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Workshop 3: Computer algebra and polynomials

Fast Algorithms for Refined Parameterized Telescoping in Difference Fields

Carsten Schneider

SFB F050 Algorithmic and Enumerative Combinatorics
Research Institute for Symbolic Computation
Johannes Kepler University Linz

Simplify

$$\sum_{k=1}^{m} H_k$$

with $H_k = \sum_{i=1}^{k} \frac{1}{i}$

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Telescoping

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for all $1 \leq k \leq m$ and $m \geq 0$

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We compute

$$g(k) = (H_k - 1)k.$$

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$$\sum_{k=1}^{m} H_k$$

with $H_k = \sum_{i=1}^{k} \frac{1}{i}$

Telescoping

Find $g(k)$ s.t,

$$\boxed{H_k = g(k+1) - g(k)}$$

for all $1 \le k \le m$ and $m \ge 0$

Summing this equation over k from 1 to m gives

$$\begin{aligned} \boxed{\sum_{k=1}^{m} H_k} &= \boxed{g(m+1) - g(1)} \\ &= (H_{m+1} - 1)(m+1). \end{aligned}$$

Telescoping in the given difference field

FIND a closed form for

$$\sum_{k=1}^{m} H_k.$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

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$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1, \qquad\qquad \mathcal{S}\, k = k + 1,$$

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FIND a closed form for

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A difference field for the summand

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$$\mathbb{F} := \mathbb{Q}(k)(h)$$

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$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{Q}, & & \\ \sigma(k) &= k + 1, & \mathcal{S}\, k &= k + 1, \\ \sigma(h) &= h + \frac{1}{k+1}, & \mathcal{S}\, H_k &= H_k + \frac{1}{k+1}. \end{aligned}$$

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This gives

$$\boxed{g(k + 1) - g(k) = H_k}$$

with

$$g(k) = (H_k - 1)k.$$

Hence,

$$(H_{m+1} - 1)(m + 1) = \sum_{k=1}^{m} H_k.$$

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}$$

- with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)$$

- with an automorphism

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{K} \\ \sigma(t_1) &= a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \qquad\qquad f_1 \in \mathbb{K} \end{aligned}$$

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)$$

- with an automorphism

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{K} \\ \sigma(t_1) &= a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, && f_1 \in \mathbb{K} \\ \sigma(t_2) &= a_2\, t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, && f_2 \in \mathbb{K}(t_1) \end{aligned}$$

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$$

- with an automorphism

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, && f_1 \in \mathbb{K} \\
\sigma(t_2) &= a_2\, t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, && f_2 \in \mathbb{K}(t_1) \\
&\vdots \\
\sigma(t_e) &= a_e\, t_e + f_e, \quad a_e \in \mathbb{K}(t_1, \dots, t_{e-1})^*, && f_e \in \mathbb{K}(t_1, \dots, t_{e-1})
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GIVEN $f \in \mathbb{F}$;
FIND $g \in \mathbb{F}$ such that

$$\sigma(g) - g = f.$$

CONSTRUCT a $\Pi\Sigma^*$-field $(\mathbb{F},\sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$$

- with an automorphism

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, && f_1 \in \mathbb{K} \\
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\end{aligned}$$

such that

$$\mathrm{const}_\sigma\mathbb{F} = \{c \in \mathbb{K}(t_1)(t_2)\dots(t_e) | \sigma(c) = c\} = \mathbb{K}.$$

GIVEN $f \in \mathbb{F}$; Karr's algorithm (1981)
FIND $g \in \mathbb{F}$ such that

$$\sigma(g) - g = f.$$

Simplified telescoping algorithm

FIND a closed form for

$$\sum_{k=1}^{m} H_k.$$

A $\Pi\Sigma^*$-field for the summand $\boxed{\mathrm{const}_\sigma \mathbb{F} = \mathbb{Q}}$

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(h)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{Q}, & & \\ \sigma(k) &= k + 1, & \mathcal{S}\, k &= k + 1, \\ \sigma(h) &= h + \frac{1}{k+1}, & \mathcal{S}\, H_k &= H_k + \frac{1}{k+1}. \end{aligned}$$

FIND $g \in \mathbb{Q}(k)(h)$:

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Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]^*$:

$$\forall g \in \mathbb{Q}(k)(h): \quad \sigma(g) - g = h \quad \Rightarrow \quad g\,d \in \mathbb{Q}(k)[h].$$

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FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\sigma(\frac{g'}{d}) - \frac{g'}{d} = h.$$

FIND $g \in \mathbb{Q}(k)(h)$:

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FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\frac{1}{\sigma(d)}\sigma(g') - \frac{1}{d}g = \sigma(\frac{g'}{d}) - \frac{g'}{d} = h.$$

FIND $g \in \mathbb{Q}(k)(h)$:

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Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]^*$:

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Degree bound: COMPUTE $m \geq 0$:

$$\forall g \in \mathbb{Q}(k)[h] \quad \sigma(g) - g = h \quad \Rightarrow \quad \deg(g) \leq m.$$

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Degree bound: COMPUTE $m \geq 0$:

$\boxed{m = 2}$

$$\forall g \in \mathbb{Q}(k)[h] \quad \sigma(g) - g = h \quad \Rightarrow \quad \deg(g) \leq m.$$

FIND $g \in \mathbb{Q}(k)(h)$:

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Polynomial Solution: FIND

$$g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h].$$

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$\boxed{\text{g=hk-k}}$

$$g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h].$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\sigma(g) - g = h$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\Big(h + \tfrac{1}{k+1}\Big)^2 + \sigma(g_1 h + g_0)\Big] \\ -\Big[g_2\, h^2 + g_1 h + g_0\Big] = h$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

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coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

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$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = h - c\big[\tfrac{2h(k+1)+1}{(k+1)^2}\big]$$

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$$\boxed{\sigma(g_1) - g_1 = 1 - c\,\frac{2}{k+1}}$$

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$$\boxed{\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{k+1}}$$

$c = 0$, $\begin{array}{l} g_1 = k + d \\ d \in \mathbb{Q} \end{array}$

$$\boxed{\sigma(g_0) - g_0 = -1 - d\frac{1}{k+1}}$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

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$$\boxed{\sigma(g_2) - g_2 = 0}$$

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$$\boxed{\sigma(g_0) - g_0 = -1 - d\frac{1}{k+1}} \longrightarrow \begin{array}{l} g_0 = -k \\ d = 0 \end{array}$$

First order difference equations in difference fields

Let $(\mathbb{F}, \sigma)$ be a $\Pi\Sigma$-field with constant field $\mathbb{K}$

Telescoping

- Given $f \in \mathbb{F}$
- Find $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = f}$$

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$$\downarrow \qquad\qquad \uparrow$$

Parameterized Telescoping

- Given $\mathbf{f} = (f_1, \ldots, f_n) \in \mathbb{F}^n$
- Find $\Big\{(c_1, \ldots, c_n, g) \in \mathbb{K}^n \times \mathbb{F} |$

$$\boxed{\sigma(g) - g = c_1\, f_1 + \cdots + c_n\, f_n}\Big\}$$

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Parameterized first order linear difference equation

- Given $\mathbf{f} = (f_1, \dots, f_n) \in \mathbb{F}^n$, $(0,0) \neq (a_0, a_1) \in \mathbb{F}^2$
- Find $\Big\{(c_1, \dots, c_n, g) \in \mathbb{K}^n \times \mathbb{F} \,|\, \boxed{a_1\sigma(g) + a_0\, g = c_1\, f_1 + \dots + c_n\, f_n}\Big\}$

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$$\boxed{a_1 \sigma(g) + a_0\, g = c_1\, f_1 + \cdots + c_n\, f_n}\Big\} =: V(\mathbf{a}, \mathbf{f}, \mathbb{F})$$

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Note:

- $V(\mathbf{a}, \mathbf{f}, \mathbb{F})$ is a subspace of $\mathbb{K}^n \times \mathbb{F}$ over $\mathbb{K}$

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- $\dim_{\mathbb{K}} V(\mathbf{a}, \mathbf{f}, \mathbb{F}) \leq n + 1$

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Note:

- $V(\mathbf{a}, \mathbf{f}, \mathbb{F})$ is a subspace of $\mathbb{K}^n \times \mathbb{F}$ over $\mathbb{K}$
- $\dim_{\mathbb{K}} V(\mathbf{a}, \mathbf{f}, \mathbb{F}) \leq n+1$
- $\boxed{\text{Task: Compute a basis of } V(\mathbf{a}, \mathbf{f}, \mathbb{F})}$

Find a basis of

$\boxed{V(\mathbf{a}, \mathbf{f}, \mathbb{G}(t))}$

$(\mathbb{G}(t), \sigma)$ $\Pi\Sigma^*$-field

Find a basis of

$$\boxed{V(\mathbf{a},\mathbf{f},\mathbb{G}(t))}$$

$$\downarrow$$

Compute a denominator bound
and search for polynomial solutions

$(\mathbb{G}(t),\sigma)$ $\Pi\Sigma^*$-field

Find a basis of

$\boxed{V(\mathbf{a}, \mathbf{f}, \mathbb{G}(t))}$

$\downarrow$

Compute a denominator bound
and search for polynomial solutions

$\downarrow$

Compute a degree bound m

$(\mathbb{G}(t), \sigma)$ $\Pi\Sigma^*$-field

Find a basis of

$$\boxed{V(\mathbf{a}, \mathbf{f}, \mathbb{G}(t))}$$

$$\downarrow$$

Compute a denominator bound
and search for polynomial solutions

$$\downarrow$$

Compute a degree bound m

$$\downarrow$$

Compute a basis of

$$\boxed{V(\mathbf{a_m}, \mathbf{f_m}, \mathbb{G}[t]_m)}$$

$(\mathbb{G}(t), \sigma)$ $\Pi\Sigma^*$-field

$$\mathbb{G}[t]_m := \{p \in \mathbb{G}[t] \mid \deg(p) \le m\}$$

$(\mathbb{G}(t),\sigma)$ $\Pi\Sigma^*$-field

Find a basis of

$$\boxed{V(\mathbf{a},\mathbf{f},\mathbb{G}(t))}$$

$$\downarrow$$

Compute a denominator bound
and search for polynomial solutions

$$\downarrow$$

Compute a degree bound m

$$\downarrow$$

Compute a basis of

$$\boxed{V(\mathbf{a_m},\mathbf{f_m},\mathbb{G}[t]_m)}$$

$$\xrightarrow{\text{coeff. comparison}}$$

Compute a basis of

$$\boxed{V(\tilde{\mathbf{a}}_{\mathbf{m}},\tilde{\mathbf{f}}_{\mathbf{m}},\mathbb{G})}$$

Find a basis of

$\boxed{V(\mathbf{a}, \mathbf{f}, \mathbb{G}(t))}$

$(\mathbb{G}(t), \sigma)$ $\Pi\Sigma^*$-field

Compute a denominator bound
and search for polynomial solutions

Compute a degree bound m

Compute a basis of

$\boxed{V(\mathbf{a_m}, \mathbf{f_m}, \mathbb{G}[t]_m)}$

coeff. comparison

Compute a basis of

$\boxed{V(\tilde{\mathbf{a}}_{\mathbf{m}}, \tilde{\mathbf{f}}_{\mathbf{m}}, \mathbb{G})}$

Plugin highest term

Compute a basis of

$\boxed{V(\mathbf{a_{m-1}}, \mathbf{f_{m-1}}, \mathbb{G}[t]_{m-1})}$

$(\mathbb{G}(t), \sigma)$ $\Pi\Sigma^*$-field

Find a basis of

$$\boxed{V(\mathbf{a}, \mathbf{f}, \mathbb{G}(t))}$$

$\downarrow$

Compute a denominator bound
and search for polynomial solutions

$\downarrow$

Compute a degree bound m

$\downarrow$

Compute a basis of

$$\boxed{V(\mathbf{a_m}, \mathbf{f_m}, \mathbb{G}[t]_m)}$$

coeff. comparison

Compute a basis of

$$\boxed{V(\tilde{\mathbf{a}}_\mathbf{m}, \tilde{\mathbf{f}}_\mathbf{m}, \mathbb{G})}$$

Plugin highest term

Compute a basis of

$$\boxed{V(\mathbf{a_{m-1}}, \mathbf{f_{m-1}}, \mathbb{G}[t]_{m-1})}$$

Combine solution

$(\mathbb{G}(t),\sigma)$ $\Pi\Sigma^*$-field

Find a basis of

$\boxed{V(\mathbf{a},\mathbf{f},\mathbb{G}(t))}$

↓

Compute a denominator bound
and search for polynomial solutions

↓

Compute a degree bound m

↓

Compute a basis of

$\boxed{V(\mathbf{a_m},\mathbf{f_m},\mathbb{G}[t]_m)}$

→ coeff. comparison →

Compute a basis of

$\boxed{V(\mathbf{\tilde{a}_m},\mathbf{\tilde{f}_m},\mathbb{G})}$

← Plugin highest term ←

Compute a basis of

$\boxed{V(\mathbf{a_{m-1}},\mathbf{f_{m-1}},\mathbb{G}[t]_{m-1})}$

↑ Combine solution (to $V(\mathbf{a_m},\mathbf{f_m},\mathbb{G}[t]_m)$)

→ coeff. comparison →

Compute a basis of

$\boxed{V(\mathbf{\tilde{a}_{m-1}},\mathbf{\tilde{f}_{m-1}},\mathbb{G})}$

← Plugin highest term ←

Compute a basis of

$\boxed{V(\mathbf{a_{m-2}},\mathbf{f_{m-2}},\mathbb{G}[t]_{m-2})}$

↑ Combine solution

Simplified version of Karr's algorithm

$(\mathbb{G}(t),\sigma)$ $\Pi\Sigma^*$-field

Find a basis of
$\boxed{V(\mathbf{a},\mathbf{f},\mathbb{G}(t))}$

Compute a denominator bound
and search for polynomial solutions

Compute a degree bound m

Compute a basis of
$\boxed{V(\mathbf{a_m},\mathbf{f_m},\mathbb{G}[t]_m)}$

coeff. comparison

Compute a basis of
$\boxed{V(\mathbf{\tilde{a}_m},\mathbf{\tilde{f}_m},\mathbb{G})}$

Plugin highest term

Combine solution

Compute a basis of
$\boxed{V(\mathbf{a_{m-1}},\mathbf{f_{m-1}},\mathbb{G}[t]_{m-1})}$

coeff. comparison

Compute a basis of
$\boxed{V(\mathbf{\tilde{a}_{m-1}},\mathbf{\tilde{f}_{m-1}},\mathbb{G})}$

Plugin highest term

Combine solution

Compute a basis of
$\boxed{V(\mathbf{a_{m-2}},\mathbf{f_{m-2}},\mathbb{G}[t]_{m-2})}$

⋮

Compute a basis of
$\boxed{V(\mathbf{a_1},\mathbf{f_1},\mathbb{G}[t]_1)}$

coeff. comparison

Compute a basis of
$\boxed{V(\mathbf{\tilde{a}_1},\mathbf{\tilde{f}_1},\mathbb{G})}$

Plugin highest term

Combine solution

Compute a basis of
$\boxed{V(\mathbf{a_0},\mathbf{f_0},\underbrace{\mathbb{G}[t]_0)}_{\mathbb{G}}}$

degree reduction
by solving problems in $\mathbb{G}$

Example 2

FIND $g(k)$:

$$\boxed{g(k+1) - g(k) = \frac{H_k}{k}}$$

for all $1 \leq k \leq m$ and $m \geq 0$.

Find $g \in \mathbb{Q}(k)(h)$ s.t.

$$\sigma(g) - g = \frac{h}{k}$$

Find $g \in \mathbb{Q}(k)[h]$ s.t.

$$\sigma(g) - g = \frac{h}{k}$$

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

$$\sigma(g) - g = \frac{h}{k}$$

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

$$\sigma(g) - g = \frac{h}{k}$$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\, h^2 + g_1 h + g_0\big] = \tfrac{h}{k}$$

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

$$\sigma(g) - g = \frac{h}{k}$$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] \\ -\big[g_2\, h^2 + g_1 h + g_0\big] = \tfrac{h}{k}$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

$$\sigma(g) - g = \frac{h}{k}$$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big]$$
$$-\big[g_2\, h^2 + g_1 h + g_0\big] = \tfrac{h}{k}$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

$$\sigma(g) - g = \frac{h}{k}$$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big]$$
$$-\big[g_2\, h^2 + g_1 h + g_0\big] = \tfrac{h}{k}$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = \tfrac{h}{k} + c\big[\tfrac{-2h(k+1)-1}{(k+1)^2}\big]$$

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

$$\sigma(g) - g = \frac{h}{k}$$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\, h^2 + g_1 h + g_0\big] = \tfrac{h}{k}$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = \tfrac{h}{k} + c\Big[\tfrac{-2h(k+1)-1}{(k+1)^2}\Big]$$

coeff. comp.

$$\boxed{\sigma(g_1) - g_1 = \frac{1}{k} + c\,\tfrac{-2}{k+1}}$$

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

$$\sigma(g) - g = \frac{h}{k}$$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\, h^2 + g_1 h + g_0\big] = \tfrac{h}{k}$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = \tfrac{h}{k} + c\Big[\tfrac{-2h(k+1)-1}{(k+1)^2}\Big]$$

coeff. comp.

$$\boxed{\sigma(g_1) - g_1 = \frac{1}{k} + c\,\tfrac{-2}{k+1}}$$

$c = \frac{1}{2}$, $\quad g_1 = -\frac{1}{k} + d$, $\; d \in \mathbb{Q}$

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

$$\sigma(g) - g = \frac{h}{k}$$

$$\left[\sigma(g_2)\left(h + \tfrac{1}{k+1}\right)^2 + \sigma(g_1 h + g_0)\right] - \left[g_2\, h^2 + g_1 h + g_0\right] = \tfrac{h}{k}$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = \tfrac{h}{k} + c\left[\tfrac{-2h(k+1)-1}{(k+1)^2}\right]$$

coeff. comp.

$$\boxed{\sigma(g_1) - g_1 = \frac{1}{k} + c\,\tfrac{-2}{k+1}}$$

$c = \frac{1}{2}$, $g_1 = -\frac{1}{k} + d$, $d \in \mathbb{Q}$

$$\boxed{\sigma(g_0) - g_0 = \frac{2}{(k+1)^2} + d\frac{-1}{k+1}}$$

Find $g = g_2h^2 + g_1h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

$$\sigma(g) - g = \frac{h}{k}$$

$$\left[\sigma(g_2)\left(h + \tfrac{1}{k+1}\right)^2 + \sigma(g_1h + g_0)\right] - \left[g_2\,h^2 + g_1h + g_0\right] = \frac{h}{k}$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\,h + g_0) - (g_1\,h + g_0) = \frac{h}{k} + c\left[\frac{-2h(k+1)-1}{(k+1)^2}\right]$$

coeff. comp.

$$\boxed{\sigma(g_1) - g_1 = \frac{1}{k} + c\,\frac{-2}{k+1}}$$

$c = \frac{1}{2}$, $\quad g_1 = -\frac{1}{k} + d$, $\quad d \in \mathbb{Q}$

$$\boxed{\sigma(g_0) - g_0 = \frac{2}{(k+1)^2} + d\frac{-1}{k+1}}$$

$\leftarrow$ no solution

FIND $g(k)$:

$$\boxed{g(k+1) - g(k) = \frac{H_k}{k}}$$

for all $1 \leq k \leq m$ and $m \geq 0$.

No solution ☹

Refined telescoping

FIND $g(k), \phi(k)$:

$$\boxed{g(k+1) - g(k) + \phi(k) = \frac{H_k}{k}}$$

for all $1 \leq k \leq m$ and $m \geq 0$.

Find $g = g_2h^2 + g_1h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

$$\sigma(g) - g = \frac{h}{k}$$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1h + g_0)\Big] - \big[g_2\,h^2 + g_1h + g_0\big] = \frac{h}{k}$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\,h + g_0) - (g_1\,h + g_0) = \frac{h}{k} + c\Big[\frac{-2h(k+1)-1}{(k+1)^2}\Big]$$

coeff. comp.

$$\boxed{\sigma(g_1) - g_1 = \frac{1}{k} + c\,\frac{-2}{k+1}}$$

$c = \frac{1}{2}$, $g_1 = -\frac{1}{k} + d$, $d \in \mathbb{Q}$

no solution $\leftarrow$ $$\boxed{\sigma(g_0) - g_0 = \frac{2}{(k+1)^2} + d\frac{-1}{k+1}}$$

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

$$\sigma(g) - g = \frac{h}{k}$$

$$\left[\sigma(g_2)\left(h + \tfrac{1}{k+1}\right)^2 + \sigma(g_1 h + g_0)\right] - \left[g_2\, h^2 + g_1 h + g_0\right] = \frac{h}{k}$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = \frac{h}{k} + c\left[\frac{-2h(k+1)-1}{(k+1)^2}\right]$$

coeff. comp.

$$\boxed{\sigma(g_1) - g_1 = \frac{1}{k} + c\,\frac{-2}{k+1}}$$

$c = \frac{1}{2}$, $g_1 = -\frac{1}{k} + d$, $d \in \mathbb{Q}$

$$\boxed{\sigma(g_0) - g_0 = \frac{2}{(k+1)^2} + d\frac{-1}{k+1}}$$

$$\sigma(0) - 0 + \frac{2}{(k+1)^2} = \frac{2}{(k+1)^2} + 0\,\frac{-1}{k+1}$$

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

$$\sigma(g) - g + \frac{1}{2(k+1)^2} = \frac{h}{k} \qquad g = \frac{1}{2}h + \frac{1}{k}$$

$$\left[\sigma(g_2)\left(h + \tfrac{1}{k+1}\right)^2 + \sigma(g_1 h + g_0)\right] - \left[g_2\, h^2 + g_1 h + g_0\right] = \frac{h}{k}$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = \frac{h}{k} + c\left[\frac{-2h(k+1)-1}{(k+1)^2}\right]$$

coeff. comp.

$$\boxed{\sigma(g_1) - g_1 = \frac{1}{k} + c\,\frac{-2}{k+1}}$$

$c = \frac{1}{2}$, $\quad g_1 = -\frac{1}{k} + d$, $d \in \mathbb{Q}$

$$\boxed{\sigma(g_0) - g_0 = \frac{2}{(k+1)^2} + d\frac{-1}{k+1}}$$

$$\sigma(0) - 0 + \frac{2}{(k+1)^2} = \frac{2}{(k+1)^2} + 0\,\frac{-1}{k+1}$$

Refined telescoping

FIND $g(k), \phi(k)$:

$$\boxed{g(k+1) - g(k) + \phi(k) = \frac{H_k}{k}}$$

for all $1 \le k \le m$ and $m \ge 0$.

We compute

$$g(k) = \frac{1}{2}H_k + \frac{1}{k}$$

$$\phi(k) = \frac{1}{2(k+1)^2}$$

Refined telescoping

FIND $g(k), \phi(k)$:

$$\boxed{g(k+1) - g(k) + \phi(k) = \frac{H_k}{k}}$$

for all $1 \leq k \leq m$ and $m \geq 0$.

Summing this equation over k from 1 to m gives

$$\boxed{\sum_{k=1}^{m} \frac{H_k}{k}} = \boxed{g(m+1) - g(1) + \sum_{k=1}^{m} \phi(k)}$$

$$= \frac{1}{2}\left(H_m^2 + \sum_{k=1}^{m} \frac{1}{k^2}\right).$$

Parameterized telescoping

Given $\Pi\Sigma^*$-field $(\mathbb{K}(t_1)\dots(t_e),\sigma)$.
Define $\mathbb{F}_i=\mathbb{K}(t_1)\dots(t_i)$, i.e.,

$$\mathbb{F}_0<\mathbb{F}_1<\dots<\mathbb{F}_i<\dots<\mathbb{F}_e$$

Parameterized telescoping

Given $\Pi\Sigma^*$-field $(\mathbb{K}(t_1)\dots(t_e),\sigma)$.
Define $\mathbb{F}_i=\mathbb{K}(t_1)\dots(t_i)$, i.e.,

$$\underbrace{\mathbb{F}_{-1}}_{=\{0\}}\subset\mathbb{F}_0<\mathbb{F}_1<\dots<\mathbb{F}_i<\dots<\mathbb{F}_e$$

Parameterized telescoping

Given $\Pi\Sigma^*$-field $(\mathbb{K}(t_1)\dots(t_e),\sigma)$.
Define $\mathbb{F}_i = \mathbb{K}(t_1)\dots(t_i)$, i.e.,

$$\underbrace{\mathbb{F}_{-1}}_{=\{0\}} \subset \mathbb{F}_0 < \mathbb{F}_1 < \cdots < \mathbb{F}_i < \cdots < \mathbb{F}_e$$

Given $f_1,\dots,f_n \in \mathbb{F}_e$.
Find $c_1,\dots c_n \in \mathbb{K}$, $g \in \mathbb{F}_e$:

$$\sigma(g) - g = c_1\, f_1 + \cdots + c_n\, f_n$$

Special cases are

- telescoping ($n = 1$)
- creative telescoping (take $f_i = F(r-1+i,k)$)

First entry telescoping

Given $\Pi\Sigma^*$-field $(\mathbb{K}(t_1)\dots(t_e),\sigma)$.
Define $\mathbb{F}_i=\mathbb{K}(t_1)\dots(t_i)$, i.e.,

$$\underbrace{\mathbb{F}_{-1}}_{=\{0\}}\subset\mathbb{F}_0<\mathbb{F}_1<\dots<\mathbb{F}_i<\dots<\mathbb{F}_e$$

Given $f_1,\dots,f_n\in\mathbb{F}_e$.
Find $c_1,\dots c_n\in\mathbb{K}$, $g\in\mathbb{F}_e$:

$$\sigma(g)-g=\underbrace{c_1}_{\neq 0}\,f_1+\dots+c_n\,f_n$$

Note: Gives rise to faster algorithms

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

$$\sigma(g) - g = \frac{h}{k}$$

$$\left[\sigma(g_2)\left(h + \tfrac{1}{k+1}\right)^2 + \sigma(g_1 h + g_0)\right] - \left[g_2\, h^2 + g_1 h + g_0\right] = \tfrac{h}{k}$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = \tfrac{h}{k} + c\left[\tfrac{-2h(k+1)-1}{(k+1)^2}\right]$$

coeff. comp.

$$\boxed{\sigma(g_1) - g_1 = \frac{1}{k} + c\,\tfrac{-2}{k+1}}$$

$c = \tfrac{1}{2}$, $g_1 = -\tfrac{1}{k} + d$, $d \in \mathbb{Q}$

no solution $\leftarrow$ $\boxed{\sigma(g_0) - g_0 = \dfrac{2}{(k+1)^2} + d\dfrac{-1}{k+1}}$

First entry telescoping

Given $\Pi\Sigma^*$-field $(\mathbb{K}(t_1)\dots(t_e),\sigma)$.
Define $\mathbb{F}_i = \mathbb{K}(t_1)\dots(t_i)$, i.e.,

$$\underbrace{\mathbb{F}_{-1}}_{=\{0\}} \subset \mathbb{F}_0 < \mathbb{F}_1 < \cdots < \mathbb{F}_i < \cdots < \mathbb{F}_e$$

Given $f_1,\dots,f_n \in \mathbb{F}_e$.
Find $c_1,\dots c_n \in \mathbb{K}$, $g \in \mathbb{F}_e$:

$$\sigma(g) - g = \underbrace{c_1}_{\neq 0} f_1 + \cdots + c_n f_n$$

Note: Gives rise to faster algorithms

Refined telescoping

Given $\Pi\Sigma^*$-field $(\mathbb{K}(t_1)\dots(t_e),\sigma)$.
Define $\mathbb{F}_i=\mathbb{K}(t_1)\dots(t_i)$, i.e.,

$$\underbrace{\mathbb{F}_{-1}}_{=\{0\}}\subset\mathbb{F}_0<\mathbb{F}_1<\dots<\mathbb{F}_i<\dots<\mathbb{F}_e$$

Given $f_1,\dots,f_n\in\mathbb{F}_e$.
Find $c_1,\dots c_n\in\mathbb{K}$, $g\in\mathbb{F}_e$
$\boxed{\phi\in\mathbb{F}_i \text{ with } i \text{ minimal}}$:

$$\sigma(g)-g+\boxed{\phi}=\underbrace{c_1}_{\neq 0}f_1+\dots+c_n f_n$$

Find $c_j \in \mathbb{K}$, $g \in \mathbb{F}_e$, $\phi \in \mathbb{F}_e$ s.t.

$$\sigma(g) - g + \phi = \underbrace{c_1}_{\neq 0} f_1 + \cdots + c_n f_n.$$

$$\boxed{V((-1,1),(f_1,\ldots,f_n),\mathbb{F}_e)}$$

$$\boxed{V((-1,1),(f_1^{(e-1)},\ldots,f_{n_{e-1}}^{(e-1)}),\mathbb{F}_{e-1})}$$

Find $g = g_2 h^2 + g_1 h + g_0 \in \mathbb{Q}(k)[h]_2$ s.t.

$$\sigma(g) - g + \frac{1}{2(k+1)^2} = \frac{h}{k} \qquad g = \frac{1}{2}h + \frac{1}{k}$$

$$V((-1,1),(\frac{h}{k}),\mathbb{Q}(k)(h))$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

coeff. comp.

$$\boxed{\sigma(g_1) - g_1 = \frac{1}{k} + c\,\frac{-2}{k+1}}$$

$$V((-1,1),(\boxed{\frac{2}{(k+1)^2}},\frac{-1}{k+1}),\mathbb{Q}(k))$$

Find $c_j \in \mathbb{K}$, $g \in \mathbb{F}_e$, $\phi \in \mathbb{F}_e$ s.t.

$$\sigma(g) - g + \phi = \underbrace{c_1}_{\neq 0} f_1 + \cdots + c_n f_n.$$

$$\boxed{V((-1,1),(f_1,\ldots,f_n),\mathbb{F}_e)}$$

$$\Downarrow$$

$$\boxed{V((-1,1),(f_1^{(e-1)},\ldots,f_{n_{e-1}}^{(e-1)}),\mathbb{F}_{e-1})}$$

Find $c_j \in \mathbb{K}$, $g \in \mathbb{F}_e$, $\phi \in \mathbb{F}_e$ s.t.

$$\sigma(g) - g + \phi = \underbrace{c_1}_{\neq 0}\, f_1 + \cdots + c_n\, f_n.$$

$$\boxed{V((-1,1),(f_1,\dots,f_n),\mathbb{F}_e)}$$

$$\rightsquigarrow$$

$$\boxed{V((-1,1),(f_1^{(e-1)},\dots,f_{n_{e-1}}^{(e-1)}),\mathbb{F}_{e-1})}$$

$$\rightsquigarrow$$

$$\vdots$$

$$\rightsquigarrow$$

$$\boxed{V((-1,1),(f_1^{(i)},\dots,f_{n_i}^{(i)}),\mathbb{F}_i)}$$

Find $c_j \in \mathbb{K}$, $g \in \mathbb{F}_e$, $\phi \in \mathbb{F}_e$ s.t.

$$\sigma(g) - g + \phi = \underbrace{c_1}_{\neq 0} f_1 + \cdots + c_n f_n.$$

$$\boxed{V((-1,1),(f_1,\ldots,f_n),\mathbb{F}_e)}$$

$$\rightsquigarrow$$

$$\boxed{V((-1,1),(f_1^{(e-1)},\ldots,f_{n_{e-1}}^{(e-1)}),\mathbb{F}_{e-1})}$$

$$\vdots$$

$$\boxed{V((-1,1),(f_1^{(i)},\ldots,f_{n_i}^{(i)}),\mathbb{F}_i)}$$

Take $\bar{c}_j \in \mathbb{K}$, $\bar{g} \in \mathbb{F}_e$, $\boxed{\bar{\phi} \in \mathbb{F}_{i-1}}$ s.t.

$$\sigma(\bar{g}) - \bar{g} + \bar{\phi} = \underbrace{\bar{c_1}}_{\neq 0} f_1 + \cdots + \bar{c_n} f_n.$$

Find $c_j \in \mathbb{K}$, $g \in \mathbb{F}_e$, $\phi \in \mathbb{F}_e$ s.t.

$$\sigma(g) - g + \phi = \underbrace{c_1}_{\neq 0} f_1 + \cdots + c_n f_n.$$

$$\boxed{V((-1,1),(f_1,\ldots,f_n),\mathbb{F}_e)}$$

$$\rightsquigarrow$$

$$\boxed{V((-1,1),(f_1^{(e-1)},\ldots,f_{n_{e-1}}^{(e-1)}),\mathbb{F}_{e-1})}$$

$$\vdots$$

$$\rightsquigarrow$$

$$\boxed{V((-1,1),(f_1^{(i)},\ldots,f_{n_i}^{(i)}),\mathbb{F}_i)}$$

Take $\bar{c}_j \in \mathbb{K}$, $\bar{g} \in \mathbb{F}_e$, $\boxed{\bar{\phi} \in \mathbb{F}_{i-1}}$ s.t.

$$\sigma(\bar{g}) - \bar{g} = \underbrace{\bar{c_1}}_{\neq 0} f_1 + \cdots + \bar{c_n} f_n - \bar{\phi}.$$

Find $c_j \in \mathbb{K}$, $g \in \mathbb{F}_e$, $\phi \in \mathbb{F}_e$ s.t.

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$$\boxed{V((-1,1),(f_1,\ldots,f_n),\mathbb{F}_e)}$$

$$\rightsquigarrow$$

$$\boxed{V((-1,1),(f_1^{(e-1)},\ldots,f_{n_{e-1}}^{(e-1)}),\mathbb{F}_{e-1})}$$

$$\rightsquigarrow$$

$$\vdots$$

$$\rightsquigarrow$$

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$$\boxed{V((-1,1),(f_1^{(i-1)},\dots,f_{n_{i-1}}^{(i-1)}),\mathbb{F}_{i-1})} \lightning$$

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Refined telescoping

Given $\Pi\Sigma^*$-field $(\mathbb{K}(t_1)\dots(t_e),\sigma)$.
Define $\mathbb{F}_i=\mathbb{K}(t_1)\dots(t_i)$, i.e.,

$$\underbrace{\mathbb{F}_{-1}}_{=\{0\}}\subset\mathbb{F}_0<\mathbb{F}_1<\dots<\mathbb{F}_i<\dots<\mathbb{F}_e$$

Given $f_1,\dots,f_n\in\mathbb{F}_e$.
Find $c_1,\dots c_n\in\mathbb{K}$, $g\in\mathbb{F}_e$
$\boxed{\phi\in\mathbb{F}_i \text{ with } i \text{ minimal}}$:

$$\sigma(g)-g+\boxed{\phi}=\underbrace{c_1}_{\neq 0}\,f_1+\dots+c_n\,f_n$$

Refined telescoping

Given $\Pi\Sigma^*$-extension $(\mathbb{G}(t_1)\dots(t_e),\sigma)$ of a $\boxed{\text{difference field } (\mathbb{G},\sigma)}$.
Define $\mathbb{F}_i = \mathbb{G}(t_1)\dots(t_i)$, i.e.,

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Requirements: an algorithm to solve first order linear difference equations in $(\mathbb{G},\sigma)$ plus technical conditions.

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Example 1: $\mathbb{G} = \mathbb{K}$

Examples

$$\sum_{k=1}^{m}(k^2+1)\,k!\,H_k^2 = \sum_{k=1}^{m}\frac{(k+1)!}{k^2} + m\,(m+1)!\,H_m^2 - 2m!\,H_m.$$

$$\sum_{k=1}^{m} k \sum_{j=1}^{k} \frac{H_j}{j^2} = \frac{1}{2}\Big(\sum_{k=1}^{m}\frac{1}{k^2} + m(m+1)\sum_{i=1}^{m}\frac{H_i}{i^2} + H_m^2 - (m+1)H_m + m\Big)$$

$$\sum_{k=1}^{m}\Big(\sum_{j=1}^{k}\binom{r}{j}\Big)^2 = - \frac{1}{2}r\sum_{k=1}^{m}\binom{r}{k}^2 + \frac{1}{2}(2m-r+2)\Big(\sum_{i=1}^{m}\binom{r}{i}\Big)^2 - (m-r)\binom{r}{m}\sum_{i=1}^{m}\binom{r}{i}$$

Refined telescoping

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Example 2 (free difference field): $\mathbb{G}=\mathbb{K}(\dots,x_{-1},x_0,x_1,\dots)$
with $\sigma(x_i)=x_{i+1}$.

Unspecified sequences (M. Kauers,CS; 2006)

$$\sum_{k=1}^{n} a^k \sum_{j=1}^{k} X_j = \frac{a^{n+1} \sum_{k=1}^{n} X_k - \sum_{k=1}^{n} a^k X_k}{a-1}, \quad a \neq 1$$

$$\sum_{k=0}^{n} k^2 \sum_{i=0}^{k} X_i = \tfrac{1}{6}\Big(n(n+1)(2n+1) \sum_{k=0}^{n} X_k - \sum_{k=0}^{n} k X_k + 3 \sum_{k=0}^{n} k^2 X_k - 2 \sum_{k=0}^{n} k^3 X_k\Big)$$

Refined telescoping

Given $\Pi\Sigma^*$-extension $(\mathbb{G}(t_1)\dots(t_e),\sigma)$ of a $\boxed{\text{difference field } (\mathbb{G},\sigma)}$.
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Example 3 (radicals of order $d>0$): $\mathbb{G}=\mathbb{K}(k)(\dots,x_{-1},x_0,x_1,\dots)$ with $\sigma(k)=k+1$ and $\sigma(x_i)=x_{i+1}$ subject to the relation $x_i^d=k$.

Radical expressions (M. Kauers, CS; 2007)

$$\sum_{k=0}^{n} \frac{1}{\sqrt{k+1}+\sqrt{k}} = \sqrt{n+1}$$

$$\begin{aligned}\sum_{k=2}^{n} H_k\Big(\frac{(\sqrt{k})^3}{k-1}+\sum_{i=1}^{k}\frac{\sqrt{i}}{i+\sqrt{i}}\Big) &= \frac{1}{2}\Big(-3-5n+(5n+3)H_n \\ &\quad -(2n+1)H_n^2+H_n^{(2)}\Big)+\sum_{k=1}^{n}\sqrt{k} \\ &\quad +\Big((n+1)H_n-(n-1)\Big)\sum_{k=2}^{n}\frac{\sqrt{k}}{k-1}\end{aligned}$$

Conclusion

simplified version of
Karr's algorithm

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simpler sum representations/
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applications
(e.g., particle physics)

Further details: Proceedings of this workshop (LNCS series) [arXiv:1307.7887]