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Workshop 3: Computer algebra and polynomials

# Fast Algorithms for Refined Parameterized Telescoping in Difference Fields

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FWF

Der Wissenschaftsfonds

Simplify

$$\sum_{k=1}^m H_k$$

with  $H_k = \sum_{i=1}^k \frac{1}{i}$

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Find  $g(k)$  s.t.,

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for all  $1 \leq k \leq m$  and  $m \geq 0$

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We compute

$$g(k) = (H_k - 1)k.$$

Simplify

$$\sum_{k=1}^m H_k$$

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Summing this equation over  $k$  from 1 to  $m$  gives

$$\begin{aligned} \sum_{k=1}^m H_k &= g(m+1) - g(1) \\ &= (H_{m+1} - 1)(m+1). \end{aligned}$$

## Telescoping in the given difference field

FIND a closed form for

$$\sum_{k=1}^m H_k.$$

### A difference field for the **summand**

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}$$

with the automorphism  $\sigma : \mathbb{F} \rightarrow \mathbb{F}$  defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

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$$\mathcal{S} k = k + 1,$$

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$$g(k + 1) - g(k) = H_k$$

with

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Hence,

$$(H_{m+1} - 1)(m + 1) = \sum_{k=1}^m H_k.$$

**CONSTRUCT** a difference field  $(\mathbb{F}, \sigma)$ :

- ▶ a rational function field

$$\mathbb{F} := \mathbb{K}$$

- ▶ with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

**CONSTRUCT** a difference field  $(\mathbb{F}, \sigma)$ :

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$$\mathbb{F} := \mathbb{K}(t_1)$$

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$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

$$\sigma(t_1) = a_1 t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \quad f_1 \in \mathbb{K}$$

**CONSTRUCT** a difference field  $(\mathbb{F}, \sigma)$ :

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$$\sigma(t_2) = a_2 t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, \quad f_2 \in \mathbb{K}(t_1)$$

**CONSTRUCT** a difference field  $(\mathbb{F}, \sigma)$ :

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$$\mathbb{F} := \mathbb{K}(t_1)(t_2) \dots (t_e)$$

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**GIVEN**  $f \in \mathbb{F}$ ;

**FIND**  $g \in \mathbb{F}$  such that

$$\sigma(g) - g = f.$$

**CONSTRUCT** a  $\Pi\Sigma^*$ -field  $(\mathbb{F}, \sigma)$ :

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such that

$$\text{const}_\sigma \mathbb{F} = \{c \in \mathbb{K}(t_1)(t_2) \dots (t_e) \mid \sigma(c) = c\} = \mathbb{K}.$$

**GIVEN**  $f \in \mathbb{F}$ ;

**FIND**  $g \in \mathbb{F}$  such that

Karr's algorithm (1981)

$$\sigma(g) - g = f.$$

## Simplified telescoping algorithm

FIND a closed form for

$$\sum_{k=1}^m H_k.$$

**A  $\Pi\Sigma^*$ -field for the summand**

$$\text{const}_\sigma \mathbb{F} = \mathbb{Q}$$

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(h)$$

with the automorphism  $\sigma : \mathbb{F} \rightarrow \mathbb{F}$  defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1,$$

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**Denominator bound:** COMPUTE a polynomial  $d \in \mathbb{Q}(k)[h]^*$ :

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FIND  $g' \in \mathbb{Q}(k)[h]$  with

$$\sigma\left(\frac{g'}{d}\right) - \frac{g'}{d} = h.$$

FIND  $g \in \mathbb{Q}(k)(h)$ :

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FIND  $g' \in \mathbb{Q}(k)[h]$  with

$$\frac{1}{\sigma(d)}\sigma(g') - \frac{1}{d}g' = \sigma\left(\frac{g'}{d}\right) - \frac{g'}{d} = h.$$

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**Degree bound:** COMPUTE  $m \geq 0$ :

$$\forall g \in \mathbb{Q}(k)[h] \quad \sigma(g) - g = h \Rightarrow \deg(g) \leq m.$$

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**Degree bound:** COMPUTE  $m \geq 0$ :

$$m = 2$$

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ANSATZ  $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]$

$$\left[ \sigma(g_2) \left( h + \frac{1}{k+1} \right)^2 + \sigma(g_1 h + g_0) \right] - [g_2 h^2 + g_1 h + g_0] = h$$



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coeff. comp.

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$$\sigma(g_1) - g_1 = 1 - c \frac{2}{k+1}$$

$$\begin{aligned} g_0 &= -k \\ d &= 0 \end{aligned}$$

$$\leftarrow \sigma(g_0) - g_0 = -1 - d \frac{1}{k+1}$$

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# First order difference equations in difference fields

Let  $(\mathbb{F}, \sigma)$  be a  $\Pi\Sigma$ -field with constant field  $\mathbb{K}$

## Telescoping

- ▶ Given  $f \in \mathbb{F}$
- ▶ Find  $g \in \mathbb{F}$ :

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## Parameterized Telescoping

- ▶ Given  $\mathbf{f} = (f_1, \dots, f_n) \in \mathbb{F}^n$
- ▶ Find  $\left\{ (c_1, \dots, c_n, g) \in \mathbb{K}^n \times \mathbb{F} \mid \right.$

$$\left. \sigma(g) - g = c_1 f_1 + \dots + c_n f_n \right\}$$

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## Parameterized first order linear difference equation

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Note:

- ▶  $V(\mathbf{a}, \mathbf{f}, \mathbb{F})$  is a subspace of  $\mathbb{K}^n \times \mathbb{F}$  over  $\mathbb{K}$

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- ▶  $\dim_{\mathbb{K}} V(\mathbf{a}, \mathbf{f}, \mathbb{F}) \leq n + 1$

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- ▶  $V(\mathbf{a}, \mathbf{f}, \mathbb{F})$  is a subspace of  $\mathbb{K}^n \times \mathbb{F}$  over  $\mathbb{K}$
- ▶  $\dim_{\mathbb{K}} V(\mathbf{a}, \mathbf{f}, \mathbb{F}) \leq n + 1$
- ▶ Task: Compute a basis of  $V(\mathbf{a}, \mathbf{f}, \mathbb{F})$

Find a basis of

$$V(\mathbf{a}, \mathbf{f}, \mathbb{G}(t))$$

$(\mathbb{G}(t), \sigma)$   $\Pi\Sigma^*$ -field

Find a basis of

$$V(\mathbf{a}, \mathbf{f}, \mathbb{G}(t))$$



Compute a denominator bound  
and search for polynomial solutions

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Compute a degree bound  $m$

$(\mathbb{G}(t), \sigma)$   $\Pi\Sigma^*$ -field

Find a basis of

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Compute a denominator bound  
and search for polynomial solutions



Compute a degree bound  $m$



Compute a basis of

$$V(\mathbf{a}_m, \mathbf{f}_m, \mathbb{G}[t]_m)$$

$(\mathbb{G}(t), \sigma)$   $\Pi\Sigma^*$ -field

$$\mathbb{G}[t]_m := \{p \in \mathbb{G}[t] \mid \deg(p) \leq m\}$$



Find a basis of

$$V(\mathbf{a}, \mathbf{f}, \mathbb{G}(t))$$

$(\mathbb{G}(t), \sigma)$   $\Pi\Sigma^*$ -field



Compute a denominator bound  
and search for polynomial solutions



Compute a degree bound  $m$



Compute a basis of

$$V(\mathbf{a}_m, \mathbf{f}_m, \mathbb{G}[t]_m)$$

coeff. comparison

→ Compute a basis of

$$V(\tilde{\mathbf{a}}_m, \tilde{\mathbf{f}}_m, \mathbb{G})$$

Find a basis of

$$V(\mathbf{a}, \mathbf{f}, \mathbb{G}(t))$$

$(\mathbb{G}(t), \sigma)$   $\Pi\Sigma^*$ -field



Compute a denominator bound  
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Compute a degree bound  $m$



Compute a basis of

$$V(\mathbf{a}_m, \mathbf{f}_m, \mathbb{G}[t]_m)$$

coeff. comparison

Compute a basis of

$$V(\tilde{\mathbf{a}}_m, \tilde{\mathbf{f}}_m, \mathbb{G})$$

Plugin highest term

Compute a basis of

$$V(\mathbf{a}_{m-1}, \mathbf{f}_{m-1}, \mathbb{G}[t]_{m-1})$$

Find a basis of

$$V(\mathbf{a}, \mathbf{f}, \mathbb{G}(t))$$

$(\mathbb{G}(t), \sigma)$   $\Pi\Sigma^*$ -field



Compute a denominator bound  
and search for polynomial solutions



Compute a degree bound  $m$



Compute a basis of

$$V(\mathbf{a}_m, \mathbf{f}_m, \mathbb{G}[t]_m)$$

coeff. comparison

Compute a basis of

Combine solution

$$V(\tilde{\mathbf{a}}_m, \tilde{\mathbf{f}}_m, \mathbb{G})$$

Plugin highest term

Compute a basis of

$$V(\mathbf{a}_{m-1}, \mathbf{f}_{m-1}, \mathbb{G}[t]_{m-1})$$

Find a basis of

$$V(\mathbf{a}, \mathbf{f}, \mathbb{G}(t))$$

 $(\mathbb{G}(t), \sigma)$   $\Pi\Sigma^*$ -field

Compute a denominator bound  
and search for polynomial solutions



Compute a degree bound  $m$



Compute a basis of

$$V(\mathbf{a}_m, \mathbf{f}_m, \mathbb{G}[t]_m)$$

coeff. comparison

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$$V(\tilde{\mathbf{a}}_m, \tilde{\mathbf{f}}_m, \mathbb{G})$$

Combine solution

Plugin highest term

Compute a basis of

$$V(\mathbf{a}_{m-1}, \mathbf{f}_{m-1}, \mathbb{G}[t]_{m-1})$$

coeff. comparison

Compute a basis of

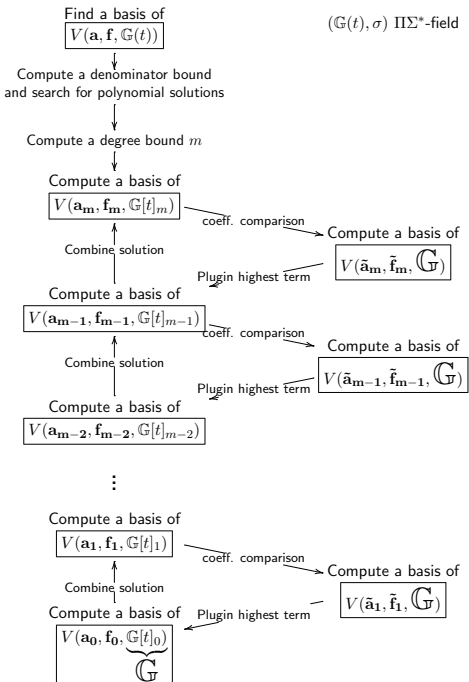
$$V(\tilde{\mathbf{a}}_{m-1}, \tilde{\mathbf{f}}_{m-1}, \mathbb{G})$$

Combine solution

Plugin highest term

Compute a basis of

$$V(\mathbf{a}_{m-2}, \mathbf{f}_{m-2}, \mathbb{G}[t]_{m-2})$$

Simplified version of  
Karr's algorithm

degree reduction  
by solving problems in  $\mathbb{G}$

## Example 2

FIND  $g(k)$ :

$$g(k+1) - g(k) = \frac{H_k}{k}$$

for all  $1 \leq k \leq m$  and  $m \geq 0$ .

Find  $g \in \mathbb{Q}(k)(h)$  s.t.

$$\sigma(g) - g = \frac{h}{k}$$

Find  $g \in \mathbb{Q}(k)[h]$  s.t.

$$\sigma(g) - g = \frac{h}{k}$$



Find  $g = g_2h^2 + g_1h + g_0 \in \mathbb{Q}(k)[h]_2$  s.t.

$$\sigma(g) - g = \frac{h}{k}$$

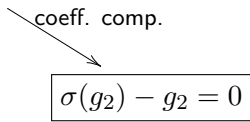
Find  $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$  s.t.

$$\sigma(g) - g = \frac{h}{k}$$

$$\begin{aligned} & [\sigma(g_2) \left(h + \frac{1}{k+1}\right)^2 + \sigma(g_1 h + g_0)] \\ & - [g_2 h^2 + g_1 h + g_0] = \frac{h}{k} \end{aligned}$$

Find  $g = g_2h^2 + g_1h + g_0 \in \mathbb{Q}(k)[h]_2$  s.t.  
$$\sigma(g) - g = \frac{h}{k}$$

$$\left[ \sigma(g_2) \left( h + \frac{1}{k+1} \right)^2 + \sigma(g_1h + g_0) \right] - [g_2h^2 + g_1h + g_0] = \frac{h}{k}$$

coeff. comp.  


$$\sigma(g_2) - g_2 = 0$$

Find  $g = g_2h^2 + g_1h + g_0 \in \mathbb{Q}(k)[h]_2$  s.t.  

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coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

$$g_2 = c \in \mathbb{Q}$$

Find  $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$  s.t.  

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coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

$$g_2 = c \in \mathbb{Q}$$

$$\sigma(g_1 h + g_0) - (g_1 h + g_0) = \frac{h}{k} + c \left[ \frac{-2h(k+1) - 1}{(k+1)^2} \right]$$

Find  $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$  s.t.  

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coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1 h + g_0) - (g_1 h + g_0) = \frac{h}{k} + c \left[ \frac{-2h(k+1)-1}{(k+1)^2} \right]$$

coeff. comp.

$$\sigma(g_1) - g_1 = \frac{1}{k} + c \frac{-2}{k+1}$$

Find  $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$  s.t.  

$$\sigma(g) - g = \frac{h}{k}$$

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coeff. comp.

$$\sigma(g_1) - g_1 = \frac{1}{k} + c \frac{-2}{k+1}$$

$$c = \frac{1}{2}, \quad g_1 = -\frac{1}{k} + d$$

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coeff. comp.

$$\sigma(g_1) - g_1 = \frac{1}{k} + c \frac{-2}{k+1}$$

$$\sigma(g_0) - g_0 = \frac{2}{(k+1)^2} + d \frac{-1}{k+1}$$

$$c = \frac{1}{2}, \quad g_1 = -\frac{1}{k} + d$$



Find  $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$  s.t.

$$\sigma(g) - g = \frac{h}{k}$$

$$\left[ \sigma(g_2) \left( h + \frac{1}{k+1} \right)^2 + \sigma(g_1 h + g_0) \right] - [g_2 h^2 + g_1 h + g_0] = \frac{h}{k}$$

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coeff. comp.

$$\sigma(g_1) - g_1 = \frac{1}{k} + c \frac{-2}{k+1}$$

no

solution

$$\leftarrow \sigma(g_0) - g_0 = \frac{2}{(k+1)^2} + d \frac{-1}{k+1}$$

$$c = \frac{1}{2}, \quad g_1 = -\frac{1}{k} + d$$

FIND  $g(k)$ :

$$g(k+1) - g(k) = \frac{H_k}{k}$$

for all  $1 \leq k \leq m$  and  $m \geq 0$ .

No solution 😞

## Refined telescoping

FIND  $g(k), \phi(k)$ :

$$g(k+1) - g(k) + \phi(k) = \frac{H_k}{k}$$

for all  $1 \leq k \leq m$  and  $m \geq 0$ .

Find  $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$  s.t.

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$$\left[ \sigma(g_2) \left( h + \frac{1}{k+1} \right)^2 + \sigma(g_1 h + g_0) \right] - [g_2 h^2 + g_1 h + g_0] = \frac{h}{k}$$

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$$\sigma(g_1) - g_1 = \frac{1}{k} + c \frac{-2}{k+1}$$

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$$\leftarrow \sigma(g_0) - g_0 = \frac{2}{(k+1)^2} + d \frac{-1}{k+1}$$

$$c = \frac{1}{2}, \quad g_1 = -\frac{1}{k} + d$$

Find  $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$  s.t.  

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$$\sigma(g_1) - g_1 = \frac{1}{k} + c \frac{-2}{k+1}$$

$$\sigma(g_0) - g_0 = \frac{2}{(k+1)^2} + d \frac{-1}{k+1}$$

$$c = \frac{1}{2}, \quad g_1 = -\frac{1}{k} + d$$

$$\sigma(0) - 0 + \frac{2}{(k+1)^2} = \frac{2}{(k+1)^2} + 0 \frac{-1}{k+1}$$

Find  $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$  s.t.

$$\sigma(g) - g + \frac{1}{2(k+1)^2} = \frac{h}{k}$$

$$g = \frac{1}{2}h + \frac{1}{k}$$

$$\left[ \sigma(g_2) \left( h + \frac{1}{k+1} \right)^2 + \sigma(g_1 h + g_0) \right] - [g_2 h^2 + g_1 h + g_0] = \frac{h}{k}$$

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## Refined telescoping

FIND  $g(k), \phi(k)$ :

$$g(k+1) - g(k) + \phi(k) = \frac{H_k}{k}$$

for all  $1 \leq k \leq m$  and  $m \geq 0$ .

We compute

$$g(k) = \frac{1}{2}H_k + \frac{1}{k}$$

$$\phi(k) = \frac{1}{2(k+1)^2}$$

## Refined telescoping

FIND  $g(k), \phi(k)$ :

$$g(k+1) - g(k) + \phi(k) = \frac{H_k}{k}$$

for all  $1 \leq k \leq m$  and  $m \geq 0$ .

Summing this equation over  $k$  from 1 to  $m$  gives

$$\sum_{k=1}^m \frac{H_k}{k} = g(m+1) - g(1) + \sum_{k=1}^m \phi(k)$$

$$= \frac{1}{2} \left( H_m^2 + \sum_{k=1}^m \frac{1}{k^2} \right).$$



## Parameterized telescoping

Given  $\Pi\Sigma^*$ -field  $(\mathbb{K}(t_1) \dots (t_e), \sigma)$ .

Define  $\mathbb{F}_i = \mathbb{K}(t_1) \dots (t_i)$ , i.e.,

$$\mathbb{F}_0 < \mathbb{F}_1 < \dots < \mathbb{F}_i < \dots < \mathbb{F}_e$$

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$$\underbrace{\mathbb{F}_{-1}}_{=\{0\}} \subset \mathbb{F}_0 < \mathbb{F}_1 < \dots < \mathbb{F}_i < \dots < \mathbb{F}_e$$

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Given  $f_1, \dots, f_n \in \mathbb{F}_e$ .

Find  $c_1, \dots, c_n \in \mathbb{K}$ ,  $g \in \mathbb{F}_e$ :

$$\sigma(g) - g = c_1 f_1 + \dots + c_n f_n$$

Special cases are

- ▶ telescoping ( $n = 1$ )
- ▶ creative telescoping (take  $f_i = F(r - 1 + i, k)$ )

## First entry telescoping

Given  $\Pi\Sigma^*$ -field  $(\mathbb{K}(t_1) \dots (t_e), \sigma)$ .

Define  $\mathbb{F}_i = \mathbb{K}(t_1) \dots (t_i)$ , i.e.,

$$\underbrace{\mathbb{F}_{-1}}_{=\{0\}} \subset \mathbb{F}_0 < \mathbb{F}_1 < \dots < \mathbb{F}_i < \dots < \mathbb{F}_e$$

Given  $f_1, \dots, f_n \in \mathbb{F}_e$ .

Find  $c_1, \dots, c_n \in \mathbb{K}$ ,  $g \in \mathbb{F}_e$ :

$$\sigma(g) - g = \underbrace{c_1}_{\neq 0} f_1 + \dots + c_n f_n$$

Note: Gives rise to faster algorithms

Find  $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$  s.t.

$$\sigma(g) - g = \frac{h}{k}$$

$$\left[ \sigma(g_2) \left( h + \frac{1}{k+1} \right)^2 + \sigma(g_1 h + g_0) \right] - [g_2 h^2 + g_1 h + g_0] = \frac{h}{k}$$

coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

$$g_2 = c \in \mathbb{Q}$$

$$\sigma(g_1 h + g_0) - (g_1 h + g_0) = \frac{h}{k} + c \left[ \frac{-2h(k+1) - 1}{(k+1)^2} \right]$$

coeff. comp.

$$\sigma(g_1) - g_1 = \frac{1}{k} + c \frac{-2}{k+1}$$

no

solution

$$\leftarrow \sigma(g_0) - g_0 = \frac{2}{(k+1)^2} + d \frac{-1}{k+1}$$

$$c = \frac{1}{2}, \quad g_1 = -\frac{1}{k} + d$$

## First entry telescoping

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Given  $f_1, \dots, f_n \in \mathbb{F}_e$ .

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Note: Gives rise to faster algorithms

## Refined telescoping

Given  $\Pi\Sigma^*$ -field  $(\mathbb{K}(t_1) \dots (t_e), \sigma)$ .

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$$\underbrace{\mathbb{F}_{-1}}_{=\{0\}} \subset \mathbb{F}_0 < \mathbb{F}_1 < \dots < \mathbb{F}_i < \dots < \mathbb{F}_e$$

Given  $f_1, \dots, f_n \in \mathbb{F}_e$ .

Find  $c_1, \dots, c_n \in \mathbb{K}$ ,  $g \in \mathbb{F}_e$

$\phi \in \mathbb{F}_i$  with  $i$  minimal :

$$\sigma(g) - g + \boxed{\phi} = \underbrace{c_1}_{\neq 0} f_1 + \dots + c_n f_n$$

Find  $c_j \in \mathbb{K}$ ,  $g \in \mathbb{F}_e$ ,  $\phi \in \mathbb{F}_e$  s.t.

$$\sigma(g) - g + \phi = \underbrace{c_1}_{\neq 0} f_1 + \cdots + c_n f_n.$$

$$V((-1, 1), (f_1, \dots, f_n), \mathbb{F}_e)$$



$$V((-1, 1), (f_1^{(e-1)}, \dots, f_{n_{e-1}}^{(e-1)}), \mathbb{F}_{e-1})$$



Find  $g = g_2h^2 + g_1h + g_0 \in \mathbb{Q}(k)[h]_2$  s.t.

$$\sigma(g) - g + \frac{1}{2(k+1)^2} = \frac{h}{k}$$

$$g = \frac{1}{2}h + \frac{1}{k}$$

$$V\left(\left(-1, 1\right), \left(\frac{h}{k}\right), \mathbb{Q}(k)(h)\right)$$

coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

$$g_2 = c \in \mathbb{Q}$$

coeff. comp.

$$\sigma(g_1) - g_1 = \frac{1}{k} + c \frac{-2}{k+1}$$

$$V\left(\left(-1, 1\right), \left(\frac{2}{(k+1)^2}, \frac{-1}{k+1}\right), \mathbb{Q}(k)\right)$$

Find  $c_j \in \mathbb{K}$ ,  $g \in \mathbb{F}_e$ ,  $\phi \in \mathbb{F}_e$  s.t.

$$\sigma(g) - g + \phi = \underbrace{c_1}_{\neq 0} f_1 + \cdots + c_n f_n.$$

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Find  $c_j \in \mathbb{K}$ ,  $g \in \mathbb{F}_e$ ,  $\phi \in \mathbb{F}_e$  s.t.

$$\sigma(g) - g + \phi = \underbrace{c_1}_{\neq 0} f_1 + \cdots + c_n f_n.$$

$$V((-1, 1), (f_1, \dots, f_n), \mathbb{F}_e)$$



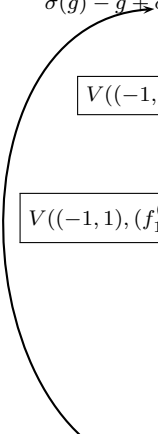
$$V((-1, 1), (f_1^{(e-1)}, \dots, f_{n_{e-1}}^{(e-1)}), \mathbb{F}_{e-1})$$



⋮



$$V((-1, 1), (f_1^{(i)}, \dots, f_{n_i}^{(i)}), \mathbb{F}_i)$$



Find  $c_j \in \mathbb{K}$ ,  $g \in \mathbb{F}_e$ ,  $\phi \in \mathbb{F}_e$  s.t.

$$\sigma(g) - g + \phi = \underbrace{c_1}_{\neq 0} f_1 + \cdots + c_n f_n.$$

$$V((-1, 1), (f_1, \dots, f_n), \mathbb{F}_e)$$



$$V((-1, 1), (f_1^{(e-1)}, \dots, f_{n_{e-1}}^{(e-1)}), \mathbb{F}_{e-1})$$



⋮



$$V((-1, 1), (f_1^{(i)}, \dots, f_{n_i}^{(i)}), \mathbb{F}_i)$$

Take  $\bar{c}_j \in \mathbb{K}$ ,  $\bar{g} \in \mathbb{F}_e$ ,  $\bar{\phi} \in \mathbb{F}_{i-1}$  s.t.

$$\sigma(\bar{g}) - \bar{g} + \bar{\phi} = \underbrace{\bar{c}_1}_{\neq 0} f_1 + \cdots + \bar{c}_n f_n.$$

Find  $c_j \in \mathbb{K}$ ,  $g \in \mathbb{F}_e$ ,  $\phi \in \mathbb{F}_e$  s.t.

$$\sigma(g) - g + \phi = \underbrace{c_1}_{\neq 0} f_1 + \cdots + c_n f_n.$$

$$V((-1, 1), (f_1, \dots, f_n), \mathbb{F}_e)$$



$$V((-1, 1), (f_1^{(e-1)}, \dots, f_{n_{e-1}}^{(e-1)}), \mathbb{F}_{e-1})$$



⋮



$$V((-1, 1), (f_1^{(i)}, \dots, f_{n_i}^{(i)}), \mathbb{F}_i)$$

Take  $\bar{c}_j \in \mathbb{K}$ ,  $\bar{g} \in \mathbb{F}_e$ ,  $\bar{\phi} \in \mathbb{F}_{i-1}$  s.t.

$$\sigma(\bar{g}) - \bar{g} = \underbrace{\bar{c}_1}_{\neq 0} f_1 + \cdots + \bar{c}_n f_n - \bar{\phi}.$$

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$$V((-1, 1), (f_1, \dots, f_n), \mathbb{F}_e)$$



$$V((-1, 1), (f_1^{(e-1)}, \dots, f_{n_{e-1}}^{(e-1)}), \mathbb{F}_{e-1})$$



⋮



$$V((-1, 1), (f_1^{(i)}, \dots, f_{n_i}^{(i)}), \mathbb{F}_i)$$

Take  $\bar{c}_j \in \mathbb{K}$ ,  $\bar{g} \in \mathbb{F}_e$ ,  $\bar{\phi} \in \mathbb{F}_{i-1}$  s.t.

$$\sigma(\bar{g}) - \bar{g} = \underbrace{\bar{c}_1}_{\neq 0} f_1 + \cdots + \bar{c}_n f_n - \bar{\phi}.$$

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## Refined telescoping

Given  $\Pi\Sigma^*$ -field  $(\mathbb{K}(t_1) \dots (t_e), \sigma)$ .

Define  $\mathbb{F}_i = \mathbb{K}(t_1) \dots (t_i)$ , i.e.,

$$\underbrace{\mathbb{F}_{-1}}_{=\{0\}} \subset \mathbb{F}_0 < \mathbb{F}_1 < \dots < \mathbb{F}_i < \dots < \mathbb{F}_e$$

Given  $f_1, \dots, f_n \in \mathbb{F}_e$ .

Find  $c_1, \dots, c_n \in \mathbb{K}$ ,  $g \in \mathbb{F}_e$

$\phi \in \mathbb{F}_i$  with  $i$  minimal :

$$\sigma(g) - g + \boxed{\phi} = \underbrace{c_1}_{\neq 0} f_1 + \dots + c_n f_n$$

## Refined telescoping

Given  $\Pi\Sigma^*$ -extension  $(\mathbb{G}(t_1) \dots (t_e), \sigma)$  of a difference field  $(\mathbb{G}, \sigma)$ .

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Requirements: an algorithm to solve first order linear difference equations in  $(\mathbb{G}, \sigma)$  plus technical conditions.

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Example 1:  $\mathbb{G} = \mathbb{K}$

## Examples

$$\sum_{k=1}^m (k^2 + 1) k! H_k^2 = \sum_{k=1}^m \frac{(k+1)!}{k^2} + m(m+1)! H_m^2 - 2m! H_m.$$

$$\sum_{k=1}^m k \sum_{j=1}^k \frac{H_j}{j^2} = \frac{1}{2} \left( \sum_{k=1}^m \frac{1}{k^2} + m(m+1) \sum_{i=1}^m \frac{H_i}{i^2} + H_m^2 - (m+1)H_m + m \right)$$

$$\begin{aligned} \sum_{k=1}^m \left( \sum_{j=1}^k \binom{r}{j} \right)^2 &= - \frac{1}{2} r \sum_{k=1}^m \binom{r}{k}^2 + \frac{1}{2} (2m - r + 2) \left( \sum_{i=1}^m \binom{r}{i} \right)^2 \\ &\quad - (m - r) \binom{r}{m} \sum_{i=1}^m \binom{r}{i} \end{aligned}$$

## Refined telescoping

Given  $\Pi\Sigma^*$ -extension  $(\mathbb{G}(t_1) \dots (t_e), \sigma)$  of a difference field  $(\mathbb{G}, \sigma)$ .

Define  $\mathbb{F}_i = \mathbb{G}(t_1) \dots (t_i)$ , i.e.,

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$\phi \in \mathbb{F}_i$  with  $i$  minimal :

$$\sigma(g) - g + \boxed{\phi} = \underbrace{c_1}_{\neq 0} f_1 + \dots + c_n f_n$$

Example 2 (free difference field):  $\mathbb{G} = \mathbb{K}(\dots, x_{-1}, x_0, x_1, \dots)$   
with  $\sigma(x_i) = x_{i+1}$ .

# Unspecified sequences (M. Kauers, CS; 2006)

$$\sum_{k=1}^n a^k \sum_{j=1}^k X_j = \frac{a^{n+1} \sum_{k=1}^n X_k - \sum_{k=1}^n a^k X_k}{a - 1}, \quad a \neq 1$$

$$\begin{aligned} \sum_{k=0}^n k^2 \sum_{i=0}^k X_i &= \frac{1}{6} \left( n(n+1)(2n+1) \sum_{k=0}^n X_k - \sum_{k=0}^n k X_k \right. \\ &\quad \left. + 3 \sum_{k=0}^n k^2 X_k - 2 \sum_{k=0}^n k^3 X_k \right) \end{aligned}$$



## Refined telescoping

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Define  $\mathbb{F}_i = \mathbb{G}(t_1) \dots (t_i)$ , i.e.,

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$\phi \in \mathbb{F}_i$  with  $i$  minimal :

$$\sigma(g) - g + \boxed{\phi} = \underbrace{c_1}_{\neq 0} f_1 + \dots + c_n f_n$$

Example 3 (radicals of order  $d > 0$ ):  $\mathbb{G} = \mathbb{K}(k)(\dots, x_{-1}, x_0, x_1, \dots)$  with  $\sigma(k) = k + 1$  and  $\sigma(x_i) = x_{i+1}$  subject to the relation  $x_i^d = k$ .

# Radical expressions (M. Kauers, CS; 2007)

$$\sum_{k=0}^n \frac{1}{\sqrt{k+1} + \sqrt{k}} = \sqrt{n+1}$$

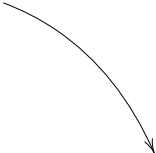
$$\begin{aligned} \sum_{k=2}^n H_k \left( \frac{(\sqrt{k})^3}{k-1} + \sum_{i=1}^k \frac{\sqrt{i}}{i + \sqrt{i}} \right) &= \frac{1}{2} \left( -3 - 5n + (5n + 3)H_n \right. \\ &\quad \left. - (2n + 1)H_n^2 + H_n^{(2)} \right) + \sum_{k=1}^n \sqrt{k} \\ &\quad + \left( (n + 1)H_n - (n - 1) \right) \sum_{k=2}^n \frac{\sqrt{k}}{k-1} \end{aligned}$$

# Conclusion

simplified version of  
Karr's algorithm

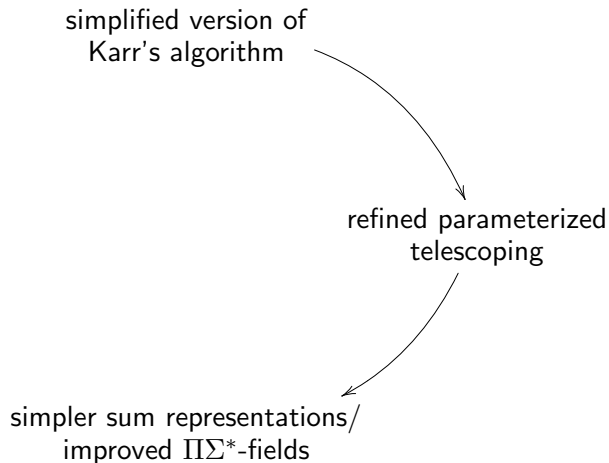
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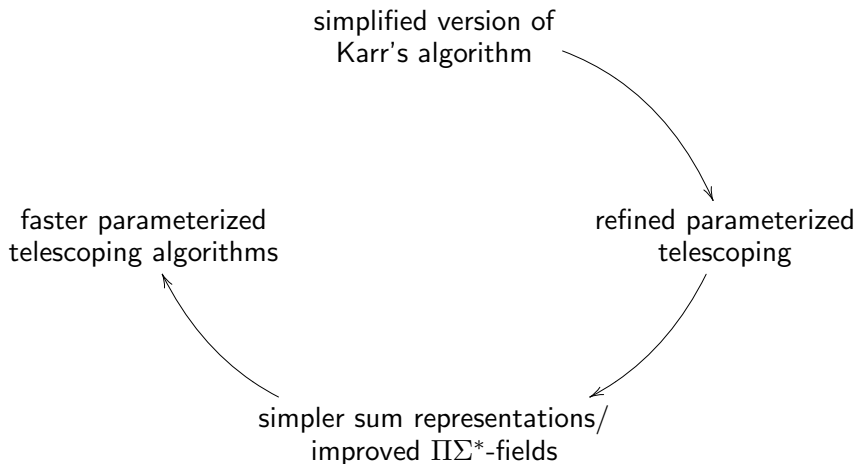


refined parameterized  
telescoping

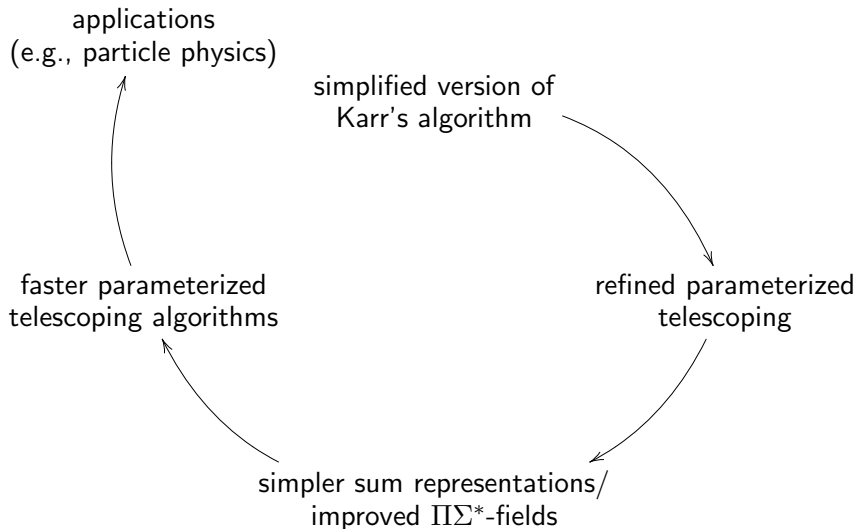
# Conclusion



# Conclusion



# Conclusion



Further details: Proceedings of this workshop (LNCS series) [arXiv:1307.7887]