

# Algorithms and Statistics for Additive Polynomials

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with

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# Polynomial Composition and Decomposition

## Functional Composition

Let  $g, h \in F[x]$ , for a field  $F$ .

**Compose**  $g, h$  as functions  $f(x) = g(h(x)) = g \circ h$

Generally non-distributive operation (not always, as we'll see!):

$$g(h_1(x) + h_2(x)) \neq g(h_1(x)) + g(h_2(x))$$

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## Decomposition

Given  $f \in F[x]$ , can it be decomposed?

Do there exist  $g, h \in F[x]$  such that  $f = g \circ h$ ?

$$f = x^4 - 2x^3 + 8x^2 - 7x + 5$$

$$g = x^2 + 3x - 5 \quad h = x^2 - x - 2$$

$$\Rightarrow f = g \circ h$$

# Tame and Wild Decomposition

Let  $F$  be a field of characteristic  $p$  and  $f \in F[x]$  monic of degree  $d$ .

Normalize  $f, g, h$  to monic and *original*:  $h(0) = 0$

- $f$  is *tame* if  $p \nmid d$

- $f$  is *wild* if  $p \mid d$

Traditionally this describes the ramification of  $F(x)$  over  $F(f(x))$ .

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## Tame decomposition

- Ritt (1922) describes all tame decompositions and “ambiguities”.
- For a fixed  $s$ , there are either 0 or 1 monic  $h \in F[x]$  of degree  $s$  with  $h(0) = 0$  such that  $f(x) = g(h(x))$ .
- See von zur Gathen (2013) for complete decompositions.

# Tame and Wild Decomposition

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## Wild decomposition

• Life is much more difficult

• (G, 1988) For a finite field  $F$  of characteristic  $p$ , there are  $f \in F[x]$  of degree  $d$  with  $> d^{\lambda \log d}$  monic, original,  $h \in F[x]$  of degree  $s \approx \sqrt{s}$  such that  $f(x) = g(h(x))$ , where  $\lambda = (6 \log p)^{-1}$ .

# Tame and Wild Decomposition

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## Wild decomposition

- On the bright side, there are at most  $(d-1)/(s-1)$  *indecomposable* monic, original  $h \in F[x]$  of degree  $s$  such that  $f(x) = g(h(x))$ . (Von zur Gathen, G, Ziegler, 2010)

## Additive Polynomials

Additive or linearized polynomials are those such that

$$f(x + y) = f(x) + f(y)$$

Non-linear additive polynomials only exist in  $F[x]$  if  $F$  has prime characteristic  $p$ , and have the form

$$f = a_0x + a_1x^p + a_2x^{p^2} + \cdots + a_nx^{p^n} \in F[x].$$



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## Example

Let  $\mathbb{F}_{125} = \mathbb{F}_5[\theta]/(\theta^3 + \theta + 1)$ .

$$f = x^{25} + (3\theta^2 + 4\theta + 2)x^5 + (3\theta^2 + 4\theta + 2)x$$

is an additive polynomial, and

$$\begin{aligned} f &= (x^5 + (\theta^2 + \theta + 4)x) \circ (x^5 + 3\theta x) \\ &= (x^5 + (2\theta^2 + 4\theta + 2)x) \circ (x^5 + (\theta^2 + 2\theta)x) \end{aligned}$$

# Ore's Legacy

In 1932-4, Oystein Ore wrote four seminal papers for finite fields, differential algebra, and computer algebra

- 1 O. Ore, *Formale Theorie der linearen Differentialgleichungen*, J. reine angew. Math., v. 168, pp. 233-252, 1932.
- 2 O. Ore, *Theory of Non-Commutative Polynomials*, "Annals of Mathematics", v. 34, no. 22, pp. 480–508, 1933.
- 3 O. Ore, *On a Special Class of Polynomials*, Trans. Amer. Math. Soc., v. 35, pp. 559-584, 1933.
- 4 O. Ore, *Contributions to the Theory of Finite Fields*, Trans. Amer. Math. Soc., v. 36, pp. 243-274, 1934.

[1,2] form the basis for modern computational theory of LODEs  
(Ore\_algebra,OreTools)

[3,4] have had great influence on theory of finite fields

# Ore Polynomials in Computational Algebra

Additive polynomials are employed in

- Error correcting codes
- HFE and other cryptosystems
- Mathematical constructions in algebraic function fields
- General fun and parlour tricks.

Despite their large (exponential) degrees we will see that we can compute very efficiently with them.

# Ore Polynomials and Additive Polynomials

Let  $q = p^e$  for prime  $p$  and integer  $e$ .

$\mathbb{F}_q$  the finite field with  $q$  elements.

- Additive polynomials over  $\mathbb{F}_q$ :

$$\mathbb{F}_q[x; p] = \left\{ \sum_{0 \leq i \leq n} a_i x^{p^i} \in \mathbb{F}_q[x] \right\}$$

Ring under usual polynomial addition (+) and functional composition( $\circ$ ), with  $x^p \circ ax = a^p x^p$ .

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$$\mathbb{F}_q[x; \sigma_p] = \left\{ \sum_{0 \leq i \leq n} a_i x^i \in \mathbb{F}_q[x] \right\}$$

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- $xa = \sigma_p(a)x$

- $\sigma_p(a) = a^p$  is the Frobenius automorphism of  $\mathbb{F}_q/\mathbb{F}_p$

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- If  $W$  an  $\mathbb{F}_p$ -subspace of  $V_f$ , and  $h \in \overline{\mathbb{F}}_q[x]$  has roots exactly  $W$  then  $h \in \overline{\mathbb{F}}_q[x; p]$  and  $\exists g \in \overline{\mathbb{F}}_q[x; p]$  such that  $f = g \circ h$ .

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**Decomposing additive polynomials  $\equiv$  finding subspaces of  $V_f$**

- Let  $\sigma_q(a) = a^q$ , the  $q$ -Frobenius automorphism.  
If  $W$  is also  $\sigma_q$ -invariant, then  $h \in \mathbb{F}_q[x; p]$

**Decomposing additive polynomial over  $\mathbb{F}_q[x]$   
 $\equiv$  finding  $\sigma_q$ -invariant subspace of  $V_f$**

## The Geometry of Additive Polynomials (2)

### Example

Again let  $\mathbb{F}_{125} = \mathbb{F}_5[\theta]/(\theta^3 + \theta + 1)$ , and

$$f = x^{25} + (3\theta^2 + 4\theta + 2)x^5 + (3\theta^2 + 4\theta + 2)x$$

Then

$$\mu = \text{RootOf}(x^4 + (\theta^2 + 3\theta + 4)x^2 + (3\theta^2 + 4\theta)x + (4\theta^2 + \theta))$$

$$\nu = \text{RootOf}(x^4 + (4\theta^2 + 2\theta + 1)x^2 + (4\theta^2 + 2\theta)x + (4\theta^2 + \theta))$$

$$V_f = \{\alpha\mu + \beta\nu : \alpha, \beta \in \mathbb{F}_p\} \subseteq \mathbb{F}_{5^{12}}$$

$$\sigma_q = \begin{pmatrix} 3 & 3 \\ 2 & 3 \end{pmatrix} \quad (\text{after some ugly calculations})$$

Probably not the best way to work with additive polynomials...

## Right Composition Factors as Eigenvectors of $\sigma_q$

Given  $f \in \mathbb{F}_q[x; p]$ , find

$$\#\left\{h = x^p + ax \in \mathbb{F}_q[x; p] : \exists g \in \mathbb{F}_q[x; p] \text{ with } f = g \circ h\right\}$$

The number of right composition factors of  $f$  degree  $p$

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= number of 1-dimensional  $\sigma_q$ -invariant subspaces of  $V_f$

= number of eigenvectors of  $\sigma_q$

Remember,  $\sigma_q : V_f \rightarrow V_f$  is a  $\mathbb{F}_p$ -linear map

➡  $\sigma_q$  acts like an  $n \times n$  matrix over  $\mathbb{F}_p$

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New questions:

- How many eigenvectors can an  $n \times n$  matrix over  $\mathbb{F}_q$  have?
- How can we compute this?



## Right Composition Factors as Eigenvectors of $\sigma_q$ (2)

How many eigenvectors can a matrix have?

Look at the (rational) Jordan form in  $\mathbb{F}_p^{n \times n}$

**Example:** degree  $p^2$  ( $n = 2$ ): the number of ways of decomposing

$$\begin{aligned} f &= x^{p^2} + a_1 x^p + a_0 x \\ &= (x^p + b_0 x) \circ (x^p + c_0 x) \end{aligned}$$

Put  $\sigma_q$  in rational Jordan form; there are only four possibilities:

$$\sigma_q \sim \begin{pmatrix} 0 & \alpha \\ 1 & \beta \end{pmatrix} \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},$$

Here  $\lambda, \mu, \alpha, \beta \in \mathbb{F}_p^*$ ,  $\lambda \neq \mu$  and  $y^2 - \beta y - \alpha \in \mathbb{F}_p[y]$  is irreducible.

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0

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0                      1

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$$\sigma_q \sim \begin{matrix} \begin{pmatrix} 0 & \alpha \\ 1 & \beta \end{pmatrix} & \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, & \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, & \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{p+1} \end{matrix}$$

Here  $\lambda, \mu, \alpha, \beta \in \mathbb{F}_p^*$ ,  $\lambda \neq \mu$  and  $y^2 - \beta y - \alpha \in \mathbb{F}_p[y]$  is irreducible.

An  $f \in \mathbb{F}_q[x; \sigma]$  of degree  $p^2$  can have only 0, 1, 2, or  $p + 1$  right composition factors of degree  $p$ .

## Right Composition Factors as Eigenvectors of $\sigma_q$ (3)

**Example:** degree  $p^3$  ( $n = 3$ ): the number of ways of decomposing

$$f = x^{p^3} + a_2 x^{p^2} + a_1 x^p + a_0 x$$

$$= (x^{p^2} + b_1 x^p + b_0 x) \circ (x^p + c_0 x)$$

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$p+2$                        $3$                        $1$                        $0$

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➔ An  $f \in \mathbb{F}_q[x; \sigma]$  of degree  $p^3$  can have only

$$0, 1, 2, 3, p + 1, p + 2, \text{ or } p^2 + p + 1$$

right composition factors of degree  $p$ .

# General categorization of number of composition factors

How many composition factors of degree  $p$  can an additive polynomial of degree  $p^n$  have?  $S_n$  is the set of possible numbers:

$$S_0 = \{0\}$$

$$S_1 = \{0, 1\}$$

$$S_2 = \{0, 1, 2, p + 1\}$$

$$S_3 = \{0, 1, 2, 3, p + 1, p + 2, p^2 + p + 1\}$$

$$S_4 = \{0, 1, 2, 3, 4, 2p + 2, p^2 + p + 2, p^3 + p^2 + p + 1\}$$

$\vdots$       $\vdots$

In general  $\#S_n = \sum_{0 \leq k \leq n} P(k)$ , where  $P(k)$  is the number of additive partitions of  $k$ .

# Efficient Counting of Composition Factors

Roots of  $f \in \mathbb{F}_q[x; p]$  of degree  $p^n$  may be in an extension field of high degree ( $O(p^{O(n^2)})$ ).

➔ Can't really compute directly with  $V_f$ .

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## Look at the ring structure of $\mathbb{F}_q[x; p]$

$\mathbb{F}_q[x; p]$  is a (non-commutative) ring under the  $+$  and  $\circ$

- Left (and right) Euclidean ring: LCLM and GCRD operations.
- No unique factorization (but Jordan-Hölder and Krull-Schmidt give a lot of structure to factorizations)
- Fast algorithms for  $+$ ,  $\circ$ , lclm and gcd (time  $O(n^3 \log^2 q)$ ).

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Example ( $\mathbb{F}_{125}[x; 5]$  again – a left Euclidean ring)

$$f = x^{25} + (3\theta^2 + 4\theta + 2)x^5 + (3\theta^2 + 4\theta + 2)x$$

$$g = x^{25} + (3\theta^2 + \theta + 3)x^5 + (4\theta^2 + 2\theta + 2)x$$

$$f + g = 2x^{25} + (3\theta^2 + 2\theta + 3)x^5 + (4\theta^2 + 3\theta + 2)x$$

$$f \circ g = x^{625} + (4\theta^2 + 2)x^{125} + \dots + (2\theta^2 + 3\theta + 1)x$$

$$\text{lclm}(f, g) = x^{125} + (\theta^2 + 3\theta + 1)x^{25} + (2\theta^2 + 3)x^5 + (2\theta^2 + 2\theta + 3)x$$

$$\text{gcd}(f, g) = x^5 + 3\theta x$$

## The Centre of It All

The **centre** of  $\mathbb{F}_q[x; p]$  is also very useful:

$$\text{centre}(\mathbb{F}_q[x; p]) = \mathbb{F}_p[x; q] = \left\{ \sum \alpha_i x^{q^i} \in \mathbb{F}_p[x] \right\}$$

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$\cong \mathbb{F}_p[y]$  the usual (commutative) polynomials!

$$\sum_{0 \leq i \leq n} \alpha_i x^{q^i} \mapsto \sum_{0 \leq i \leq n} \alpha_i y^i \quad \text{for } a_0, \dots, a_n \in \mathbb{F}_p$$



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## A cool trick

Given any  $f \in \mathbb{F}_q[x; p]$  we can find a left multiple in the center with  $O(n^3 \log^2 q)$  operations in  $\mathbb{F}_q$ .

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$$f = x^{25} + (3\theta^2 + 4\theta + 2)x^5 + (3\theta^2 + 4\theta + 2)x \in \mathbb{F}_q[x; 5]$$

$$f^* = x^{125^2} + 4x^{125} + 3x \in \mathbb{F}_p[x; 125]$$

*$f^*$  is the minimal central left multiple (mclm) of  $f$*

## The Centre of It All (2)

Basis of the factoring algorithm in G (1992, 1998):

Factor the minimal central left multiple and take GCRDs:

$$f = x^{25} + (3\theta^2 + 4\theta + 2)x^5 + (3\theta^2 + 4\theta + 2)x \in \mathbb{F}_q[x; 5]$$

$$f^* = x^{125^2} + 4x^{125} + 3x \in \mathbb{F}_p[x; 125]$$

$$\mapsto y^2 + 4y + 3 = (y + 1)(y + 3)$$

$$f^* = \underbrace{(x^{125} + x)}_{f_1} \circ \underbrace{(x^{125} + 3x)}_{f_2} = (x^{125} + 3x) \circ (x^{125} + x)$$

$$\left. \begin{aligned} \text{gcd}(f, f_1) &= x^5 + (\theta^2 + 2\theta)x \\ \text{gcd}(f, f_2) &= x^5 + 3\theta x \end{aligned} \right\} \text{right composition factors of } f$$

Can't completely decompose with this technique...

## Decomposition in $\mathbb{F}_q[x; p]$

### Theorem (G 1992, 1998)

*Given  $f = \sum_{0 \leq i \leq n} a_i x^{p^i} \in \mathbb{F}_q[x]$ , can find  $g, h \in \mathbb{F}_q[x]$ , if they exist, such that  $f = g \circ h$ . Requires expected time  $O(n^4 \log^2 q)$  operations in  $\mathbb{F}_q$  (Las Vegas).*

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Hardest when minimal central left multiple is irreducible in  $\mathbb{F}_p[y]$ .

- Construct a finite algebra  $\mathcal{A}$  from  $f$ , called the *eigenring*; show that zero-divisors in  $\mathcal{A}$  yields composition factors of  $f$ .
- Show how to find zero divisors in a finite algebra quickly.
- Build very explicit Krull-Schmidt and Jordan-Hölder like decompositions, which show structure of all decompositions

# Enter the Eigenring

Decompose an algebra over  $\mathbb{F}_q$  associated with  $f$ .

## Definitions

- Idealizer:  $\mathcal{I}_f \subseteq R$  largest subring in which  $Rf$  a two-sided ideal

$$\mathcal{I}_f = \{u \in R : fu \in Rf\}$$

- Eigenring:  $E_f = \mathcal{I}_f/Rf$  an associative algebra over  $\mathbb{F}_q$

## Theorems

- $E_f$  has zero divisors  $uv = 0$  iff  $f$  is decomposable.
- Zero divisors “split”  $f$ :  $\text{gcd}(f, v) \neq 1$ .
- $E_f$  has orthogonal idempotents  $v^2 = 1, w^2 = 1$  with  $v + w = 1$  iff  $f = \text{lcm}(f_1, f_2)$ , with  $\text{gcd}(f_1, f_2) = 1$
- Can find eigenring with  $O(n^3)$  operations in  $\mathbb{F}_q$

## Detour: Decomposing Associative Algebras over $\mathbb{F}_q$

Describe associative algebra  $\mathcal{A}$  by a  $\mathbb{F}_q$ -basis  $\alpha_1, \dots, \alpha_\ell \in \mathbb{F}_q^{m \times m}$

- $\mathcal{A} = \langle \alpha_1, \dots, \alpha_\ell \rangle \subseteq \mathbb{F}_q^{m \times m}$

How do we

- Find zero-divisors or certify there are none
- If  $\mathcal{A}$  semisimple, decompose  $\mathcal{A} = \mathcal{A}_1 \oplus \dots \oplus \mathcal{A}_r$ , for  $\mathcal{A}_i$  simple
- If  $\mathcal{A}$  is simple, find the explicit isomorphism with  $\mathbb{F}_{q^n}^{s \times s}$

Friedl & Rónyai (1985) show how to do all this in polynomial time (up to factoring polynomials in  $\mathbb{F}_p[x]$ )

G & Eberly (2000, 2004): “nearly optimal”  $O(m^3 \log m + m^2 \ell)$  operations in  $\mathbb{F}_q$

- Las Vegas for semisimple algebras over  $\mathbb{F}_q$
- Monte Carlo for general algebras over  $\mathbb{F}_q$

# Detour: Decomposing Associative Algebras over $\mathbb{F}_q$

## Density Theorems

Let  $\mathcal{A} = \langle a_1, \dots, a_\ell \rangle \subseteq \mathbb{F}_q^{n \times n}$  be an associative algebra over  $\mathbb{F}_q$ ,  $\alpha$  randomly chosen from  $\mathcal{A}$ , and  $f = \text{minpoly}(\alpha) \in \mathbb{F}_q[x]$ .

- For *any*  $\mathcal{A}$  over  $\mathbb{F}_q$  with zero divisors,

$$\text{Prob}\left\{f \text{ reducible}\right\} \geq 1/9$$

- If  $f = f_1 f_2$  then  $f_1(\alpha) f_2(\alpha) = 0$ , so  $f_1(\alpha), f_2(\alpha)$  zero divisors
- When  $\mathcal{A}$  is a field,  $\text{Prob}\{\deg f = n\} \geq 1/4$ .



# Central Multiples and Frobenius Automorphisms

Theorem (von zur Gathen, G, and Ziegler 2010)

- $f \in \mathbb{F}_q[x; p]$  squarefree of degree  $p^n$  with roots  $V_f$
- $\sigma_q : V_f \rightarrow V_f$  the Frobenius automorphism.
- $f^* \in \mathbb{F}_p[x; q]$  be the minimal central left multiple of  $f$ .

$$f^* = \sum_{0 \leq i \leq n} \alpha_i x^{q^i} \Rightarrow f^+ = \sum_{0 \leq i \leq n} \alpha_i y^i \text{ is min poly of } \sigma_q.$$

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- ➔ Can find the minimal polynomial of  $\sigma_q$  quickly
- ➔ Can compute the complete rational Jordan form of  $\sigma_q$
- ➔ Given  $f \in \mathbb{F}_q[x; p]$  of degree  $p^n$ , we can compute the number of right composition factors of degree  $p$  with  $O(n^3 \log^2 q)$  operations in  $\mathbb{F}_q$ .

## Back to our example in $\mathbb{F}_{125}[x; 5]$

$$f = x^{25} + (3\theta^2 + 4\theta + 2)x^5 + (3\theta^2 + 4\theta + 2)x \in \mathbb{F}_q[x; 5]$$

$$\begin{aligned} f^* &= x^{125^2} + 4x^{125} + 3x \in \mathbb{F}_p[x; 125] \\ &= (x^{125} - 4x) \circ (x^{125} - 2x) \end{aligned}$$

So  $\sigma_q \sim \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$  and  $\begin{cases} \sigma_q \text{ has two eigenvectors} \\ f \text{ has two right factors of degree 5} \\ h_1 = x^5 + \theta^2 x + 2\theta x, h_2 = x^5 + 3\theta x \end{cases}$

## Subadditive/Projective Polynomials

Subadditive polynomials: Cohen (1990), Abhyankar (1997):

$$\Psi = \sum_{0 \leq i \leq n} a_i x^{(p^i - 1)/(p - 1)} \in \mathbb{F}_q[x] \text{ for } b \neq 0$$

Numerous applications: strong Davenport pairs, difference sets, cryptographically secure sequences, error-correcting codes...

Blüher (2004) showed that  $x^{p+1} + ax + b$  has either 0, 1, 2, or  $p + 1$  roots in  $\mathbb{F}_q$ . This looks familiar!

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## Lemma

$\Psi$  has a root  $c \in \mathbb{F}_q \iff \sum a_i x^{p^i} = g \circ (x^p - cx)$  for  $g \in \mathbb{F}_q[x; p]$

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## Theorem

*We can compute the number of roots in  $\mathbb{F}_q$  of a subadditive  $\Psi \in \mathbb{F}_q[x]$  with  $O(n^3 \log^2 q)$  operations in  $\mathbb{F}_q$  (even though  $\deg \Psi \approx p^{n-1}$ ).*

## Inverse Problem – degree $p^2$ case

- For each possible number of right components, how many additive polynomials of degree  $n$  have that many right components?
- **Equivalently:** how many subadditive polynomials in  $\mathbb{F}_q[x]$  have each possible number of roots in  $\mathbb{F}_q$ ?

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Bluher (2004): For  $f = x^{p^2} + a_1 x^p + a_0 x \in \mathbb{F}_q[x; p]$  ( $a_0 \neq 0$ )

Right components of degree $p$	# additive polynomials of degree $p^2$ with that many right components
0	$\frac{p^2-p}{2} \cdot \frac{q^2-1}{p-1}$
1	$(p-1) \cdot \frac{q^2-q}{p^2-p}$
2	$\frac{(p-1)(p-2)}{2} \cdot \frac{(q-1)^2}{(p-1)^2}$
$p+1$	$(p-1) \cdot \frac{(q-1)(q-p)}{p(p-1)^2(p+1)}$



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Get an elementary  
proof, algorithm for  
enumeration

# Inverse Problem – the degree $p^3$ case

Von zur Gathen & G (2011): degree  $p^3$  in  $\mathbb{F}_q[x; p]$

Right components of degree $p$	Number of $f \in \mathbb{F}_q[x; p]$ of degree $p^3$ with specified number of right components
0	$\frac{p^3-p}{3} \cdot \frac{q^3-1}{p^3-1}$
1	$(p-1) \cdot \frac{p^2-p}{2} \cdot \frac{q-1}{p-1} \cdot \frac{q^2-1}{p^2-1} + (p-1) \cdot \frac{q^3-q^2}{p^3-p^2}$
2	$(p-1)(p-2) \cdot \frac{q^2-q}{p^2-p} \cdot \frac{q-1}{p-1}$
3	$\frac{(p-1)(p-2)(p-3)}{6} \cdot \frac{(q-1)^3}{(p-1)^3}$
$p+1$	$(p-1) \cdot \frac{q^2-q}{p^2-p} \cdot \frac{q-p}{p-1} \cdot \frac{1}{p^2}$
$p+2$	$(p-1)(p-2) \cdot \frac{q-1}{p-1} \cdot \frac{(q-1)(q-p)}{p(p-1)^2(p+1)}$
$p^2+p+1$	$(p-1) \cdot \frac{(q-1)(q-p)(q-p^2)}{(p^3-1)(p^3-p)(p^3-p^2)}$

# Indecomposable Additive Polynomials

Indecomposable additive polynomials  $f \in \mathbb{F}_q[x; p]$

- $\sigma_q$  has a single, irreducible Jordan block
- $f^* \in \mathbb{F}_p[y]$  is irreducible of degree  $n$

$$\text{Let } N_p(n) = \sum_{k|n} \mu(n/k) p^k = \begin{cases} \# \text{ irreducibles in } \mathbb{F}_p[y] \\ \text{of degree } d \end{cases}$$

Number of indecomposable additive polynomials is then

$$\frac{q^n - 1}{p^n - 1} \cdot N_p(n) \approx \frac{q^n}{n}$$

- Gives a very compact proof of a theorem of Odoni (1999)
- A random additive polynomial of degree  $p^n$  in  $\mathbb{F}_q[x; p]$  will be indecomposable with probability about  $1/n$
- Randomized polynomial-time algorithm for generating.

## Maximizing collisions

For  $f \in \mathbb{F}_q[x; r]$  of degree  $p^n$ , the number of distinct right components of degree  $p$  is at most  $(p^n - 1)/(p - 1)$ .

**Goal:** Generate  $f \in \mathbb{F}_q[x; r]$  with this maximal number

- Let  $V_f \subseteq \overline{\mathbb{F}}_q$  and  $\sigma_q : V_f \rightarrow V_f$
- Components maximized when  $\sigma = c \cdot \text{Id}$  for some  $c \in \mathbb{F}_p$
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**Algorithm:**

- Find right components of  $x^q - cx$  of degree  $p^n$ 
  - all have  $(p^n - 1)/(p - 1)$  distinct right components of degree  $p$
- Cost is  $O(n^4)$  operations in  $\mathbb{F}_q$

## How many maximal collisions?

Count the number of  $n$ -dimensional subspaces of  $V_{f^*}$  :

$$S(q, p, n) = \frac{(q-1)(q-p) \cdots (q-p^{n-1})}{(p^n-1)(p^n-p) \cdots (p^n-p^{n-1})}$$

assuming  $p^{n-1} < q$ .

There are  $p-1$  non-zero values for  $c \in \mathbb{F}_p$  in  $f^* = x^q - cx$

Total number of "maximal collision polynomials" is thus:

$$(p-1) \cdot S(q, p, n)$$

# Open Questions

- Inverse theory for number of right factors of degree  $p$  of any polynomial in  $\mathbb{F}_q[x; p]$
- Automatically generate inverse formulas
- Compute number of right factors of any given degree of a polynomial in  $\mathbb{F}_q[x; \sigma]$
- Resolve conjecture: how many decompositions possible for a general polynomial?