t_0^2+t_1^2-t_1$, that is the circle centered at $(0,1/2)$ and radius $1/2$. Moreover, this circle is parametrized as

$$\left(\frac{t}{t^2+1}, \frac{1}{2}\,\frac{t^2-1}{t^2+1}+\frac{1}{2}\right).$$

Therefore, we get the real parametrization

$$\Phi\left(\frac{t}{t^2+1}+\mathtt{i}\left(\frac{1}{2}\,\frac{t^2-1}{t^2+1}+\frac{1}{2}\right)\right) = \left(\frac{1}{t^2+1}, \frac{t^2}{t^2+1}, -\frac{1}{t\,(t^2+1)}\right)$$

2. Introduction: $\mathtt{i}$-Ultracuadrics

Let θ be an automorphism defined over $\mathbb{K}(\mathtt{i})$, $\theta : \mathbb{F}^2 \to \mathbb{F}^2$:

$$(t, s) \mapsto \theta(t, s) = (\theta_1(t, s), \theta_2(t, s)).$$

Each $\theta_i \in \mathbb{K}(\mathtt{i})(t, s)$ and can be written uniquely as

$$\theta_i(t, s) = \theta_{i0}(t, s) + \mathtt{i}\theta_{i1}(t, s)$$

where $\theta_{i0}, \theta_{i1} \in \mathbb{K}(t, s)$.

Then, the $\mathtt{i}$-ultraquadric generated by θ is the rational variety in $\mathbb{F}^4$ parametrized by

$$(\theta_{10}(t, s), \theta_{11}(t, s), \theta_{20}(t, s), \theta_{21}(t, s)).$$

- Let $\theta : \mathbb{C}(t, s) \to \mathbb{C}(t, s)$ be the authomorphism $\theta(t, s) = \left(\frac{s}{s + it}, \frac{t - i}{s + it}\right)$.
- $\theta(t, s) = \left(\frac{s^2 - ist}{s^2 + t^2}, \frac{st - t - i(s + t^2)}{s^2 + t^2}\right)$.
- Then the $\mathtt{i}$-ultraquadric $\mathcal{U}$ defined by θ is the rational surface parametrized by:

$$\bar{\theta}(t, s) = \left(\frac{s^2}{s^2 + t^2}, -\frac{st}{s^2 + t^2}, \frac{st - t}{s^2 + t^2}, -\frac{s + t^2}{s^2 + t^2}\right).$$

2. Preliminaires: $\mathtt{i}$-Weil(descent) Parametric Variety of a Surface

- Let $\Phi(T_1, T_2) = (\xi_1(T_1, T_2), \xi_2(T_1, T_2), \xi_3(T_1, T_2)) \in \mathbb{K}(\mathtt{i})(T_1, T_2)^3$ be a proper parametrization of a surface $\mathcal{S}$ in $\mathbb{F}^3$.

- We introduce new variables T_{ij}, $1 \leq i \leq 2$, $0 \leq j \leq 1$, and we consider the formal substitution $\Phi(T_{10} + \mathtt{i}T_{11}, T_{20} + \mathtt{i}T_{21})$. We express $\xi_i(T_{10} + \mathtt{i}T_{11}, T_{20} + \mathtt{i}T_{21})$ as

$$\xi_i(T_{10} + \mathtt{i}T_{11}, T_{20} + \mathtt{i}T_{21}) = \frac{\psi_{i0}(T_{10}, T_{11}, T_{20}, T_{21}) + \mathtt{i}\psi_{i1}(T_{10}, T_{11}, T_{20}, T_{21})}{\delta(T_{10}, T_{11}, T_{20}, T_{21})}$$

  for $i = 1, 2, 3$, where $\psi_{ij}, \delta \in \mathbb{K}[T_{10}, T_{11}, T_{20}, T_{21}]$ and $\gcd(\psi_{10}, \ldots, \psi_{31}, \delta) = 1$.

- Let Y be the algebraic variety in $\mathbb{F}^4$ defined by the polynomials $\{\psi_{i1}(T_{10}, \ldots, T_{21})\}_{i=1,2,3}$ (i.e. the imaginary parts of the numerators of $\xi_i(T_{10} + \mathtt{i}T_{11}, T_{20} + \mathtt{i}T_{21})$).

- Let Δ be the algebraic variety in $\mathbb{F}^4$ defined by $\delta(T_{10}, \ldots, T_{21})$.

Then, the $\mathtt{i}$-Weil (descent) parametric variety of $\mathcal{S}$ via $\Phi(T_1, T_2)$ is defined as the Zariski closure of $Y \setminus \Delta$. We denote it by $\mathrm{Weil2D}(\Phi)$.

2. Preliminaires: $\mathtt{i}$-Weil(descent) Parametric Variety of a Surface

The following theorem establishes the main properties of $\mathrm{Weil2D}(\Phi)$.

Theorem 3 *It holds that*

1. *$\mathcal{S}$ is defined over $\mathbb{K}$ if and only if $\mathrm{Weil2D}(\Phi)$ contains a component defined over $\mathbb{K}$ that is $\mathbb{K}$-birational to $\mathcal{S}$.*
2. *$\mathcal{S}$ is parametrizable over $\mathbb{K}$ if and only if $\mathrm{Weil2D}(\Phi)$ contains an $\mathtt{i}$-ultraquadric that is $\mathbb{K}$-birational to $\mathcal{S}$. If this ultraquadric is properly parametrized by*

$$(\theta_{10}(t,s), \theta_{11}(t,s), \theta_{20}(t,s), \theta_{21}(t,s))$$

then the authomorphism

$$\theta(t,s) = (\theta_{10}(t,s) + \mathtt{i}\theta_{11}(t,s), \theta_{20}(t,s) + \mathtt{i}\theta_{21}(t,s))$$

verifies

$$\Phi(\theta(t,s)) \in \mathbb{K}(t,s)^3.$$

3. Reparametrization of ruled surfaces: Standard Form

- Let $\mathcal{S}$ be a rational ruled surface in $\mathbb{C}^3$, and let

$$\mathcal{P}(T_1, T_2) = (\varphi_1(T_1) + T_2\psi_1(T_1), \varphi_2(T_1) + T_2\psi_2(T_1)\varphi_3(T_1) + T_2\psi_3(T_1))$$

  be a proper parametrization of $\mathcal{S}$ where

$$\mathcal{P}(T_1, T_2) \in \mathbb{K}(\mathtt{i})(T_1, T_2)^3.$$

- Assuming that $\psi_3 \neq 0$, $\mathcal{S}$ always admits a proper parametrization of the form

$$(\phi_1(T_1) + T_2\chi_1(T_1), \phi_2(T_1) + T_2\chi_2(T_1), T_2) \in \mathbb{K}(\mathtt{i})(T_1, T_2)^3.$$

- Such a parametrization can be achieved by performing the $\mathbb{K}(\mathtt{i})$-birational transformation

$$(T_1, T_2) \mapsto \left(T_1, \frac{T_2 - \varphi_3(T_1)}{\psi_3(T_1)}\right)$$

  to $\mathcal{P}(T_1, T_2)$.

3. Reparametrization of ruled surfaces: Standard Form

- **The parametrization**

$$\mathcal{Q}(T_1, T_2) = \mathcal{P}\left(T_1, \frac{T_2 - \varphi_3(T_1)}{\psi_3(T_1)}\right)$$

is called a standard parametrization of $\mathcal{S}$.

- **Moreover, if**

$$\mathcal{Q}(T_1, T_2) = (\phi_1(T_1) + T_2\chi_1(T_1), \phi_2(T_1) + T_2\chi_2(T_1), T_2),$$

the curve in $\mathbb{C}^4$ given by the parametrization

$$(\phi_1(T_1), \chi_1(T_1), \phi_2(T_1), \chi_2(T_1))$$

is the reparametrizing curve associated to $\mathcal{Q}(T_1, T_2)$, **denoted by $\mathcal{C}_{\mathcal{Q}}$.**

- **The parametrization $(\phi_1(T_1), \chi_1(T_1), \phi_2(T_1), \chi_2(T_1))$ of $\mathcal{C}_{\mathcal{Q}}$ is proper.**

Theorem of Reparametrization of Ruled Surfaces

- Let $\mathcal{Q}(T_1, T_2) = (\phi_1(T_1) + T_2\chi_1(T_1), \phi_2(T_1) + T_2\chi_2(T_1), T_2) \in \mathbb{K}(\mathrm{i})(T_1, T_2)^3$ be a proper parametrization in standard form of $\mathcal{S}$
- and let $\mathcal{C}_{\mathcal{Q}} = (\phi_1(T_1), \chi_1(T_1), \phi_2(T_1), \chi_2(T_1))$ be the parametrization of the associated curve.

$\mathcal{S}$ can be properly parametrized over a finite real algebraic extension $\mathbb{L}$ of $\mathbb{K}$ if and only if one of the following conditions hold:

(1) $\mathcal{C}_{\mathcal{Q}}$ is $\mathbb{L}$-parametrizable.

(2) The rational functions $\frac{\mathrm{Im}(\phi_i(T_1+\mathrm{i}T_2))}{\mathrm{Im}(\chi_i(T_1+\mathrm{i}T_2))}$, $i = 1, 2$, are well-defined, equal, non-constant, and the transformation of $\mathbb{C}^2$

$$(T_1, T_2) \mapsto \left(T_1 + \mathrm{i}T_2, -\frac{\mathrm{Im}(\phi_i(T_1 + \mathrm{i}T_2))}{\mathrm{Im}(\chi_i(T_1 + \mathrm{i}T_2))}\right),$$

is birational.

Moreover, in the affirmative case,

(i) If (1) holds and $T_1 \mapsto f(T_1)$ reparametrizes $\mathcal{C}_{\mathcal{Q}}$ over $\mathbb{L}$, then

$$\mathcal{Q}(f(T_1), T_2)$$

is a real proper parametrization of $\mathcal{S}$.

(ii) If (2) holds,

$$\mathcal{Q}\left(T_1 + \mathtt{i}T_2, -\frac{\mathrm{Im}(\phi_1(T_1 + \mathtt{i}T_2))}{\mathrm{Im}(\chi_1(T_1 + \mathtt{i}T_2))}\right)$$

is a proper reparametrization of $\mathcal{S}$ over $\mathbb{K}$.

Algorithm of Reparametrization of Ruled Surfaces

Input: A proper parametrization $\mathcal{P}(T_1, T_2)$ of a ruled surface $\mathcal{S}$.
Output: "$\mathcal{S}$ is not properly parametrizable over the reals" or a change of variables g such that $\mathcal{P} \circ g$ is a real parametrization with coefficients over a extension of degree at most two over $\mathbb{K}$.

1. Let $i \in \{1, 2, 3\}$ such that $\psi_i \neq 0$. Permute i and 3 so that $\psi_3 \neq 0$.
2. $\mathcal{Q}(T_1, T_2) := \mathcal{P}(T_1, \frac{T_2 - \varphi_3(T_1)}{\psi_3(T_1)})$
3. Write $\mathcal{Q}(T_1, T_2) = (\phi_1(T_1) + T_2\chi_1(T_1), \phi_2(T_1) + T_2\chi_2(T_1), T_2)$
4. Apply Algorithm 1 to $\Phi(T_1) := (\phi_1(T_1), \chi_1(T_1), \phi_2(T_1), \chi_2(T_1))$
5. If the output in Step 4 is $f(T_1)$, then return $g : (T_1, T_2) \mapsto (f(T_1), T_2)$

Algorithm of Reparametrization of Ruled Surfaces

6. $h1 := \mathrm{Im}(\chi_1(T_1 + \mathtt{i}T_2)), h_2 := \mathrm{Im}(\chi_2(T_1 + \mathtt{i}T_2))$
7. If $h_1 = 0$ or $h_2 = 0$, then return "$\mathcal{S}$ is not properly parametrizable over the reals".
8. $A := -\frac{\mathrm{Im}(\phi_1(T_1+\mathtt{i}T_2))}{\mathrm{Im}(\chi_1(T_1+\mathtt{i}T_2))}, B := -\frac{\mathrm{Im}(\phi_2(T_1+\mathtt{i}T_2))}{\mathrm{Im}(\chi_2(T_1+\mathtt{i}T_2))}$
9. If $A \neq B$ or A is constant then return "$\mathcal{S}$ is not properly parametrizable over the reals"
10. $g := (T_1, T_2) \mapsto (T_1 + \mathtt{i}T_2, A)$
11. If g is birational then return g. Else return "$\mathcal{S}$ is not properly parametrizable over the reals".

We consider the ruled surface $\mathcal{S}$ given by the proper parametrization over $\mathbb{Q}(\mathtt{i})$

$$\mathcal{P}(T_1,T_2)=\left(\frac{T_1+\mathtt{i}}{\mathtt{i}T_1+1}+\frac{\left(T_1^2+1\right)T_2}{T_1},\frac{\mathtt{i}T_1+1}{T_1+\mathtt{i}}+\frac{T_1T_2}{T_1^2+1},\frac{\left(T_1+\mathtt{i}\right)^2}{\left(\mathtt{i}T_1+1\right)^2}+\frac{\left(T_1^2+1\right)^2T_2}{T_1^2}\right).$$

The associated standard parametrization is

$$\mathcal{Q}(T_1,T_2)=\left(\frac{\mathtt{i}T_1^4+T_1-T_1^3+\mathtt{i}-T_2T_1^3+2\mathtt{i}T_2T_1^2+T_1T_2}{\left(T_1^2+1\right)\left(\mathtt{i}T_1+1\right)^2},\right.$$

$$\left.\frac{\mathtt{i}T_1^9+3T_1^8+(T_2+9)T_1^6-\mathtt{i}(T_2+3)T_1^5+(T_2+3)T_1^4-\mathtt{i}(T_2+9)T_1^3-3\mathtt{i}T_1-1}{-\left(T_1+\mathtt{i}\right)\left(T_1^2+1\right)^3\left(\mathtt{i}T_1+1\right)^2},T_2\right).$$

The curve $\mathcal{C}_{\mathcal{Q}}$ is parametrized as

$$\Phi(T_1) = \left(\frac{\mathrm{i}T_1^4 + T_1 - T_1^3 + \mathrm{i}}{(T_1^2+1)(\mathrm{i}T_1+1)^2}, \frac{-T_1^3 + 2\,\mathrm{i}T_1^2 + T_1}{(T_1^2+1)(\mathrm{i}T_1+1)^2},\right.$$

$$\left. -\frac{3T_1^8 + 9T_1^6 + 3T_1^4 - 1 + \mathrm{i}(T_1^9 - 3T_1^5 - 9T_1^3 - 3T_1)}{(T_1+\mathrm{i})(T_1^2+1)^3(\mathrm{i}T_1+1)^2}, -\frac{-\mathrm{i}T_1^5 - \mathrm{i}T_1^3 + T_1^4 + T_1^6}{(T_1+\mathrm{i})(T_1^2+1)^3(\mathrm{i}T_1+1)^2}\right).$$

In addition, $\mathtt{Weil1D}(\Phi)$ contains the real circle $T_{10}^2 + T_{11}^2 = 1$. Therefore,

$$\mathcal{Q}\left(\frac{2T_1}{T_1^2+1} + \mathrm{i}\frac{T_1^2-1}{T_1^2+1}, T_2\right)$$

is a proper parametrization of $\mathcal{S}$. Indeed, it is

$$\left(\frac{T_1^4 - 3\,T_1^2 - T_2T_1^2 - T_2}{-4T_2}, \frac{T_1^8 - (T_2-3)T_1^6 - 3(T_2-1)T_1^4 - 3(T_2+21)T_1^2 - T_2}{-64T_1^3}, T_2\right).$$

3. Reparametrization of ruled surfaces: Example 2

We consider the ruled surface (the plane) $\mathcal{S}$ given by the standard proper parametrization

$$\mathcal{Q}(T_1, T_2) = (1 + \mathrm{i}T_1 + 2\mathrm{i}T_2, \mathrm{i}T_1 + (1+2\mathrm{i})T_2, T_2).$$

Observe that

$$\mathcal{C}_{\mathcal{Q}} = (\phi_1(T_1), \chi_1(T_1), \phi_2(T_1), \chi_2(T_1)) = (1 + \mathrm{i}T_1, 2\mathrm{i}, \mathrm{i}T_1, 1 + 2\mathrm{i})$$

is not real. However the rational functions $\frac{\mathrm{Im}(\phi_i(T_1+\mathrm{i}T_2))}{\mathrm{Im}(\chi_i(T_1+\mathrm{i}T_2))}$, $i = 1, 2$, in Theorem of reparametrization (2) are both equal to $\dfrac{T_1}{2}$. Moreover, the map

$$(T_1, T_2) \mapsto \left(T_1 + \mathrm{i}T_2, -\frac{T_1}{2}\right)$$

is birational. Therefore,

$$\mathcal{Q}\left(T_1 + \mathrm{i}T_2, -\frac{T_1}{2}\right) = \left(1 - T_2, T_2 - \frac{T_1}{2}, -\frac{T_1}{2}\right)$$

parametrizes $\mathcal{S}$.

We consider the ruled surface $\mathcal{S}$ given by the standard proper parametrization

$$\mathcal{Q}(T_1, T_2) = (T_1^2 + \mathrm{i}T_1T_2, \mathrm{i}T_1 + \mathrm{i}T_1^2T_2, T_2).$$

The reparametrizing curve $\mathcal{C}$ is given by

$$\mathcal{C}_{\mathcal{Q}} = (\phi_1(T_1), \chi_1(T_1), \phi_2(T_1), \chi_2(T_1)) = (T_1^2, \mathrm{i}T_1, \mathrm{i}T_1, \mathrm{i}T_1^2)$$

and $\mathsf{Weil1D}(\Phi)$ does not contain 1-dimensional components. Therefore, $\mathcal{C}_{\mathcal{Q}}$ is not real. On the other hand, the rational functions $\frac{\mathrm{Im}(\phi_i(T_1+\mathrm{i}T_2))}{\mathrm{Im}(\chi_i(T_1+\mathrm{i}T_2))}$, $i = 1, 2$, in Theorem of reparametrization (2), although well-defined and non-constant, they are different. Therefore, $\mathcal{S}$ cannot be parametrized properly over a real finite field extension of $\mathbb{Q}$.

4. Reparametrization of Swung surfaces: Swung Surfaces

- **A rational Swung surface is a surface $\mathcal{S}$ parametrized as**

$$\mathcal{P}(t,s) = (\phi_1(t)\psi_1(s), \phi_1(t)\psi_2(s), \phi_2(t))$$

- **A rational Revolution surface is a surface $\mathcal{S}$ parametrized as**

$$\mathcal{P}(t,s) = (\phi_1(t)\frac{s^2-1}{s^2+1}, \phi_1(t)\frac{2s}{s^2+1}, \phi_2(t))$$

4. Reparametrization of Swung surfaces: Theorem of Reparametrization

- Let $\mathcal{S}$ be a rational complex surface, other than a plane, parametrized by

$$\mathcal{P}(t,s) = (\phi_1(t)\psi_1(s), \phi_1(t)\psi_2(s), \phi_2(t)) \in \mathbb{K}(\mathtt{i})(t,s)^3$$

- where $(\phi_1(t), \phi_2(t)) \in \mathbb{K}(\mathtt{i})(t)^2$ and $(\psi_1(s), \psi_2(s)) \in \mathbb{K}(\mathtt{i})(s)^2$ are curves properly parametrized

Then, the following statements are equivalent:

- $\mathcal{S}$ is $\mathbb{K}$-parametrizable.
- There exists $\lambda \in \mathbb{K}(\mathtt{i}) \setminus \{0\}$ such that the curves defined by the parametrizations $\phi_\lambda = (\lambda\phi_1(t), \phi_2(t))$ and $\psi_\lambda = (\frac{1}{\lambda}\psi_1(s), \frac{1}{\lambda}\psi_2(s))$ are $\mathbb{K}$-parameterizable.
- There exists a change of variables:

$$\begin{array}{rccc} \xi: & \mathbb{K}(\mathtt{i})^2 & \rightarrow & \mathbb{K}(\mathtt{i})^2 \\ & (t,s) & \mapsto & \left(\frac{a_1t+b_1}{c_1t+d_1}, \frac{a_2s+b_2}{c_2s+d_2}\right) \end{array}$$

where $a_ib_i - c_id_i \neq 0$, $i = 1, 2$, such that $\mathcal{P}(\xi(t,s)) \in \mathbb{K}(t,s)^3$.

- $\lambda = 1$ if $(\phi_1(t), \phi_2(t))$ and $(\psi_1(s), \psi_2(s))$ are parametrizables over $\mathbb{K}$.
- $\lambda = \mathtt{i}$ if $(\mathtt{i}\phi_1(t), \phi_2(t))$ and $(\mathtt{i}\psi_1(s), \mathtt{i}\psi_2(s))$ are parametrizables over $\mathbb{K}$.
- $\lambda = -r + \mathtt{i} \in \mathbb{K}(\mathtt{i})$ where

$$r = \frac{Re(\phi_1(t_0 + \mathtt{i}t_1)}{Im(\phi_1(t_0 + \mathtt{i}t_1)} = -\frac{Re(\psi_1(s_0 + \mathtt{i}s_1)}{Im(\psi_1(s_0 + \mathtt{i}s_1)} = -\frac{Re(\psi_2(s_0 + \mathtt{i}s_1)}{Im(\psi_2(s_0 + \mathtt{i}s_1)} \in \mathbb{K}$$

- Let $\mathcal{S}$ be a rational revolution surface, parametrized by

$$\mathcal{P}(t,s) = \left(\phi_1(t)\frac{s^2-1}{s^2+1}, \phi_1(t)\frac{2s}{s^2+1}, \phi_2(t)\right) \in \mathbb{K}(\mathtt{i})(t,s)^3,$$

where $(\phi_1(t), \phi_2(t))$ is a proper parametrization of a curve.

The following statements are equivalent:

- $\mathcal{S}$ is $\mathbb{K}$-parametrizable (but, perhaps, not necessarily with a proper parametrization)
- The curve defined by $\phi(t) = (\phi_1(t), \phi_2(t))$ is $\mathbb{K}$-parametrizable.
- There exists a change of parameters with complex coefficients. $\xi : \mathbb{K}(\mathtt{i}) \longrightarrow \mathbb{K}(\mathtt{i})$, where $\xi(t) = \dfrac{at+b}{ct+d}$ and $ad - bc \neq 0$, such that $\mathcal{P}(\xi(t), s) \in \mathbb{K}(t,s)^3$.

Algorithm of Reparametrization of Swung Surfaces

- **Input:** A complex parametrization $\mathcal{P}$ of a swung surface $\mathcal{S}$, other than a plane,

$$\mathcal{P}(t,s) = (\phi_1(t)\psi_1(s), \phi_1(t)\psi_2(s), \phi_2(t)) \in \mathbb{K}(\mathtt{i})(t,s)^3$$

  where $(\phi_1(t), \phi_2(t)) \in \mathbb{K}(\mathtt{i})(t)^2$ and $(\psi_1(s), \psi_2(s)) \in \mathbb{K}(\mathtt{i})(s)^2$ are curves properly parametrized

- **Output:** A real parametrization, $\mathcal{P}'(t,s) \in \mathbb{K}(t,s)^3$ of $\mathcal{S}$ or *"The surface is not $\mathbb{K}$-parametrizable"*

4. Reparametrization of Swung surfaces: Algorithm of Reparametrization

1. Write $\phi_2(t_0 + \mathtt{i}t_1) = \frac{B_0(t_0,t_1)+\mathtt{i}B_1(t_0,t_1)}{B(t_0,t_1)}$
2. Compute the factors of degree 1 and/or of degree 2 (that correspond to circles) of $B_1(t_0, t_1)$ in $\mathbb{K}[t_0, t_1]$.
3. For each factor f from step 4. do
   (a) Compute a real parametrization $(v_0(t), v_1(t))$ of the line or circle defined by f.
   (b) Let $v(t) = v_0(t) + \mathtt{i}v_1(t)$
   (c) If there exists a $\lambda_f \in \mathbb{C}^*$ such that $(\lambda_f \cdot \phi_1(v(t)), \phi_2(v(t)))$ is real then:
      i. Apply the real reparametrization algorithm for curves to $\psi_{\lambda_f} = (1/\lambda_f\psi_1, 1/\lambda_f\psi_2)$.
      ii. If ψ_{λ_f} is real and $u(s)$ is an invertible linear fraction such that $\psi_{\lambda_f}(u(s))$ is real then return $(v(t), u(s))$.
4. If no factor f works then return *"The surface is not real"*.

Example 1 *Let $\mathcal{S}_{\mathbb{C}}$ be the classical revolution surface given by the parametrization*

$$\left(\frac{3-t^2}{4-2t}\frac{s^2-1}{s^2+1}, \frac{3-t^2}{4-2t}\frac{2s}{s^2+1}, \frac{-\mathtt{i}t^2+4\mathtt{i}t-3\mathtt{i}}{2t-4}\right)$$

If we take the ϕ-curve parametrized by

$$\left(\frac{3-t^2}{4-2t}, \frac{-\mathtt{i}t^2+4\mathtt{i}t-3\mathtt{i}}{2t-4}\right)$$

we obtain that we have to parametrize the circle $t_0^2+t_1^2-4t_0+3=0$ (and, thus, the given curve is real), yielding the associated unit

$$\xi(t) = (t+3\mathtt{i})/(t+\mathtt{i})$$

If we apply this unit to the original parametrization we get the following real parametrization of $\mathcal{S}_{\mathbb{C}}$:

$$\left(\frac{t^2+3}{t^2+1}\frac{s^2-1}{s^2+1}, \frac{t^2+3}{t^2+1}\frac{2s}{s^2+1}, \frac{2t}{t^2+1}\right)$$