

Symmetric Tensor Decompositions

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homogeneous of degree d . A presentation

$$F = l_1^d + \dots + l_r^d, \text{ with } l_i \in S_1,$$

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Can we find $r(F)$, and if not, why?

example: conic

A smooth conic in $\mathbb{C}\mathbb{P}^2$ have equation

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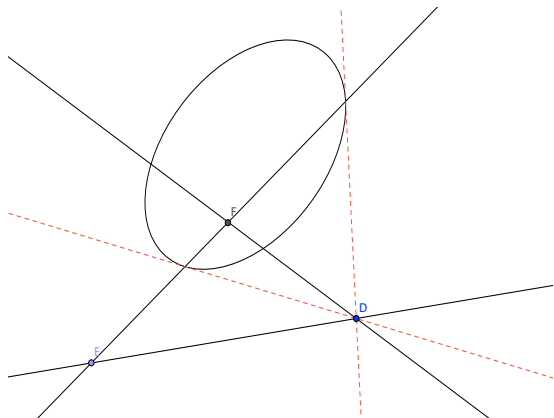
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The rank is 3, but in how many ways?

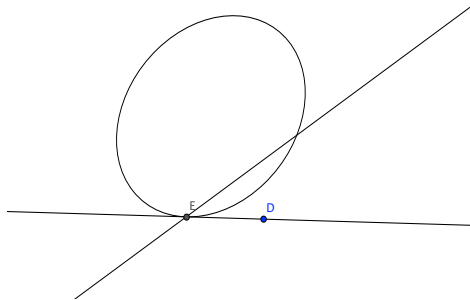
$$x_0x_1x_2 = 0$$

defines a polar triangle.

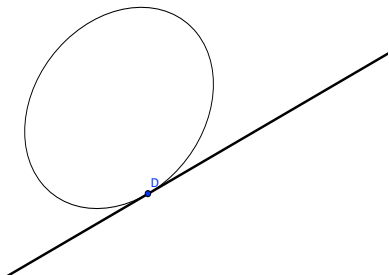
Polar triangle



Degenerate polar triangle



Further degenerate polar triangle



Apolarity

Sylvester et al. introduced apolarity to find decompositions.
Let $T = \mathbb{C}[y_0, \dots, y_n]$ act on S by differentiation:

Then
$$y_i(F) = \frac{\partial}{\partial x_i} F.$$

$$I = \sum a_i x_i \text{ and } g \in T_d \Rightarrow g(I^d) = \lambda g(a_0, \dots, a_n),$$

for some $\lambda \neq 0$.

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Our key object: $F^\perp = \{g \in T \mid g(F) = 0\} \subset T$.

The quotient T/F^\perp is Artinian and *Gorenstein*.

Apolarity lemma

Let $\mathbb{P}(S_1)$ and $\mathbb{P}(T_1)$ denote the projective spaces of 1-dimensional subspaces of S_1 (resp. T_1). By apolarity,

$$\mathbb{P}(S_1) = \mathbb{P}(T_1)^* \quad \text{and} \quad \Gamma \subset \mathbb{P}(S_1) \Rightarrow \mathcal{I}_\Gamma \subset T.$$

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Definition

$\Gamma \subset \mathbb{P}(S_1)$ is an *apolar subscheme* to F if $\mathcal{I}_\Gamma \subset F^\perp$.

Lemma

Let $\Gamma = \{[l_1], \dots, [l_r]\} \subset \mathbb{P}(S_1)$, a collection of r points. Then

$$F = \lambda_1 l_1^d + \dots + \lambda_r l_r^d \quad \text{with } \lambda_i \in \mathbb{C}$$

if and only if

$$\mathcal{I}_\Gamma \subset F^\perp \subset T.$$

example: conic revisited

$$(x_0^2 + x_1^2 + x_2^2)^\perp = (y_0y_1, y_0y_2, y_1y_2, y_0^3 - y_1^3, y_1^3 - y_2^3) \subset T$$

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$$\Gamma = \{[x_0], [x_1], [x_2]\} \subset \mathbb{P}(S_1), \quad \mathcal{I}_\Gamma = (y_0y_1, y_0y_2, y_1y_2),$$

so Γ is apolar to $x_0^2 + x_1^2 + x_2^2$.

Binary forms

$$F \in \mathbb{C}[x_0, x_1] \Rightarrow F^\perp = (g_1, g_2) \subset \mathbb{C}[y_0, y_1] \quad [\text{Sylvester}]$$

where $\deg g_1 + \deg g_2 = \deg F + 2$.

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Assume $\deg g_1 \leq \deg g_2$.

$$r(F) = \begin{cases} \deg g_1 & \text{when } g_1 \text{ is squarefree} \\ \deg g_2 & \text{else} \end{cases}$$

Cactus rank

Definition

The *cactus rank* (or *length*) of F is the minimal length of a 0-dimensional apolar subscheme Γ to F , i.e.

$$cr(F) := \min\{\text{length } \Gamma \mid \dim \Gamma = 0, \mathcal{I}_\Gamma \subset F^\perp\}.$$

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Clearly

$$cr(F) \leq r(F)$$

and the inequality may be strict: If $d > 2$,

$$(x_0 x_1^{d-1})^\perp = (y_0^2, y_1^d) \Rightarrow cr(x_0 x_1^{d-1}) = 2 < r(x_0 x_1^{d-1}) = d.$$

Border rank

Another much studied rank is the *border rank*:

$$br(F) = \min\{r \mid F \text{ is the limit of forms of rank } r\}$$

The border rank may be smaller than the cactus rank.

Bounds on the rank

The most famous bound on the rank is not a bound

Theorem (Alexander-Hirschowitz 1995)

Let $F \in \mathbb{C}[x_0, \dots, x_n]$ be a **general** form of degree d , then

- $r(F) = AH(d, n) := \lceil \frac{1}{n+1} \binom{n+d}{n} \rceil$, **except**
- $r(F) = n+1$ if $d=2$,
- $r(F) = 6, 10, 15$ if $d=4$, $n=2, 3, 4$,
- $r(F) = 8$ if $d=3$, $n=4$.

Remark

Special forms may have larger rank, a sharp upper bound is only known in a few special cases

$$((n, d) = (n, 2), (1, d), (2, 3), (2, 4), (3, 3)).$$

The simplest lower bound for the rank is explained by **differentiation**. If

$$F = I_1^d + \dots + I_r^d \quad \text{and} \quad g \in \mathbb{C}[y_0, \dots, y_n]_s$$

then

$$g(F) = \lambda_1 I_1^{d-s} + \dots + \lambda_r I_r^{d-s} \quad \text{for some } \lambda_1, \dots, \lambda_r \in \mathbb{C}$$

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So, the **dimension** $h(F, s)$ of the vector space of partials of order s is a lower bound for $r(F)$.

In fact

$$r(F) \geq \max\{h(F, s) \mid 0 < s < d\}$$

The same lower bound is valid for the cactus rank and the border rank.

An improvement by Landsberg and Teitler depending on the **singular locus** of the hypersurface $V(F)$.

Let $d(F, s)$ be the **dimension** of the locus of points on $V(F)$ of multiplicity at least s .

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Theorem (Landsberg-Teitler 2009)

Let $F \in \mathbb{C}[x_0, \dots, x_n]_d$. Assume that $V(F)$ is not a cone, and let $0 < s < d$. Then

$$r(F) \geq h(F, s) + d(F, s) + 1.$$

The proof uses apolarity in a very essential way.

Bounds on the cactus rank

For $n > 2$ and $d > 6$ the cactus rank of a general form is smaller than the rank:

Proposition (Bernardi, R 2011)

Let $F \in \mathbb{C}[x_0, \dots, x_n]_d$ be any form of degree d , then

$$cr(F) \leq N(d, n) := \begin{cases} 2 \binom{n+k}{n} & \text{when } d = 2k + 1 \\ \binom{n+k}{n} + \binom{n+k+1}{n} & \text{when } d = 2k + 2 \end{cases}$$

Notice that $N(d, n) \approx O((d/2)^n)$, while $AH(d, n) \approx O(d^n)$.

Question

Is this bound sharp? (Yes, at least for cubics (nov 13))

The proof of the Proposition uses:

Theorem (Emsalem 1978)

Let $\Gamma \subset \mathbb{C}^n$ be a local 0-dimensional scheme with $\mathcal{I}_\Gamma \subset (y_1, \dots, y_n)$.

Γ is Gorenstein $\iff \exists f \in \mathbb{C}[x_1, \dots, x_n]$ s.t. $\mathcal{I}_\Gamma = f^\perp$.

Furthermore, in this case,

$$\text{length } \Gamma = \dim_{\mathbb{C}} D(f)$$

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This is relevant for the cactus rank:

Lemma (Buczynska-Buczynski, Brachat et al.)

If Γ is apolar to F and $cr(F) = \text{length } \Gamma$, then every component of Γ is a local Gorenstein scheme.

Proof of Proposition

There is a local Gorenstein scheme $\Gamma_{x_0}(F)$ for F supported on $[x_0] \in \mathbb{P}(S_1)$:

Let $f = F(1, x_1, \dots, x_n)$ be the dehomogenization. Then

$$f^\perp \subset \mathbb{C}[y_1, \dots, y_n]$$

defines a local Gorenstein scheme

$\Gamma_{x_0}(F) \subset \mathbb{C}^n = \{y_0 \neq 0\} \subset \mathbb{P}(S_1)$. It is apolar to F and has

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Remark

For any linear form $l \in S_1$, the homogeneous ideal obtained by saturation of the degree d part of the annihilator $(l^{d+1}F)^\perp$ defines a local Gorenstein scheme Γ_l of length bounded above by $N(d, n)$ and supported at $[l] \in \mathbb{P}(S_1)$.

Recent improvement

Proposition (Jelisiejew, nov 13)

The local Gorenstein scheme $\Gamma_{x_0}(F)$ has minimal length among local Gorenstein schemes supported on $[x_0] \in \mathbb{P}(S_1)$ that are apolar to F .

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Corollary

The cactus rank of F is computed by a decomposition $F = F_1 + \dots + F_r$, with linear forms l_1, \dots, l_r , such that

$$cr(F) = \text{length } \Gamma_{l_1}(F_1) + \dots + \text{length } \Gamma_{l_r}(F_r)$$

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Notice, if $F = l^d$, then $\Gamma_l(F) = [l] \in \mathbb{P}(S_1)$.

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Then $f = F(1, x_1, x_2, x_3) = x_3 + x_1 x_2 + x_1^3$ and

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But $h(F, 1) = 4$ so $cr(F), br(F) \geq 4$, hence

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On the other hand, one may show that $r(F) = 7$.

Bounds for monomials

Proposition (R-Schreyer 2011)

Let $1 \leq d_0 \leq d_1 \leq \dots \leq d_n$, then

$$cr(x_0^{d_0} x_1^{d_1} \cdots x_n^{d_n}) = (d_0 + 1) \cdots (d_{n-1} + 1)$$

and

$$r(x_0^{d_0} x_1^{d_1} \cdots x_n^{d_n}) \leq (d_1 + 1) \cdots (d_n + 1).$$

In particular

$$cr((x_0 x_1 \cdots x_n)^d) = r((x_0 x_1 \cdots x_n)^d) = (d + 1)^n.$$

Remark

Enrico Carlini, Maria Virginia Catalisano and Anthony V. Geramita proved equality also in the middle.

In how many ways?

Given r , the set of Waring decompositions

$$\{ \{ [I_1], \dots, [I_r] \} \mid F = I_1^d + \dots + I_r^d \} \subset \text{Hilb}_r \mathbb{P}(S_1)$$

is a subscheme of the Hilbert scheme. Its closure is called the *V*(ariety) of *S*(ums) of *P*(owers), and denoted $VSP(F, r)$.

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For some F and r this variety is a single point, i.e.

there is a unique shortest Waring decomposition of F .

For a general F and $r = r(F)$,

$$\dim VSP(F, r) = AH(n, d)(n + 1) - \binom{n + d}{n}.$$

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Theorem (Sylvester, Hilbert, Palatini, Richmond, 1851-1902,
Mukai, Dolgachev-Kanev, Schreyer, Iliev, R 1989-2000)

If F is general and $r = r(F)$, then $VSP(F, r)$ is

- a point if $(n, d) = (1, 2r - 1), (2, 5), (3, 3)$,
- \mathbb{P}^1 if $(n, d) = (1, 2r - 2)$,
- \mathbb{P}^2 when $(n, d) = (2, 3)$,
- a K3 surface when $(n, d) = (2, 6)$,
- a Fano threefold when $(n, d) = (2, 4)$,
- a Fano fivefold when $(n, d) = (4, 3)$,
- a Hyperkähler fourfold when $(n, d) = (5, 3)$.

Some methods have been developed, building on apolarity, to find Waring decompositions of F .

(cf. [Brachat et al.] and [Oeding-Ottaviani])

For small n and d or when $r(F) \ll AH(n, d)$, then these methods are effective.

In computations, one normally works over \mathbb{Q} or a finite field. For general F , the **first obstruction** is therefore to find a point on the variety $VSP(F, r)$ with the additional property that each l is defined over the ground field.

Question

$VSP(x_0^3x_1 + x_1^3x_2 + x_2^3x_0, 6)$ is Fano threefold. Does there exist $l_1, \dots, l_6 \in \mathbb{Q}[x_0, x_1, x_2]$ and rational numbers $\lambda_1, \dots, \lambda_6$ such that

$$x_0^3x_1 + x_1^3x_2 + x_2^3x_0 = \lambda_1 l_1^4 + \dots + \lambda_6 l_6^4?$$

VSP and VAPS

$VSP(F, r)$ is a natural subscheme of

$$VAPS(F, r) = \{\Gamma \subset \mathbb{P}(S_1) \mid \mathcal{I}_\Gamma \subset F^\perp\} \subset \text{Hilb}_r \mathbb{P}(S_1).$$

$VSP(F, r) \subset VAPS(F, r)$ is the closure of the set of smooth apolar subschemes.

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In general $VSP(F, r)$ is a proper subscheme of $VAPS(F, r)$, in particular when $r(F) > N(n, d) \geq cr(F)$.

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In general $VSP(F, r)$ is a proper subscheme of $VAPS(F, r)$, in particular when $r(F) > N(n, d) \geq cr(F)$.

The difference $VAPS(F, r) \setminus VSP(F, r)$ is a **second obstruction** to finding a decomposition of F .

VSP and VAPS

$VSP(F, r)$ is a natural subscheme of

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The difference $VAPS(F, r) \setminus VSP(F, r)$ is a **second obstruction** to finding a decomposition of F .

Even when F is a quadric of rank n ,

$$VAPS(F, n) \setminus VSP(F, n) \neq \emptyset \text{ when } n \gg 0$$

Quadrics

Let $Q \in \mathbb{C}[x_0, \dots, x_n]$ be a quadric of maximal rank $n + 1$. If

$$Q = l_0^2 + \dots + l_n^2, \text{ then } \{l_0 \cdots l_n = 0\} \subset \mathbb{P}(T_1)$$

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Theorem (R-Schreyer 2011)

Let $Q \in \mathbb{C}[x_0, \dots, x_n]$ be a quadric of rank $n + 1$.

- $VSP(Q, n + 1)$ is rational variety of dimension $\binom{n+1}{2}$.
- It is a smooth Fano variety of index 2 with Picard group isomorphic to \mathbb{Z} if $n < 5$.
- $VSP(Q, n + 1)$ is singular if $n \geq 5$.

When $n \geq 23$, $VAPS(Q, n + 1)$ is reducible.

Thank You!

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