RICAM Special Semester on
Applications of Algebra and Number Theory
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Computer Algebra, Ramanujan Congruences, and Polynomials

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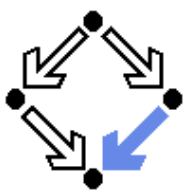

Introduction

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42	56	77	101	135
176	231	297	385	490
627	792	1002	1255	1575
1958	2436	3010	3718	4565
5604	6842	8349	10143	12310
14883	17977	21637	26015	31185
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89134	105558	124754	147273	173525
204226	239943	281589	329931	386155
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966467	1121505	1300156	1505499	1741630
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Theorem

Let S be a compact Riemann surface and z a meromorphic function on S having n poles (incl. multiplicity). Let f be any other meromorphic function on S. Then there exist rational functions $c_k(x) \in \mathbb{C}(x)$ such that

$$f^n + c_1(z)f^{n-1} + \cdots + c_{n-1}(z)f + c_n(z) = 0.$$

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Moreover, if

$$\mathrm{PoleSet}(z) \subseteq \mathrm{PoleSet}(f),$$

then the $c_k(x)$ are polynomials; i.e., $c_k(x) \in \mathbb{C}[x]$.

Partition Numbers

Example: $p(4) = 5$: $4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1.$

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Proof. The generating function is

$$\begin{aligned}\sum_{n=0}^{\infty} p(n)q^n &= \prod_{n=1}^{\infty}(1-q^n)^{-1}\\ &= (1+q^1+q^{1+1}+q^{1+1+1}+\dots)\\ &\times (1+q^2+q^{2+2}+q^{2+2+2}+\dots)\\ &\times \text{etc.}\\ &= \dots + q^{1+1+1}q^{2+2}\dots+\dots\end{aligned}$$

This implies non-holonomicity, because holonomic functions have only finitely many singularities.

Growth of $p(n)$

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$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \quad \textit{as} \quad n \to \infty.$$

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Does this pattern continue?

DEEPER INSIGHT: The Subbarao Conjecture (1966)

- `Ono [1996]:` Given integers $0 \leq B < A$:

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Remark. Radu's algorithmic insight enabled him to invoke a result by Deligne and Rapoport [1979]. The same applies to his proof [2012] of a conjecture of Ahlgren and Ono [2002].

Ahlgren and Ono determined:

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WHY? Let's look again at our table:

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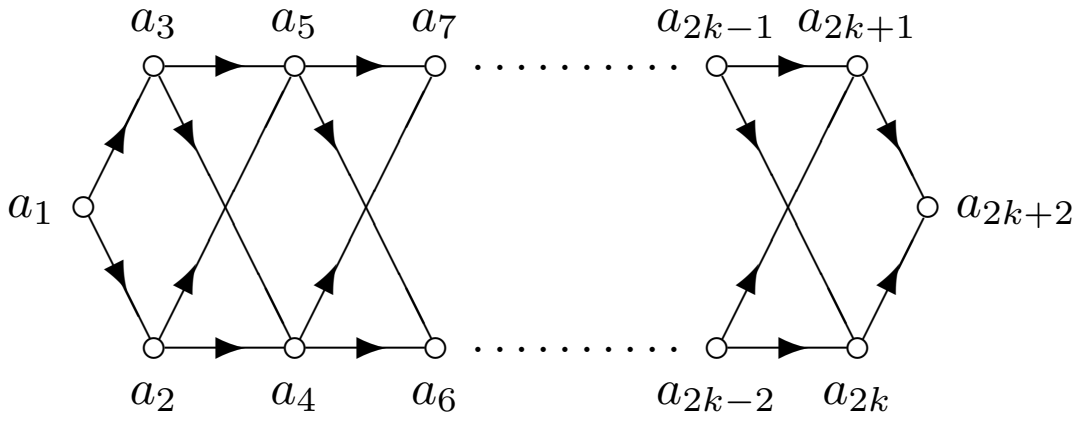
Partition Congruences and Combinatorics

- There is a combinatorial explanation for the divisibility of $p(5n+4)$ by 5 (Dyson's rank statistics). But concerning further connections between combinatorics and modular forms, not too much is known so far.

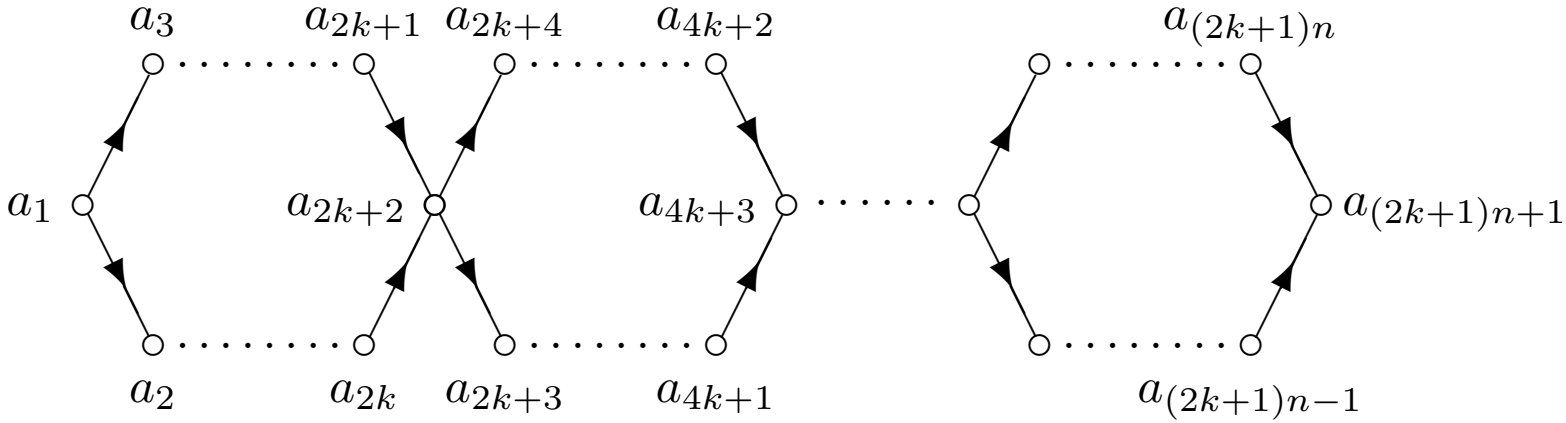
As an example, we present a recent combinatorial construction of modular forms which involves Dedekind's eta function:

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n) \quad (q = e^{2\pi i\tau}).$$

Partition Diamonds



A k-elongated partition diamond of length 1



A k-elongated partition diamond of length n

Generating function:

$$h_{n,k}(q) = \frac{\prod_{j=0}^{n-1}(1+q^{(2k+1)j+2})(1+q^{(2k+1)j+4})\cdots(1+q^{(2k+1)j+2k})}{\prod_{j=1}^{(2k+1)n+1}(1-q^j)}$$

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From: George E Andrews<andrews@math.psu.edu>
Date: Jan 2005
To: Peter.Paule@risc.uni-linz.ac.at,
Axel.Riese@risc.uni-linz.ac.at

Subject: I had a brainstorm

Dear Axel and Peter,

I have had a BRAINSTORM! (I can almost see the smile on Peter's face ...) To begin with let me draw your attention to Thm.2 in PAXI and Cor.2.1 in PAVIII ... However, THIS IS JUST THE BEGINNING.

Namely, I always had a slight disappointment in both the earlier results because the generating function ... was not a modular form. However, I now know how to produce the appropriate directed graph to provide ... a generating function that is not only a modular form but, in fact, a product of instances of Dedekind's eta-function.

Recall the generating function:

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The result of Andrews' brainstorm: delete the source:

$$h^*_{n,k}(q) = \frac{\prod_{j=0}^{n-1}(1+q^{(2k+1)j+1})(1+q^{(2k+1)j+3})\cdots(1+q^{(2k+1)j+2k-1})}{\prod_{j=1}^{(2k+1)n}(1-q^j)}$$

and glue the diamonds together:

$$\sum_{n=0}^{\infty} \Delta_k(n) q^n := h_{n,k}(q)\, h^*_{n,k}(q)$$

$$= \prod_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^{(2k+1)n})}{(1-q^n)^3(1-q^{(4k+2)n})}$$

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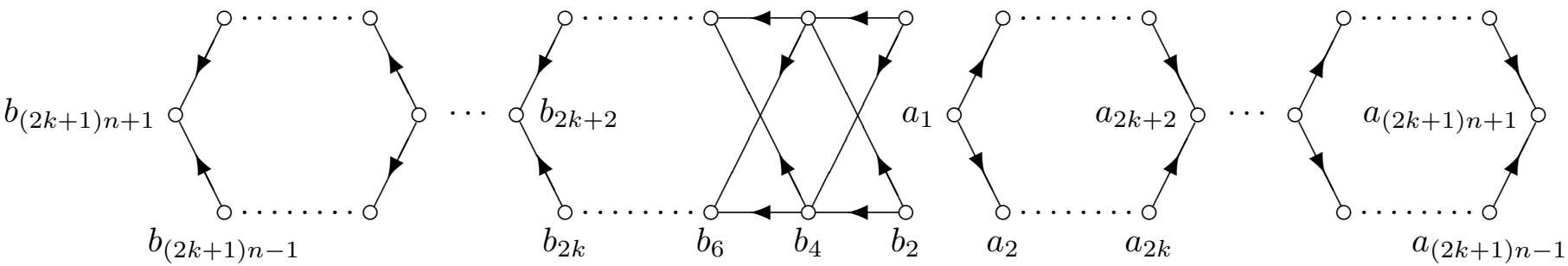
$$= q^{(k+1)/12} \frac{\eta(2\tau)\eta((2k+1)\tau)}{\eta(\tau)^3\eta((4k+2)\tau)}$$



A broken k-diamond of length $2n$

Numerous Congruences for $\Delta_k(n)$

Example [Sellers & Radu, 2012]

$$\sum_{n=0}^{\infty} \Delta_2(3n+1)q^n \equiv 2q \prod_{n=1}^{\infty} \frac{(1-q^{10n})^4}{(1-q^{5n})^2} \pmod 3$$

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This implies for all $n \geq 0$:

$$\begin{aligned} \Delta_2(15n+1) &\equiv 0 \pmod 3, \\ \Delta_2(27n+16) &\equiv 0 \pmod 3, \\ \Delta_2(147n+58) &\equiv 0 \pmod 3, \end{aligned}$$

etc.

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Note. More people became interested in such congruences, e.g., S.H. Chan, W.Y.C. Chen, S. Cui, A.R.B. Fan, S.S. Fu, N. Gu, M.D. Hirschhorn, M. Jameson, E. Mortenson, X. Xiong, R.T. Yu.

Ramanujan's Congruences

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$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5, \\ p(7n+5) &\equiv 0 \pmod 7, \\ p(11n+6) &\equiv 0 \pmod{11} \end{aligned}$$

Ramanujan's Congruences

$$
\begin{aligned}
p(5n+4) &\equiv 0 \pmod 5,\\
p(5^2n+24) &\equiv 0 \pmod{5^2},\\
p(5^3n+99) &\equiv 0 \pmod{5^3},
\end{aligned}
$$

etc.

Ramanujan's Conjecture (1919)

For $\ell \in \{5, 7, 11\}$ and $\alpha \in \{1, 2, 3, \dots\}$:

$$p(\ell^\alpha n + \mu_{\alpha,\ell}) \equiv 0 \pmod{\ell^\alpha},$$

where $\mu_{\alpha,\ell} \in \{0, \dots, \ell^\alpha - 1\}$ is uniquely defined by

$$24\,\mu_{\alpha,\ell} \equiv 1 \pmod{\ell^\alpha}.$$

A Bit of History

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- Watson [1938]: $\ell = 5$ and $\ell = 7$ *(corrected version)*
- Atkin [1967]: $\ell = 11$
- Gordon [1983]:

$$p(11^{\alpha} n + \mu_{\alpha,11}) \equiv 0 \pmod{11^{\alpha+\epsilon}},$$

where ϵ is a fixed non-positive integer to be determined. Following Gordon's proof one can easily determine $\epsilon = 0$.

Our proof for $\ell = 11$ and $\ell = 5, 7$:

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- This way we obtain a unified framework for all three cases $\ell = 5, 7, 11$.
- The price to pay: our approach is more expensive computationally.

The Modular Function Setting

Basic Notions

The level ℓ congruence subgroup

$$\Gamma_0(\ell) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{\ell} \right\}$$

acts on $\mathbb{H}^* := \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$ (extended upper half plane) via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau) \mapsto \frac{a\tau + b}{c\tau + d}.$$

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This induces an action on functions $f : \mathbb{H}^* \to \mathbb{C} \cup \{i\infty\}$ by

$$(f|\gamma)(\tau) := f\left(\frac{a\tau + b}{c\tau + d}\right) \quad \text{where} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Compact Riemann Surfaces $X_0(\ell)$

$$X_0(\ell) := \text{orbit space of the action of } \Gamma_0(\ell) \text{ on } \mathbb{H}^*$$
$$= \{[\tau] : \tau \in \mathbb{H}^*\}$$

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$$\frac{a\tau + b}{c\tau + d} \longrightarrow \frac{a}{c} \quad \text{when} \quad \tau \longrightarrow i\infty.$$

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Set of cusps: For any prime ℓ,

$$\{[x] \in X_0(\ell) : x \in \mathbb{Q} \cup \{i\infty\}\} = \{[0], [i\infty]\}.$$

Modular Functions (MF) on Compact Riemann Surfaces $X_0(\ell)$

$f : \mathbb{H}^* \to \mathbb{C} \cup \{i\infty\}$ induces a MF $f^* : X_0(\ell) \to \mathbb{C} \cup \{i\infty\}$ if

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$f : \mathbb{H}^* \to \mathbb{C} \cup \{i\infty\}$ induces a MF $f^* : X_0(\ell) \to \mathbb{C} \cup \{i\infty\}$ if

- f is holomorphic on $\mathbb{H}$;
- f is constant on the orbits $[\tau]$, $\tau \in \mathbb{H}^*$;
- f is meromorphic at points in $\mathbb{Q} \cup \{i\infty\}$.

Fourier expansions of f^* *at* cusps $[a/c] \in X_0(\ell)$

For $\tau \in \text{neighborhood}(i\infty)$:

$$f^*\left(\left[\frac{a\tau+b}{c\tau+d}\right]\right)) = (f|\gamma)(\tau) = \sum_{n \geq \text{ord}_{[a/c]}(f^*)} a_{[a/c]}(n) q_{[a/c]}^n.$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$, $q_{[a/c]} = \left(e^{2\pi i \tau}\right)^{\frac{\gcd(c^2,\ell)}{\ell}}$.

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Note. If $[a/c] = [a/0] = [i\infty]$ then $q_{[a/c]} = q$.

Back to Ramanujan's congruences

$$L_{2\beta-1,\ell} := q\prod_{n=1}^{\infty}(1-q^{\ell n})\sum_{n=0}^{\infty}p(\ell^{2\beta-1}n+\mu_{2\beta-1,\ell})q^n,$$

and

$$L_{2\beta,\ell} := q\prod_{n=1}^{\infty}(1-q^{n})\sum_{n=0}^{\infty}p(\ell^{2\beta}n+\mu_{2\beta,\ell})q^n.$$

We consider these series as Fourier expansions at $[i\infty]$ of modular functions $f^* : X_0(\ell) \to \mathbb{C} \cup \{i\infty\}$ from a suitable subring $R(\ell)$.

The Rings $R(5), R(7)$, and $R(11)$

Recall $$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n) \quad (q = e^{2\pi i\tau}).$$

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From the literature it is well known that we can take

$$R(5) = \mathbb{Z}[z_5], \quad R(7) = \mathbb{Z}[z_7],$$

and

$$R(11) = \langle J_0, J_1, \ldots, J_4 \rangle_{\mathbb{Z}[z_{11}]}$$

where

$$z_5 := \frac{\eta^6(5\tau)}{\eta^6(\tau)}, \quad z_7 := \frac{\eta^4(7\tau)}{\eta^4(\tau)}, \quad z_{11} := \frac{\eta^{12}(11\tau)}{\eta^{12}(\tau)},$$

and with the J_k such that $\mathrm{ord}_{[i\infty]}(J_k^*) = k$ as in Atkin (1967).

The U-operator

$$U_\ell(f^*) := \sum_n a(\ell n) q^n$$

where $f^*([\tau]) = \sum_n a(n) q^n$ is the Fourier expansion of f^* at $[i\infty]$.

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Definition. For prime ℓ,

$$U_\ell^{(1)}(f^*) := U_\ell\left(\frac{\eta(\ell^2\tau)}{\eta(\tau)} f^*\right) \quad \text{and} \quad U_\ell^{(2)}(f^*) := U_\ell(f^*).$$

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Lemma.

$$f^* \in R(\ell) \Rightarrow U_\ell^{(i)}(f^*) \in R(\ell).$$

Recursive representation of $L_{\alpha,\ell}$ in $R(\ell)$

Lemma. For $\ell \in \{5, 7, 11\}$,

$$L_{\alpha,\ell} \in R(\ell);$$

the $L_{\alpha,\ell}$ satisfy a recursion in $R(\ell)$ as follows:

$$L_{0,\ell} = 1,$$

$$L_{2\beta-1,\ell} = U_\ell^{(1)}(L_{2\beta-2,\ell}) \text{ and } L_{2\beta,\ell} = U_\ell^{(2)}(L_{2\beta-1,\ell}).$$

This recursion is used in the induction proof of our "Main Theorem":

Main Theorem

For suitably defined subsets $X_{s,\ell}$ of $R(\ell)$ we have:

Theorem. For $\ell \in \{5, 7, 11\}$ and all positive integers β there exist $f_{\beta,\ell} \in X_{1,\ell}$ if β is odd, and $f_{\beta,\ell} \in X_{2,\ell}$ if β is even, such that

$$L_{\beta,\ell} = \ell^{\beta} f_{\beta,\ell} \qquad \text{for} \quad \ell = 5, 11,$$

and

$$L_{\beta,\ell} = \ell^{\lceil \frac{\beta+1}{2} \rceil} f_{\beta,\ell} \qquad \text{for} \quad \ell = 7.$$

Note. Ramanujan's congruences are obtained as an immediate corollary.

The $R(\ell)$-subsets $X_{s,\ell}$ ($\ell = 11$: Atkin)

Set

$$A_\ell := \frac{12}{\gcd(\ell-1,12)}\frac{\ell}{\ell+1} \quad \text{and} \quad z_\ell := \left(\frac{\eta_\ell}{\eta}\right)^{\frac{24}{\gcd(\ell-1,12)}}.$$

Definition. For $\ell \in \{5,7,11\}$ and $s = 1, 2$:

$$X_{s,\ell} := \left\{ \sum_{i=0}^{n_\ell - 1} J_{i,\ell} \sum_{j=0}^{\infty} a_i(j) \ell^{\lceil \frac{A_\ell}{\ell}(\ell j + \xi_i^{(s,\ell)})\rceil} z_\ell^j \;:\; a_0(0) = 0 \text{ and } \right.$$
$$\left. \text{with } a_i(j) \in \mathbb{Z} \text{ nonzero for only finitely many } j \right\}$$

The integer exponents $\xi_i^{(s,\ell)}$ are defined as follows:

The Exponents $\xi_i^{(s,\ell)}$

Definition. For $\ell \in \{5, 7, 11\}$ and $s = 1, 2$ define

$$\xi^{(s,\ell)} : \{0, \ldots, n_\ell - 1\} \to \mathbb{Z},\ i \mapsto \xi_i^{(s,\ell)}$$

by

$$\begin{aligned}(\xi_0^{(1,11)}, \ldots, \xi_4^{(1,11)}) &:= (-5, -1, 1, 2, 6),\\ (\xi_0^{(2,11)}, \ldots, \xi_4^{(2,11)}) &:= (-4, 0, 2, 3, 7);\end{aligned}$$

and

$$\xi_0^{(1,7)} := -7 \text{ and } \xi_0^{(2,7)} := -10;$$

and

$$\xi_0^{(1,5)} := -6 \text{ and } \xi_0^{(2,5)} := -5.$$

Main Tool: The Fundamental Lemma

Fundamental Lemma. Let $\ell \in \{5, 7, 11\}$. For any analytic $w : \mathbb{H} \to \mathbb{C}$ and $j \in \mathbb{Z}$:

$$U_\ell(w z_\ell^j) = -\sum_{i=0}^{d_\ell - 1} a_i^{(\ell)}(z_\ell) U_\ell(w z_\ell^{j+i-d_\ell}).$$

Setting w to the generators of the module $R(\ell)$, we obtain the $U_\ell^{(i)}$ actions on all the module elements — provided

we know the $2 \cdot d_\ell \cdot n_\ell$ $(2 \cdot 5 \cdot 1, 2 \cdot 7 \cdot 1, 2 \cdot 55 \cdot 5)$ initial actions.

This action then serves to express $L_{\beta,\ell}$ in terms of the generators which reveal the divisibilty properties of these elements.

Proof of the Main Theorem

The Fundamental Polynomial for any prime $\ell \geq 5$

The main ingredient in our induction proof is

Theorem. For any prime $\ell \geq 5$ there exists a monic irreducible polynomial $F_\ell(X, Y) \in \mathbb{Q}[Y][X]$ of degree $d_\ell := \frac{\ell(\ell-1)}{\gcd(\ell-1,12)}$ in X (and Y) such that

$$F_\ell(z_\ell(\tau), z_\ell(\ell\tau)) = 0.$$

Recall that

$$z_\ell(\tau) = \left(\frac{\eta(\ell\tau)}{\eta(\tau)}\right)^{\frac{24}{\gcd(\ell-1,12)}}.$$

Proving the Existence of the Fundamental Polynomial

Proof.

The proof uses a variant of a classical fact from Riemann surfaces; namely, that on a compact Riemann surface two meromorphic functions are connected by an algebraic relation. □

The Fundamental Polynomial for $\ell \in \{5, 7, 11\}$

Theorem. For $\ell \in \{5, 7, 11\}$ the fundamental polynomial

$$F_\ell(X, Y) = X^{d_\ell} + \sum_{i=0}^{d_\ell - 1} a_i^{(\ell)}(Y) X^i$$

has coefficient polynomials $a_i^{(\ell)}(Y) \in \mathbb{Q}[Y]$ of the form

$$a_i^{(\ell)}(Y) = \sum_{j=\lceil \frac{d_\ell - i}{\ell} \rceil}^{d_\ell} s_\ell(i, j) \ell^{\lceil \frac{A_\ell}{\ell}(\ell j + i - d_\ell) \rceil} Y^j$$

with $s_\ell(i, j) \in \mathbb{Z}$ (determined using computer algebra).

Corollary: The Fundamental Lemma

Fundamental Lemma. Let $\ell \in \{5, 7, 11\}$. For any $w : \mathbb{H} \to \mathbb{C}$ and $j \in \mathbb{Z}$:

$$U_\ell(w z_\ell^j) = -\sum_{i=0}^{d_\ell - 1} a_i^{(\ell)}(z_\ell) U_\ell(w z_\ell^{j+i-d_\ell}).$$

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Proof.

Applying U_ℓ to $wz_\ell^{j-d_\ell} F_\ell(z_\ell(\tau), z_\ell(\ell\tau)) = 0$ gives the desired result. □