Almost Vanishing Polynomials and an Application to Hough Transforms

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Introduction

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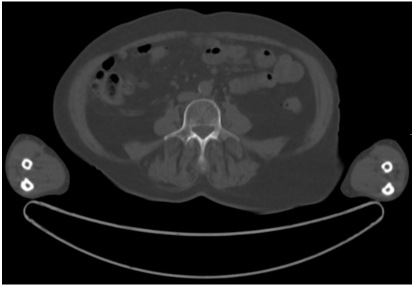
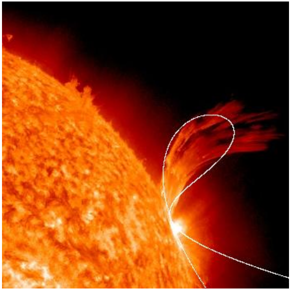
In the analysis of digital images, e.g. medical and astronomical images, the problem of automated recognition of special curves is very important.





The Hough Transform

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- Originally (1962, Hough) HT was concerned with identification of lines in images; later on (1972 Duda & Hart, 1981 Ballard) HT was extended to identify circles and ellipses; many refinements have been investigated since then.
- HT exploits the duality between image space and parameter space; result is achieved through a voting procedure in the parameter space.

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Aligned Points

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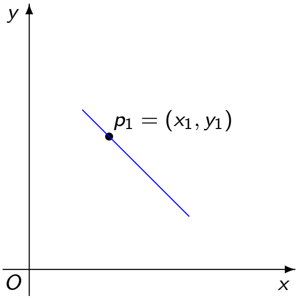

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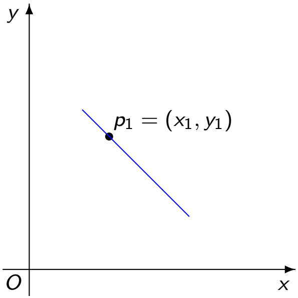
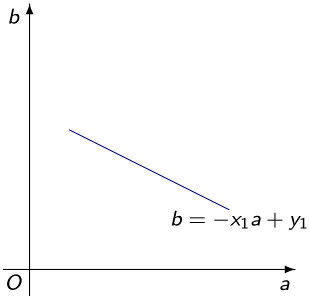


Parameter space



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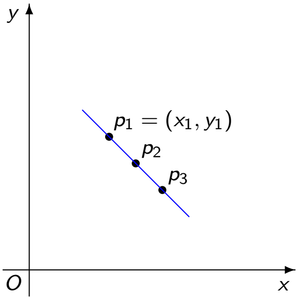

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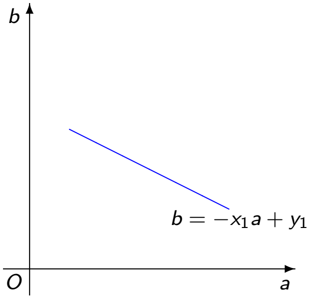

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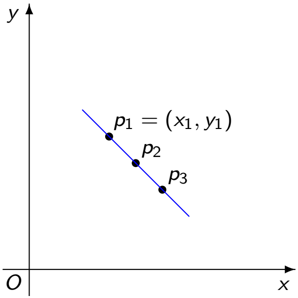

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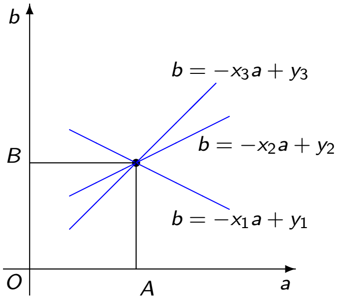

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  a straight line containing it satisfies $y_1 = ax_1 + b$.
- **MAIN IDEA:** move to the parameter space,
  so $y_1 = ax_1 + b_1$ is a straight line in this space.
- Repeat this process for every point $p_2, p_3, \ldots$ in the picture.
- Let (A, B) be the intersection point of many lines, it means
  that the corresponding points in the picture lie on $y = Ax + B$!!!

Aligned Points

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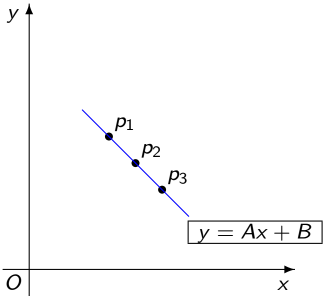

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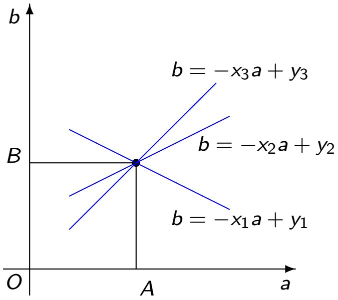

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- $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{a} = (a_1, \ldots, a_t)$;
- (Beltrametti, Robbiano 2012) $F(\mathbf{x}, \mathbf{a}) \in K[x_1, \ldots, x_n, a_1, \ldots, a_t] = K[\mathbf{x}, \mathbf{a}]$ such that for each $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_t) \in \mathbb{A}^t(K)$ (parameter space) and for each $p = (p_1, \ldots, p_n) \in \mathbb{A}^n(K)$ (image space) we have:

$$\mathcal{H}_{\boldsymbol{\alpha}} : F(\mathbf{x}, \boldsymbol{\alpha}) := f_{\boldsymbol{\alpha}}(\mathbf{x}) = 0 \quad \text{irreduc. hypersurface degree } d$$

$$\Gamma_p : F(p, \mathbf{a}) := \Gamma_p(\mathbf{a}) = 0 \quad \text{Hough transform of the point } p$$

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Proposition (Regularity Property): the following conditions are equivalent:

a) for any $\mathcal{H}_{\boldsymbol{\alpha}}, \mathcal{H}_{\boldsymbol{\alpha}'}$, we have $\mathcal{H}_{\boldsymbol{\alpha}} = \mathcal{H}_{\boldsymbol{\alpha}'} \Rightarrow \boldsymbol{\alpha} = \boldsymbol{\alpha}'$;

b) for any $\mathcal{H}_{\boldsymbol{\alpha}}$, we have $\bigcap_{p \in \mathcal{H}_{\boldsymbol{\alpha}}} \Gamma_p(\mathbf{a}) = \{\boldsymbol{\alpha}\}$.

Detection procedure

- Consider the case $n = 2$ (detection of curves in images).
- Let $\mathcal{F} = \{\mathcal{H}_{\alpha}\}$ be a suitable (irreducible, with fixed degree...) family of curves. Assume that Regularity Property (RP) holds.

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Recognition Algorithm

1. Choose a set $\mathbb{X} = \{p_1, \ldots, p_\nu\}$ of points of interest in $\mathbb{A}^2(\mathbb{R})$.
2. In $\mathbb{A}^t(\mathbb{R})$ find the (unique) intersection of the HT corresponding to the points p_i, that is compute $\{\alpha\} = \bigcap_{i=1,\ldots,\nu} \Gamma_{p_i}(\mathbf{a})$.
3. Return the parameter α, and the curve $\mathcal{H}_\alpha$ uniquely determined by α.

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- RP applies to *infinite* intersection;
- in general $\bigcap_{i=1,\ldots,\nu} \Gamma_{p_i}(\mathbf{a}) = \emptyset$!!!!

Approximate computation of $\bigcap_{i=1,\dots,\nu} \Gamma_{p_i}(\mathbf{a})$

In practice, we want to compute an approximation of $\bigcap_{i=1,\dots,\nu} \Gamma_{p_i}(\mathbf{a})$, that is $\{\boldsymbol{\alpha}\} \approx \bigcap_{i=1,\dots,\nu} \Gamma_{p_i}(\mathbf{a})$. We proceed as follows:

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1. Consider a bounded region $\mathcal{T} = [a_1, b_1] \times \ldots \times [a_t, b_t] \subset \mathbb{A}^t(\mathbb{R})$ and a discretization step $d = (d_1, \ldots, d_t)$.
2. Construct d-discretization of $\mathcal{T} \Leftrightarrow$ multi-grid of size $J_1 \times \ldots \times J_t$. Construct the corresponding multi-matrix $\mathcal{A}$ (**accumulator function**) of size $J_1 \times \ldots \times J_t$. Initially $\mathcal{A}$ is zero.
3. For each $i = 1, \ldots, \nu$ and each $\mathbf{j} = (j_1, \ldots, j_t)$ assign to $\mathcal{A}(\mathbf{j})$
$$\mathcal{A}(\mathbf{j}) = \begin{cases} \mathcal{A}(\mathbf{j}) + 1 & \text{if} \quad \Gamma_{p_i}(\mathbf{a}) \cap \mathbf{C}(\mathbf{j}) \neq \emptyset \\ \mathcal{A}(\mathbf{j}) & \text{if} \quad \Gamma_{p_i}(\mathbf{a}) \cap \mathbf{C}(\mathbf{j}) = \emptyset \end{cases}$$
where $\mathbf{C}(\mathbf{j})$ denotes the $\mathbf{j}$-th cell of the discretization of $\mathcal{T}$.
4. Find cell $\mathbf{C}(\mathbf{j}^*)$ such that $\mathcal{A}(\mathbf{j}^*) = \max_{\mathbf{j}} \mathcal{A}(\mathbf{j})$; return its center $\boldsymbol{\alpha}$.

The problem

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- let $\varepsilon_1, \ldots, \varepsilon_n$ be positive real numbers and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$;
- let $B(p)$ be the (∞, ε)-*unit ball centered at* p, that is:

$$B(p) = \{q \in \mathbb{R}^n : \|(q-p)^t\|_{\infty,\mathcal{E}} \leq 1\}$$

  where $\mathcal{E} = \mathrm{diag}(1/\varepsilon_1, \ldots, 1/\varepsilon_n)$ and

$$\|v\|_{\infty,\mathcal{E}} = \|\mathcal{E}v\|_{\infty} = \max_{i=1}^{n} |(\mathcal{E}v)_i|, \qquad \text{where } v \in \mathbb{R}^n.$$

- $B(p)$ represents the generic cell $\mathbf{C}(\mathbf{j})$ of the discretization.

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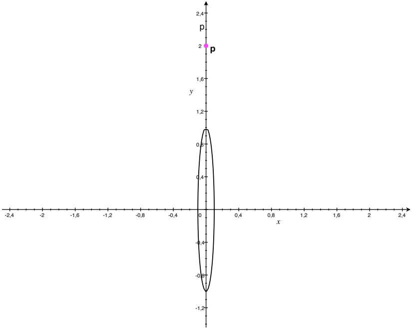
  - $|f(p)|$ large $\not\Rightarrow \{f = 0\} \cap B(p) = \emptyset$.
- The crossing criteria will depend on:
  - the tolerance ε (equivalently the discretization step);
  - the local differential geometry of $z = f(x_1, \ldots, x_n)$ in $\mathbb{A}^{n+1}(\mathbb{R})$.

Example 1

- Let $f(x,y) = x^2 + \frac{1}{100}y^2 - \frac{1}{100}$ and $p = (0,2)$.
- We have:



$$|f(p)| = 0.03 \text{ small}$$

$$\mathrm{dist}(p, \{f=0\}) = 1$$

Figure : $x^2 + \frac{1}{100}y^2 - \frac{1}{100} = 0$ and point $p = (0,2)$

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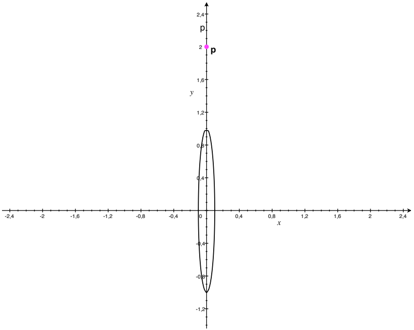

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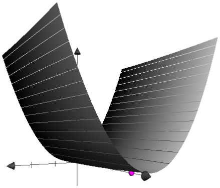
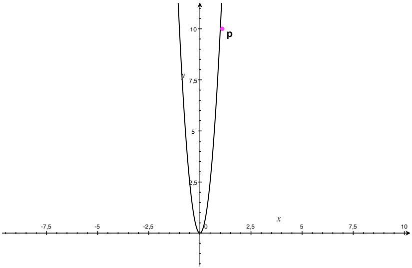


Figure : $z = x^2 + \frac{1}{100} y^2 - \frac{1}{100} = 0$ and point $p' = (0, 2, 0)$

Example 2

- Let $f(x, y) = y - 10x^2$ and $p = (1.1, 10)$.
- We have:

$$|f(p)| = 2.1 \text{ big}$$
$$\operatorname{dist}(p, \{f = 0\}) \approx 0.1$$



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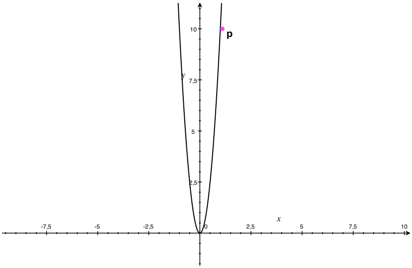

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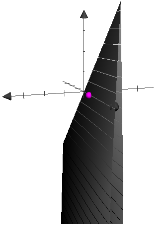

Figure : $z = y - 10x^2 = 0$ and point $p' = (1.1, 10, 0)$

Necessary crossing cell conditions

Prop. 1. Notation as above.
Let $\mathsf{H} = \max_{\mathbf{x} \in B(p)} \|H_f(\mathbf{x})\|_\infty$ and $\varepsilon_{\max} = \|\varepsilon\|_\infty$. If

$$|f(p)| > \|\operatorname{Jac}_f(p)\|_1 \varepsilon_{\max} + \frac{\mathsf{H}}{2}\varepsilon_{\max}^2 \quad := \mathbf{B_1}$$

then the hypersurface of equation $f = 0$ does not cross $B(p)$.

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Prop. 2. Notation as above. If

$$|f(p)| > \|\operatorname{Jac}_f(p)\|_1 \varepsilon_{\max} + \frac{1}{2}\|H_f(p)\|_\infty \varepsilon_{\max}^2 \quad := \mathbf{B_1'}$$

then the hypersurface of equation $f = 0$ does not cross $B(p)$ (neglecting contributions of order $O(\varepsilon_{\max}^3)$).

Sufficient crossing cell conditions

Prop. 3. Notation as above.
Suppose that $\mathrm{Jac}_f(p)$ is not the zero vector. Let $0 < R < \min\{\varepsilon_{\min}, \frac{\|\mathrm{Jac}_f(p)\|_1}{H}\}$, $\mathrm{J} = \sup_{\mathbf{x}\in B(p,R)} \|\mathrm{Jac}_f^{\dagger}(\mathbf{x})\|_\infty$ and $c = \max\{2, \sqrt{n}\}$. If

$$|f(p)| < \frac{2R}{\mathrm{J}(c + \sqrt{n}\mathrm{HJ}R)} \quad := \mathbf{B_2}$$

then the hypersurface of equation $f = 0$ crosses $B(p)$.

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Prop. 4. Assumptions as in Prop. 3. Let $0 < R < \min\{\varepsilon_{\min}, \frac{\|\mathrm{Jac}_f(p)\|_1}{n^2\|H_f(p)\|_\infty}\}$ and $\Theta = \|\mathrm{Jac}_f^\dagger(p)\|_\infty + n^2(1 + 2\sqrt{n})\frac{\|H_f(p)\|_\infty}{\|\mathrm{Jac}_f(p)\|_1^2}R$. If

$$|f(p)| < \frac{2R}{\Theta(c + n^{5/2}\|H_f(p)\|_\infty \Theta R)} \quad := \mathbf{B'_2}$$

then the hypersurface of equation $f = 0$ crosses $B(p)$ (neglecting contributions of order $O(R^3)$).

Comparison among the bounds

Prop. 5. Notation as above. Let R be a real number s.t.

$$0 < R < \min\left\{\varepsilon_{\min}, \frac{\|\operatorname{Jac}_f(p)\|_1}{H}, \frac{\|\operatorname{Jac}_f(p)\|_1}{n^2\|H_f(p)\|_\infty}\right\}$$

Then:

$$B_2' + O(R^3) \leq B_2 < B_1' \leq B_1$$

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Then:

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We observe that:

- Prop. 5. yields $B_2 < B_1$, so it may happen that $|f(p)| \in (B_1, B_2)$. In this cases, using the previous results, nothing can be concluded regarding the crossing problem.
- Because of the local nature of the results, a more accurate analysis, by considering smaller subregions of $B(p)$, may overcome the problem.

The CROSSING CELL algorithm

Input: polynomial f, point p s.t. $\mathrm{Jac}_f(p) \neq \mathbf{0}$, tolerance $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$.

Output: an element of $\{0, 1, \xi\}$.

1. Compute $|f(p)|$, and the bounds B_1 and B_2.
2. If $|f(p)| > B_1$ return 0;
   if $|f(p)| > B_2$ return 1;
   else return ξ.

Algorithms II

The CROSSING AREA algorithm

Input: polynomial f, region $\mathcal{T} = [a_1, b_1] \times \ldots \times [a_n, b_n]$, discretization step $d = (d_1, \ldots, d_n)$.

Output: multi-matrix $\mathcal{A}$ with values in $\{0, 1, \xi\}$.

1. Construct the discretization of $\mathcal{T}$ and the multi-matrix $\mathcal{A}$.
2. For each multi-index $\mathbf{j}$ set $\mathcal{A}(\mathbf{j}) = \text{CROSSING CELL}(f, \mathbf{x_j}, \frac{d}{2})$.
3. Return $\mathcal{A}$.

Algorithms III

The RECOGNITION algorithm

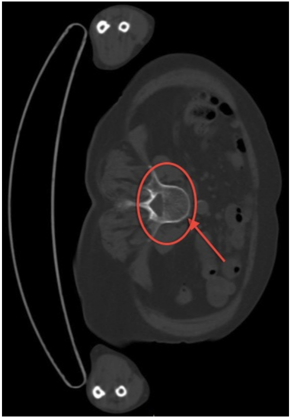
Input: regular family $\mathcal{F}$ of irreducible curves with same degree, set $\mathbb{X} = \{p_1, \ldots, p_\nu\}$ of points of interest, region $\mathcal{T} = [a_1, b_1] \times \ldots \times [a_n, b_n]$, discretization step $d = (d_1, \ldots, d_n)$.

Output: point $\mathbf{x}^*$.

1. For each $i = 1, \ldots, \nu$, set $\mathcal{A}_i =$CROSSING AREA$(\Gamma_{p_i}(\mathcal{F}), \mathcal{T}, d)$.
2. Compute $\mathcal{A} = \sum_{i=1,\ldots,\nu} \mathcal{A}_i$.
3. Compute $\mathbf{j}^* = \mathrm{argmax}_{\mathbf{j}} \mathcal{A}(\mathbf{j})$; return the point $\mathbf{x}^* = \mathbf{x}(\mathbf{j}^*)$.

Example I

- In this (toy) example we aim to detect the external profile of vertebral column using a rational cubic curve (Conchoid of Slüse).



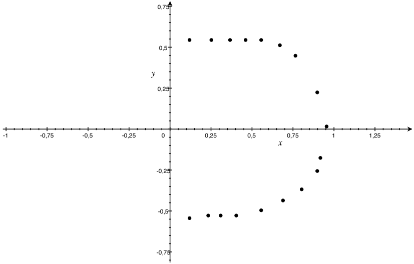


Figure : Points of interest

Example II

- The Conchoid of Slüse of parameters a, b has the following equation:

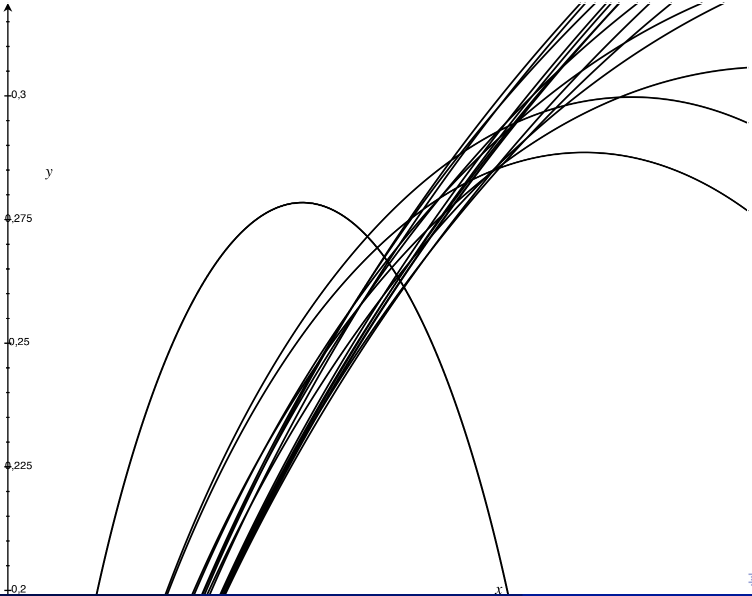
$$\mathcal{C}_{a,b} : a(x-a)(x^2+y^2) = b^2x^2$$

- For each $p = (x_p, y_p)$ in the image space, the HT is a conic in the parameter space $\mathbb{A}^2_{a,b}(\mathbb{R})$ whose equation is:

$$\Gamma_p(a,b) : (x_p^2+y_p^2)a^2 + x_p^2b^2 - x_p(x_p^2+y_p^2)a = 0$$

- $\Gamma_p(a,b)$ are represented in the following figure.

Example III



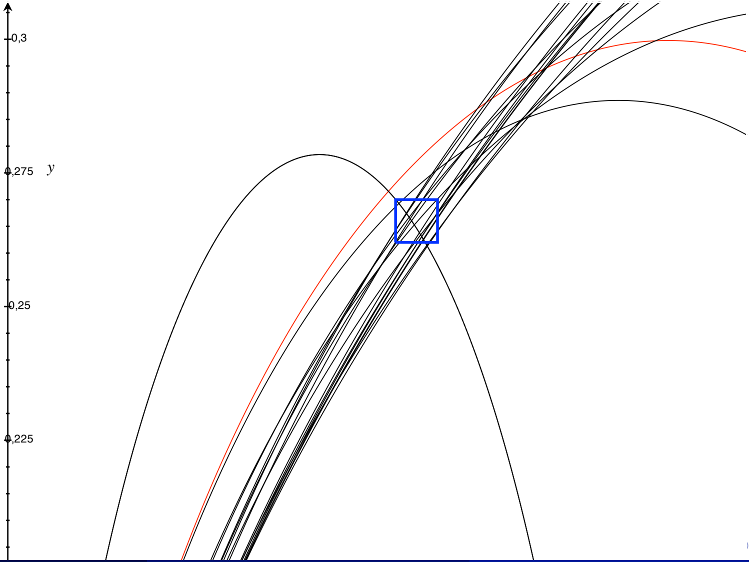
Example IV

- We consider the region $\mathcal{T} = [0.03, 0.5] \times [0.05, 0.4] \subset \mathbb{A}^2_{(a,b)}(\mathbb{R})$, and a discretization step $d = 2\varepsilon = (0.008, 0.008)$.
- Accumulator matrix $\mathcal{A} \in \mathrm{Mat}_{47\times 45}$ with values in $\{0, 1\}$
- The maximum value of $\mathcal{A}$ is **17** and corresponds to the cell centered in $(A, B) = (0.078, 0.266)$

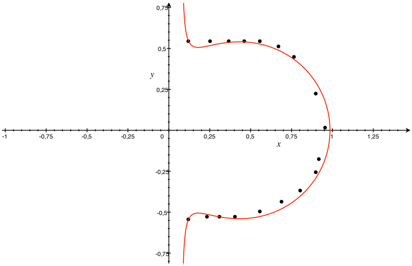
$$\begin{pmatrix}
\dots & 0 & 0 & 0 & 0 & 7 & 12 & 14 & 8 & 2 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 14 & 11 & 5 & 3 & 2 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 10 & 17 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 8 & 15 & 9 & 2 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 5 & 13 & 11 & 3 & 0 & 0 & 0 & \dots
\end{pmatrix}$$

Example V

- The blue cell is the one which realizes the maximum for $\mathcal{A}$.



- Finally, the detected curve is:



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Thank you!