Almost Vanishing Polynomials and an Application to Hough Transforms

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Introduction

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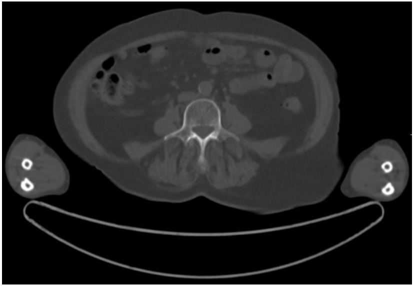
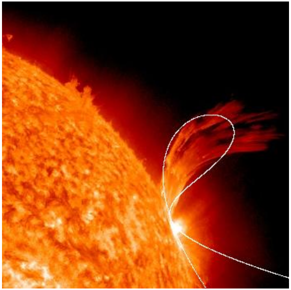
In the analysis of digital images, e.g. medical and astronomical images, the problem of automated recognition of special curves is very important.





The Hough Transform

The main tool is based on the Hough Transform technique.

- HT is a technique mainly used in image analysis, computer vision, and digital image processing.
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- The purpose of HT is to identify, in a given image, (approximate) instances of a certain class of shapes.
- Originally (1962, Hough) HT was concerned with identification of lines in images; later on (1972 Duda & Hart, 1981 Ballard) HT was extended to identify circles and ellipses; many refinements have been investigated since then.
- HT exploits the duality between image space and parameter space; result is achieved through a voting procedure in the parameter space.

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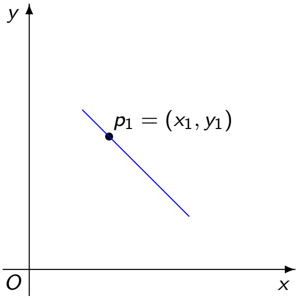

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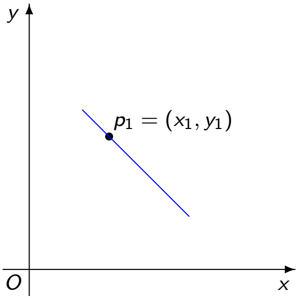
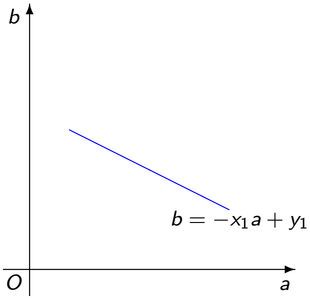


Parameter space



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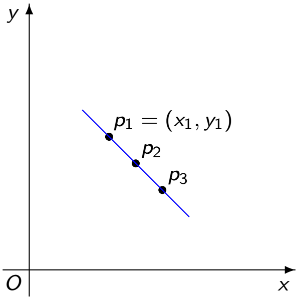

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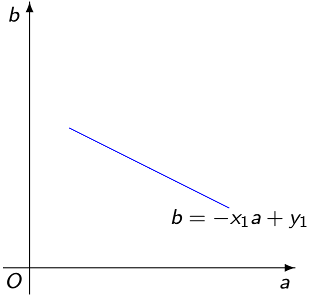

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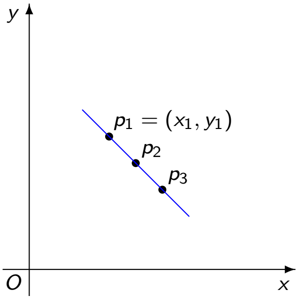

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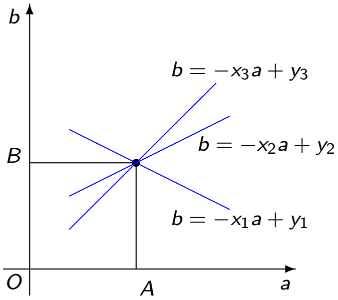

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  a straight line containing it satisfies $y_1 = ax_1 + b$.
- **MAIN IDEA:** move to the parameter space,
  so $y_1 = ax_1 + b_1$ is a straight line in this space.
- Repeat this process for every point $p_2, p_3, \dots$ in the picture.
- Let (A, B) be the intersection point of many lines, it means
  that the corresponding points in the picture lie on $y = Ax + B$!!!

Aligned Points

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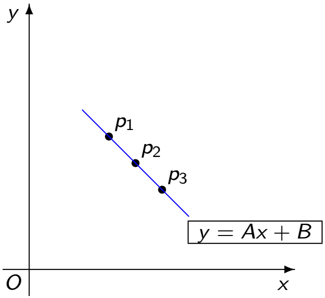

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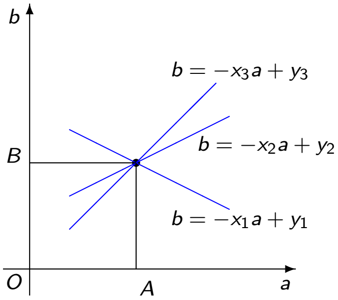

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- $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{a} = (a_1, \ldots, a_t)$;
- (Beltrametti, Robbiano 2012)
  $F(\mathbf{x}, \mathbf{a}) \in K[x_1, \ldots, x_n, a_1, \ldots, a_t] = K[\mathbf{x}, \mathbf{a}]$ such that
  for each $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_t) \in \mathbb{A}^t(K)$ (parameter space) and
  for each $p = (p_1, \ldots, p_n) \in \mathbb{A}^n(K)$ (image space) we have:

$$\mathcal{H}_{\boldsymbol{\alpha}} : F(\mathbf{x}, \boldsymbol{\alpha}) := f_{\boldsymbol{\alpha}}(\mathbf{x}) = 0 \quad \text{irreduc. hypersurface degree } d$$

$$\Gamma_p : F(p, \mathbf{a}) := \Gamma_p(\mathbf{a}) = 0 \quad \text{Hough transform of the point } p$$

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Proposition (Regularity Property): the following conditions are equivalent:

a) for any $\mathcal{H}_{\boldsymbol{\alpha}}, \mathcal{H}_{\boldsymbol{\alpha}'}$, we have $\mathcal{H}_{\boldsymbol{\alpha}} = \mathcal{H}_{\boldsymbol{\alpha}'} \Rightarrow \boldsymbol{\alpha} = \boldsymbol{\alpha}'$;

b) for any $\mathcal{H}_{\boldsymbol{\alpha}}$, we have $\bigcap_{p \in \mathcal{H}_{\boldsymbol{\alpha}}} \Gamma_p(\mathbf{a}) = \{\boldsymbol{\alpha}\}$.

Detection procedure

- Consider the case $n = 2$ (detection of curves in images).
- Let $\mathcal{F} = \{\mathcal{H}_\alpha\}$ be a suitable (irreducible, with fixed degree...) family of curves. Assume that Regularity Property (RP) holds.

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Recognition Algorithm

1. Choose a set $\mathbb{X} = \{p_1, \ldots, p_\nu\}$ of points of interest in $\mathbb{A}^2(\mathbb{R})$.
2. In $\mathbb{A}^t(\mathbb{R})$ find the (unique) intersection of the HT corresponding to the points p_i, that is compute $\{\boldsymbol{\alpha}\} = \bigcap_{i=1,\ldots,\nu} \Gamma_{p_i}(\mathbf{a})$.
3. Return the parameter $\boldsymbol{\alpha}$, and the curve $\mathcal{H}_{\boldsymbol{\alpha}}$ uniquely determined by $\boldsymbol{\alpha}$.

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Note that, in practice, $\{\boldsymbol{\alpha}\} = \bigcap_{i=1,\ldots,\nu} \Gamma_{p_i}(\mathbf{a})$ is **NOT CORRECT** since:

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- RP applies to *infinite* intersection;
- in general $\bigcap_{i=1,\dots,\nu} \Gamma_{p_i}(\mathbf{a}) = \emptyset$!!!!

Approximate computation of $\bigcap_{i=1,\dots,\nu} \Gamma_{p_i}(\mathbf{a})$

In practice, we want to compute an approximation of $\bigcap_{i=1,\dots,\nu} \Gamma_{p_i}(\mathbf{a})$, that is $\{\boldsymbol{\alpha}\} \approx \bigcap_{i=1,\dots,\nu} \Gamma_{p_i}(\mathbf{a})$. We proceed as follows:

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1. Consider a bounded region $\mathcal{T} = [a_1, b_1] \times \ldots \times [a_t, b_t] \subset \mathbb{A}^t(\mathbb{R})$ and a discretization step $d = (d_1, \ldots, d_t)$.
2. Construct d-discretization of $\mathcal{T} \Leftrightarrow$ multi-grid of size $J_1 \times \ldots \times J_t$. Construct the corresponding multi-matrix $\mathcal{A}$ (**accumulator function**) of size $J_1 \times \ldots \times J_t$. Initially $\mathcal{A}$ is zero.
3. For each $i = 1, \ldots, \nu$ and each $\mathbf{j} = (j_1, \ldots, j_t)$ assign to $\mathcal{A}(\mathbf{j})$

$$\mathcal{A}(\mathbf{j}) = \begin{cases} \mathcal{A}(\mathbf{j}) + 1 & \text{if} \quad \Gamma_{p_i}(\mathbf{a}) \cap \mathbf{C}(\mathbf{j}) \neq \emptyset \\ \mathcal{A}(\mathbf{j}) & \text{if} \quad \Gamma_{p_i}(\mathbf{a}) \cap \mathbf{C}(\mathbf{j}) = \emptyset \end{cases}$$

   where $\mathbf{C}(\mathbf{j})$ denotes the $\mathbf{j}$-th cell of the discretization of $\mathcal{T}$.
4. Find cell $\mathbf{C}(\mathbf{j}^*)$ such that $\mathcal{A}(\mathbf{j}^*) = \max_{\mathbf{j}} \mathcal{A}(\mathbf{j})$; return its center α.

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- let $\varepsilon_1, \ldots, \varepsilon_n$ be positive real numbers and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$;
- let $B(p)$ be the (∞, ε)-*unit ball centered at* p, that is:

  $$B(p) = \{q \in \mathbb{R}^n : \|(q-p)^t\|_{\infty,\mathcal{E}} \leq 1\}$$

  where $\mathcal{E} = \mathrm{diag}(1/\varepsilon_1, \ldots, 1/\varepsilon_\mathrm{n})$ and

  $$\|v\|_{\infty,\mathcal{E}} = \|\mathcal{E}v\|_\infty = \max_{i=1}^{n} |(\mathcal{E}v)_i|, \qquad \text{where } v \in \mathbb{R}^n.$$

- $B(p)$ represents the generic cell $\mathbf{C}(\mathbf{j})$ of the discretization.

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- The crossing criteria will depend on:
  - the tolerance ε (equivalently the discretization step);
  - the local differential geometry of $z = f(x_1, \ldots, x_n)$ in $\mathbb{A}^{n+1}(\mathbb{R})$.

Example 1

- Let $f(x,y) = x^2 + \frac{1}{100}y^2 - \frac{1}{100}$ and $p = (0,2)$.
- We have:

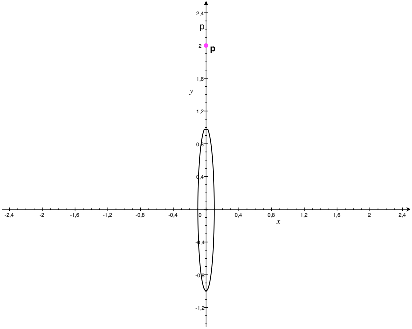

Figure : $x^2 + \frac{1}{100}y^2 - \frac{1}{100} = 0$ and point $p = (0,2)$

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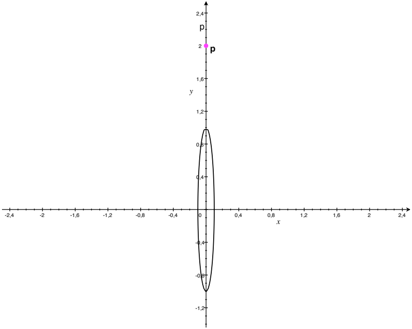

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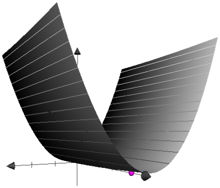

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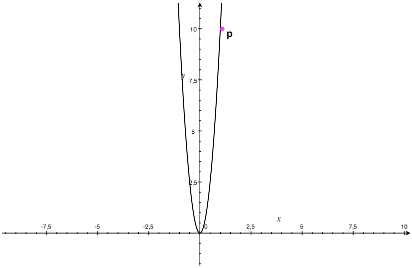
Figure : $z = x^2 + \frac{1}{100}y^2 - \frac{1}{100} = 0$ and point $p' = (0,2,0)$

Example 2

- Let $f(x, y) = y - 10x^2$ and $p = (1.1, 10)$.
- We have:

$$|f(p)| = 2.1 \text{ big}$$
$$\mathrm{dist}(p, \{f = 0\}) \approx 0.1$$



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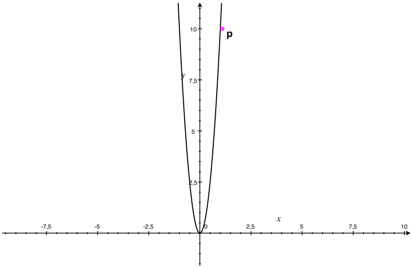

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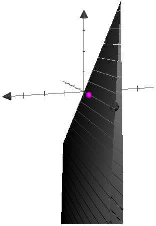

Figure : $z = y - 10x^2 = 0$ and point $p' = (1.1, 10, 0)$

Necessary crossing cell conditions

Prop. 1. Notation as above.
Let $\mathsf{H} = \max_{\mathbf{x}\in B(p)} \|H_f(\mathbf{x})\|_\infty$ and $\varepsilon_{\max} = \|\varepsilon\|_\infty$. If

$$|f(p)| > \|\,\mathrm{Jac}_f(p)\|_1 \varepsilon_{\max} + \frac{\mathsf{H}}{2}\varepsilon^2_{\max} \quad := \mathbf{B_1}$$

then the hypersurface of equation $f = 0$ does not cross $B(p)$.

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Prop. 2. Notation as above. If

$$|f(p)| > \|\operatorname{Jac}_f(p)\|_1 \varepsilon_{\max} + \frac{1}{2}\|H_f(p)\|_\infty \varepsilon_{\max}^2 \quad := \mathbf{B_1'}$$

then the hypersurface of equation $f = 0$ does not cross $B(p)$ (neglecting contributions of order $O(\varepsilon_{\max}^3)$).

Sufficient crossing cell conditions

Prop. 3. Notation as above.
Suppose that $\mathrm{Jac}_f(p)$ is not the zero vector. Let $0 < R < \min\{\varepsilon_{\min}, \frac{\|\mathrm{Jac}_f(p)\|_1}{H}\}$, $\mathrm{J} = \sup_{\mathbf{x}\in B(p,R)} \|\mathrm{Jac}_f^\dagger(\mathbf{x})\|_\infty$ and $c = \max\{2, \sqrt{n}\}$. If

$$|f(p)| < \frac{2R}{\mathrm{J}(c + \sqrt{n}\mathrm{HJ}R)} \quad := \mathbf{B_2}$$

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Prop. 4. Assumptions as in Prop. 3. Let $0 < R < \min\{\varepsilon_{\min}, \frac{\|\mathrm{Jac}_f(p)\|_1}{n^2\|H_f(p)\|_\infty}\}$ and $\Theta = \|\mathrm{Jac}_f^{\dagger}(p)\|_\infty + n^2(1+2\sqrt{n})\frac{\|H_f(p)\|_\infty}{\|\mathrm{Jac}_f(p)\|_1^2}R$. If

$$|f(p)| < \frac{2R}{\Theta(c + n^{5/2}\|H_f(p)\|_\infty \Theta R)} \quad := \mathbf{B}_2'$$

then the hypersurface of equation $f = 0$ crosses $B(p)$ (neglecting contributions of order $O(R^3)$).

Comparison among the bounds

Prop. 5. Notation as above. Let R be a real number s.t.

$$0 < R < \min\left\{\varepsilon_{\min}, \frac{\|\operatorname{Jac}_f(p)\|_1}{H}, \frac{\|\operatorname{Jac}_f(p)\|_1}{n^2\|H_f(p)\|_\infty}\right\}$$

Then:

$$B_2' + O(R^3) \leq B_2 < B_1' \leq B_1$$

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Then:

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We observe that:

- Prop. 5. yields $B_2 < B_1$, so it may happen that $|f(p)| \in (B_1, B_2)$. In this cases, using the previous results, nothing can be concluded regarding the crossing problem.
- Because of the local nature of the results, a more accurate analysis, by considering smaller subregions of $B(p)$, may overcome the problem.

The CROSSING CELL algorithm

Input: polynomial f, point p s.t. $\mathrm{Jac}_f(p) \neq \mathbf{0}$, tolerance $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$.

Output: an element of $\{0, 1, \xi\}$.

1. Compute $|f(p)|$, and the bounds B_1 and B_2.
2. If $|f(p)| > B_1$ return 0;
   if $|f(p)| > B_2$ return 1;
   else return ξ.

The CROSSING AREA algorithm

Input: polynomial f, region $\mathcal{T} = [a_1, b_1] \times \ldots \times [a_n, b_n]$, discretization step $d = (d_1, \ldots, d_n)$.

Output: multi-matrix $\mathcal{A}$ with values in $\{0, 1, \xi\}$.

1. Construct the discretization of $\mathcal{T}$ and the multi-matrix $\mathcal{A}$.
2. For each multi-index $\mathbf{j}$ set $\mathcal{A}(\mathbf{j}) = \text{CROSSING CELL}(f, \mathbf{x_j}, \frac{d}{2})$.
3. Return $\mathcal{A}$.

Algorithms III

The RECOGNITION algorithm

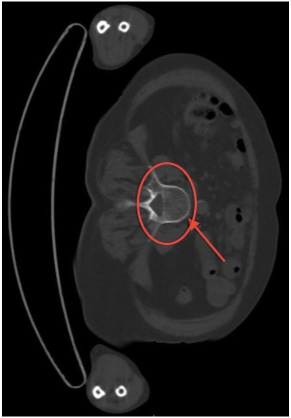
Input: regular family $\mathcal{F}$ of irreducible curves with same degree, set $\mathbb{X} = \{p_1, \ldots, p_\nu\}$ of points of interest, region $\mathcal{T} = [a_1, b_1] \times \ldots \times [a_n, b_n]$, discretization step $d = (d_1, \ldots, d_n)$.

Output: point $\mathbf{x}^*$.

1. For each $i = 1, \ldots, \nu$, set $\mathcal{A}_i =$CROSSING AREA$(\Gamma_{p_i}(\mathcal{F}), \mathcal{T}, d)$.
2. Compute $\mathcal{A} = \sum_{i=1,\ldots,\nu} \mathcal{A}_i$.
3. Compute $\mathbf{j}^* = \mathrm{argmax}_{\mathbf{j}} \mathcal{A}(\mathbf{j})$; return the point $\mathbf{x}^* = \mathbf{x}(\mathbf{j}^*)$.

Example I

- In this (toy) example we aim to detect the external profile of vertebral column using a rational cubic curve (Conchoid of Slüse).



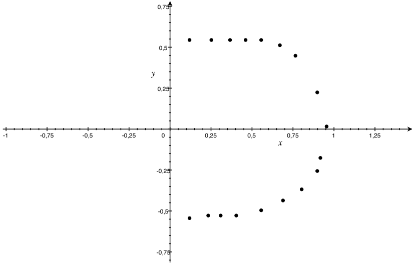


Figure : Points of interest

Example II

- The Conchoid of Slüse of parameters a, b has the following equation:

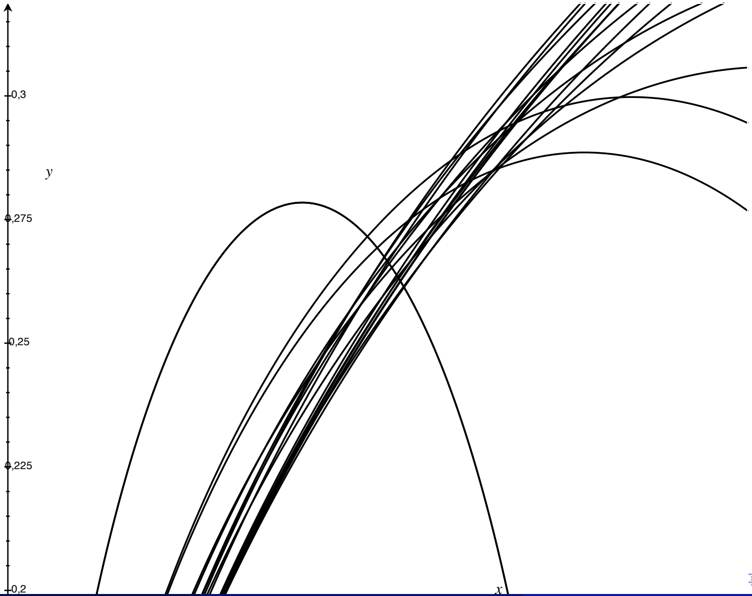
$$\mathcal{C}_{a,b} : a(x-a)(x^2+y^2) = b^2x^2$$

- For each $p = (x_p, y_p)$ in the image space, the HT is a conic in the parameter space $\mathbb{A}^2_{a,b}(\mathbb{R})$ whose equation is:

$$\Gamma_p(a,b) : (x_p^2+y_p^2)a^2 + x_p^2b^2 - x_p(x_p^2+y_p^2)a = 0$$

- $\Gamma_p(a,b)$ are represented in the following figure.

Example III



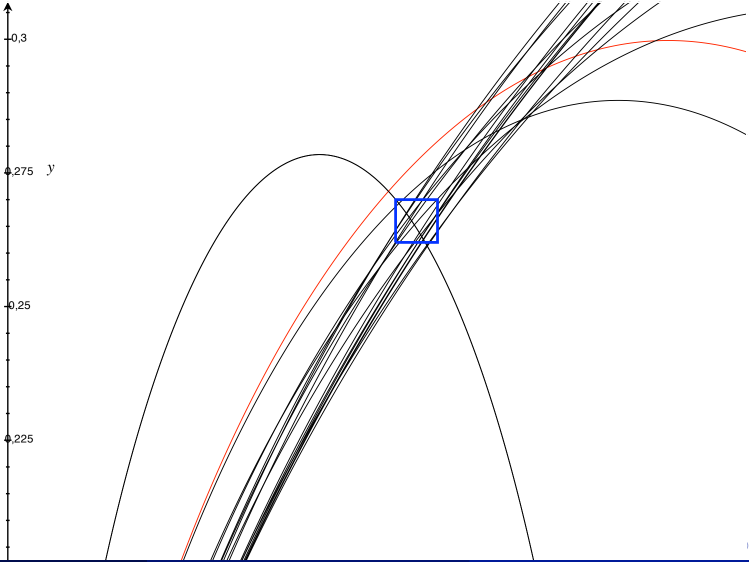
Example IV

- We consider the region $\mathcal{T} = [0.03, 0.5] \times [0.05, 0.4] \subset \mathbb{A}^2_{(a,b)}(\mathbb{R})$, and a discretization step $d = 2\varepsilon = (0.008, 0.008)$.
- Accumulator matrix $\mathcal{A} \in \mathrm{Mat}_{47\times 45}$ with values in $\{0, 1\}$
- The maximum value of $\mathcal{A}$ is **17** and corresponds to the cell centered in $(A, B) = (0.078, 0.266)$

$$\begin{pmatrix}
\dots & 0 & 0 & 0 & 0 & 7 & 12 & 14 & 8 & 2 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 14 & 11 & 5 & 3 & 2 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 10 & \mathbf{17} & 10 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 8 & 15 & 9 & 2 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 5 & 13 & 11 & 3 & 0 & 0 & 0 & \dots
\end{pmatrix}$$

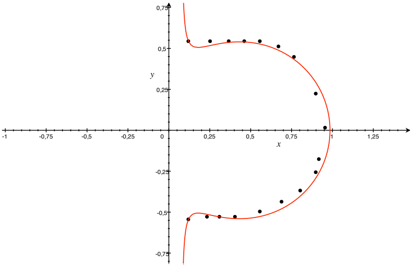
Example V

- The blue cell is the one which realizes the maximum for $\mathcal{A}$.



Example VI

- Finally, the detected curve is:



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Thank you!