# Towers of function fields over finite fields and their sequences of zeta functions


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**joint work with Alexey Zykin**

# Towers

## Definition

A *tower* of function fields over $\mathbb{F}_q$ is an infinite sequence

$$\mathcal{F} = (F_1, F_2, \ldots)$$

of function fields $F_i/\mathbb{F}_q$ with properties

- $F_1 \subset F_2 \subset F_3 \subset \ldots,$
- $[F_i : F_{i-1}] > 1$ for $i > 1$,
- the genus $g(F_j) > 0$ for some $j$.

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## remark

1. $g(F_i) \to \infty$ as $i \to \infty$,
2. $\lim \frac{N(F_n)}{g(F_n)}$ exits and called $\lambda(\mathcal{F})$.

## Definition

Let $\mathcal{F} = (F_n)_{n\geq 1}$ be a tower of function fields over $\mathbb{F}_q$. Then

- $\mathcal{F}$ is asymptotically good, if $\lambda(\mathcal{F}) > 0$,
- $\mathcal{F}$ is asymptotically bad, if $\lambda(\mathcal{F}) = 0$,
- $\mathcal{F}$ is optimal, if $\lambda(\mathcal{F}) = A(q)$.

# Garcia-Stichtenoth optimal tower

Let $T_1$ be a rational function field $\mathbb{F}_4(x_1)$. Then we define the function field $T_n$ as following

$$T_n = T_{n-1}(x_n), \qquad \text{where} \qquad x_n^2 + x_n = \frac{x_{n-1}^3}{x_{n-1}^2 + x_{n-1}}.$$

$$F(X, Y) = (Y^2 + Y)(X + 1) + X^2.$$

- it is optimal, in other words

$$\lim_{n \to \infty} \frac{N_1(T_n)}{g(T_n)} = \sqrt{4} - 1 = 1,$$

- genus of function field $T_n$ is

$$g(T_n) = \begin{cases} (2^{n/2} - 1)^2 & \text{if } i \text{ even}, \\ (2^{(n+1)/2} - 1)(2^{(n-1)/2} - 1) & \text{if } i \text{ odd}, \end{cases}$$

# Geer-Vlugt tower

Let $\mathcal{F} = (F_n)_{n\geq 1}$ be a tower of function field over $\mathbb{F}_8$ where $F_1 = \mathbb{F}_8(x_1)$ and

$$F_n = F_{n-1}(x_n), \qquad \text{where} \qquad x_n^2 + x_n = x_{n-1} + 1 + 1/x_{n-1}.$$

So the tower $\mathcal{F}$ is a recursive tower given by an irreducible polynomial

$$F(X,Y) = (Y^2+Y)X - X^2 - X - 1 \in \mathbb{F}_8[X,Y].$$

The following proposition describes the behavior of the tower and its ramification locus.

Let $\mathcal{F}$ be a tower over finite field $\mathbb{F}_8$ defined by the polynomial $F(X,Y)$. Then the following properties hold:

- it is a good tower with limit attaining the Ihara bound

$$\lim_{n\to\infty} \frac{N_1(F_n)}{g(F_n)} = \frac{2(p^2-1)}{p+2} = 3/2,$$

- if $Q \in \mathbb{P}(F_n)$ is a ramification place of an extension $F_n/F_1$ then $Q \cap F_1$ is either a pole of $x_1$ or a zero $x_1 - a$, where $a \in \{\pm 1, \rho, \rho^2\}$, with $\rho^2 + \rho + 1 = 0$,
- genus of $F_n$ equals

$$g(F_n) = 2^{n+2} + 1 - \begin{cases} (n+10)2^{i/2-1} & \text{for} \quad i \text{ even} \\ (n+2[i/4]+15)2^{(i-3)/2} & \text{for} \quad i \text{ odd} \end{cases}$$

Let $\mathcal{K} = (K_n)_{n\geq 1}$ be a tower of function fields over $\mathbb{F}_9$ where $F_1 = \mathbb{F}_9(x_1)$ and

$$K_n = K_{n-1}(x_n), \qquad \text{where} \qquad x_n^2 = (x_{n-1}^2 + 1)/(2x_{n-1}).$$

So the tower $\mathcal{K}$ is a recursive optimal tower given by an absolutely irreducible polynomial

$$F(X,Y) = 2XY^2 - (X^2+1) \in \mathbb{F}_9[X,Y].$$

# Goal

Let $T$ be a function field over $\mathbb{F}_q$ then the zeta function of $T$ is

$$\log Z_T(x) = \sum_{m\geq 1} \frac{N_m(T)}{m} x^m = \frac{\mathrm{L}_T(x)}{(1-x)(1-qx)}$$

where $N_m(T)$ is a number of $\mathbb{F}_{q^m}-$rational points of $T$ and

$$\mathrm{L}_T(x) = a_0 + a_1 x + \cdots + a_{2g(T)} x^{2g(T)}$$

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For each function field in a tower $\mathcal{T} = (T_n)_{n\geq 1}$

$$\mathrm{L}_{T_n}(x) = a(0,n) + a(1,n)x + \cdots + a(2g(T_n),n)x^{2g(T_n)}$$

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$$\mathrm{L}_{T_n}(x) = a(0,n) + a(1,n)x + \cdots + a(2g(T_n), n) x^{2g(T_n)}$$

**Question**

Can we find explicitly functions $a(i,n)$ as functions in $i$, $n$ for at least one given good tower?

# Asymptotic zeta function

Let $\mathcal{T} = (T_n)_{n\geq 1}$ be a tower. Then one can define an asymptotic zeta function

$$\mu_n = \lim_{m\to\infty} \frac{N_n(T_m)}{g(T_m)}$$

$$\log \mathcal{Z}_{\mathcal{T}}(x) = \sum_{n\geq 1} \frac{\mu_n}{n} x^n$$

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## Garcia-Stictenoth tower

$$Z_{\mathcal{T}}(t) = \frac{1}{(1-t)}$$

## tower of Kummer extensions

$$Z_{\mathcal{T}}(t) = \frac{1}{(1-t)^2}$$

Base on Lenstre relation Peter Beelen proved that locus of split completely places is bounded and lies in $V(G_1\mathcal{F})$. Then according to the Perron-Frobenius theorem it follows that number of paths of length $m$ in the graph $G_i(\mathcal{F})$ is completely determined by a maximum eigenvalue. Therefore $\mu_i(\mathcal{F})$ is a constant.

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Hence

**Geer-Vlugt tower**

$$Z_{\mathcal{F}}(t) = \frac{1}{(1-t)^{3/2}}.$$

# L-polynomials of Garcia-Stichtenoth tower

| | |
|---|---|
| $L_{T_1}$ | $1$ |
| $L_{T_2}$ | $1+3T+4T^2$ |
| $L_{T_3}$ | $(1+3T+4T^2)^3$ |
| $L_{T_4}$ | $(1-T+4T^2)^2(1+3T+4T^2)^7$ |
| $L_{T_5}$ | $(1-T+4T^2)^4(1+3T+4T^2)^{11}(1+T+4T^2)^2$ $(1+2T+T^2+8T^3+16T^4)^2$ |
| $L_{T_6}$ | $(1-T+4T^2)^4(1+T+4T^2)^{10}(1+2T+T^2+8T^3+16T^4)^6$ $(1+3T+4T^2)^{17}(1+T-T^2+3T^3-4T^4+16T^5+64T^6)^2$ |

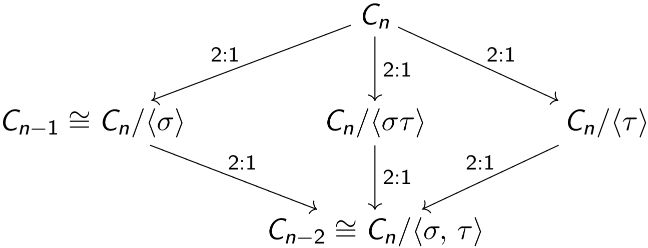
**Proposition**

If $n \geq 3$, then the extension $\mathrm{T}_n$ over $\mathrm{T}_{n-2}$ is Galois and

$$\mathrm{Gal}(\mathrm{T}_n/\mathrm{T}_{n-2}) \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}.$$

We will always let $C_n$ denote a curve with function field $\mathrm{T}_n$.
The Galois covering $C_n \to C_{n-2}$ implies a decomposition of the Jacobian of the curve $C_n$. If we denote Galois automorphism group by $\langle \sigma, \tau \rangle$ then we have the following diagram of coverings

# Galois Group and Kani-Rosen decomposition




and the following isogeny of Jacobians

$$\mathrm{Jac}(C_n) \times \mathrm{Jac}(C_{n-2})^2 \sim \mathrm{Jac}(C_{n-1}) \times \mathrm{Jac}(C_n/\langle\sigma\tau\rangle) \times \mathrm{Jac}(C_n/\langle\tau\rangle),$$

which implies decomposition of $\mathrm{L}-$polynomials

$$\mathrm{L}_{C_n}(T)\, \mathrm{L}_{C_{n-2}}(T)^2 = \mathrm{L}_{C_{n-1}}(T)\, \mathrm{L}_{C_n/\langle\sigma\tau\rangle}(T)\, \mathrm{L}_{C_n/\langle\tau\rangle}(T).$$

Decomposition of $\mathrm{Pic}^0(\mathrm{T}_n)$ and the $\mathrm{L}$-polynomial of $\mathrm{T}_n$.

**Corollary**

*The $\mathrm{L}$-polynomial of the function field $\mathrm{T}_n$ has the following factorization*

$$\mathrm{L}_{\mathrm{T}_n} = \mathrm{L}_{X_1}^{2n-3} \times \mathrm{L}_{X_{2,1}}^{2n-6} \times \mathrm{L}_{\mathrm{Y}_{3,1}}^{2n-8} \times \cdots \times \mathrm{L}_{\mathrm{Y}_{n-2,1}}^{2},$$

*or more precisely*

$$\mathrm{L}_{\mathrm{T}_n} = (T^2+T+4)^{2n-8}(T^2+3T+4)^{12n-49}(T^2-T+4)^{6n-26}$$
$$(T^4+2T^3+T^2+8T+16)^{6n-24}$$
$$(T^6+T^5-T^4+3T^3-4T^2+16T+64)^{2n-10}\mathrm{L}_{Y_{5,1}}^{2n-12}\cdots\mathrm{L}_{Y_{n-2,1}}^{2}$$

The order of the finite group

$$\#\mathrm{Pic}^0(\mathrm{T}_n)(\mathbb{F}_4) = 2^{58n-243}3^{2n-8}5^{2n-10}\mathrm{L}_{Y_{5,1}}^{2n-12}(1)\ldots\mathrm{L}_{Y_{n-2,1}}^{2}(1).$$

## Graphs and recursive tower

Let $\mathcal{T} := \{T_n\}$ be a recursive tower of function fields with the full constant field $\mathbb{F}_q$, given by an absolutely irreducible polynomial in two variables $F(X, Y) \in \mathbb{F}_q(X, Y)$.
Then one can associate a sequence of directed graphs $(\Gamma_n)_{n\geq 1}$ in the following way:

- the set of vertices $V_n$ are elements of $\mathbb{F}_{q^n}$ with property not being a coordinate of a ramification point,
- there is a directed edge from $a \in V$ to $b \in V$ if $F(a, b) = 0$.

Similar we can define a directed graph of ramification locus, namely it is a directed graph $R$ with

- $V(R)$ vertices are elements of $\bar{\mathbb{F}}_p \cup \{\infty\}$ such that each vertex is a coordinate of a ramification point,
- there is a directed edge from $a \in V(R)$ to $b \in V(R)$ if $F(a, b) = 0$.

# Example, $G_1(\mathcal{T})$

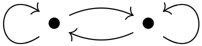
Let $\mathbb{F}_4 = \{0, 1, \alpha, \alpha^2 = \alpha + 1\}$. Then $\alpha \to \alpha$ and $\alpha \to \alpha + 1$, since

$$\alpha^2 + \alpha = \frac{\alpha^2}{\alpha + 1} = 1$$

and

$$(\alpha + 1)^2 + (\alpha + 1) = \frac{\alpha^2}{\alpha + 1} = 1.$$

Similarly $\alpha + 1 \to \alpha$ and $\alpha + 1 \to \alpha + 1$



# Characteristic polynomials

| $\mathbb{F}_{4^n}$ | Characteristic polynomial |
|---|---|
| $\mathbb{F}_4$ | $x(x-2)$ |
| $\mathbb{F}_{4^2}$ | $x^{13}(x-2)$ |
| $\mathbb{F}_{4^3}$ | $x^{61}(x-2)$ |
| $\mathbb{F}_{4^4}$ | $x^{237}(x-2)(x-1)(x^2+1)(x^4+1)^2$ |
| $\mathbb{F}_{4^5}$ | $x(x-2)^2x^{1011}(x^4+x^3+x^2+x+1)$ |
| $\mathbb{F}_{4^6}$ | $x^{3949}(x-2)(x+1)^2(x-1)^{10}(x^2-x+1)^2(x^2+1)^2$ $(x^2+x+1)^{10}(x^4-x^2+1)^2(x^6+x^3+1)^{10}(x^{12}-x^6+1)^2$ |
| ... | ... |
| $\mathbb{F}_{4^{10}}$ | $x^{1015482}(x^{60}-1)^{22}(x^{28}-1)^{10}(x^{100}-1)^4$ $(x^5-1)^2(x^{310}-1)^8(x^{140}-1)^{18}(x^{420}-1)^2(x^{820}-1)^2$ $(x^{370}-1)^4(x^{980}-1)^2(x^{460}-1)^2(x^{220}-1)^4(x^{660}-1)^2$ $(x^{300}-1)^4(x^{500}-1)^2(x^{200}-1)^6(x^{580}-1)^2(x^{25}-1)^{24}$ $(x^{40}-1)^{36}(x^{110}-1)^4(x^{44}-1)^{40}(x^{760}-1)^2(x^{70}-1)^8$ $(x^{280}-1)^2(x^{180}-1)^2(x^{170}-1)^8(x^{90}-1)^8(x^{340}-1)^4$ $(x^{260}-1)^4(x^{150}-1)^4(x^{80}-1)^2(x^2-2x)$ |

# Root of unities

| $n$ | roots of unity |
|---|---|
| 1 | — |
| 2 | — |
| 3 | — |
| 4 | $(2)^3$ |
| 5 | $(5)$ |
| 6 | $(2)^2(3)^2$ |
| 7 | $(2)^2(3)^2(5)(7)$ |
| 8 | $(2)^7(5)(7)(11)$ |
| 9 | $(2)^3(3)^3(5)(7)(11)(13)(17)(31)$ |
| 10 | $(2)^4(3)^2(5)^3(7)^2(11)(13)(17)(19)(23)(29)(31)(37)(41)$ |

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| 9 | $(2)^3(3)^3(5)(7)(11)(13)(17)(31)$ |
| 10 | $(2)^4(3)^2(5)^3(7)^2(11)(13)(17)(19)(23)(29)(31)(37)(41)$ |

**Observation-I**

Each eigenvalue of the graph is either 2 or zero or **a root of unity**.

# Generating function

Let $\Gamma$ be a directed graph with an adjacency matrix $A$.
$f(n) :=$ number of all paths of lengths $n = \sum_{a_{i,j} \in A^n} a_{i,j}$.

$$G(x) = \sum_{m \geq 0} f(n)x^n = \sum_{m \geq 0} \frac{\sum_{i,j} \det(I - xA; i, j)}{\det(I - xA)}$$

# generating function of graphs and $\mathbb{F}_{4^n}$-points

| | |
|---|---|
| $G_1(\mathcal{T})$ | $\frac{-1}{x-1/2}$ |
| $G_2(\mathcal{T})$ | $8x+12+\frac{-1}{x-1/2}$ |
| $G_3(\mathcal{T})$ | $24x^2+48x+60+\frac{-1}{x-1/2}$ |
| $G_4(\mathcal{T})$ | $128x^5+192x^4+160x^3+80x^2+24x-4+\frac{-256}{x-1}+\frac{-1}{x-1/2}$ |
| $G_5(\mathcal{T})$ | $320x^3+680x^2+800x-\frac{160}{x-1}-\frac{2}{2x-1}$ |

$$\frac{-1}{x-1/2} = 2 + 4x + \ldots + 2^{n+1}x^n + \ldots$$

$$8x + 12 + \frac{-1}{x-1/2} = 14 + 12x + 8x^2 + \ldots + 2^{n+1}x^n + \ldots$$

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$N_1(T_n) = 2^n + n + 2$

$N_2(T_1) = 17,\ N_2(T_2) = 16,\ N_2(T_n) = 2^n + n + 2,\ n > 2$

$N_3(T_1) = 65,\ N_3(T_2) = 56,\ N_3(T_n) = 37,$

$N_3(T_n) = 2^n + n + 2,\ n > 3$

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$$N_3(T_n) = 2^n + n + 2,\ n > 3$$

## Observation-II

$N_m(T_n) = 2^n$+polynomial in $n$

| | |
|---|---|
| $a(1,n) =$ | $2^n + (n-3)$ |
| $a(2,n) =$ | $\frac{1}{2}(2^n)^2 + (n-3)$ |
| $a(3,n) =$ | $\frac{1}{6}(2^n)^3 + (\frac{n}{2}-1)(2^n)^2 + (\frac{1}{2}n^2 - \frac{3}{4}n - \frac{61}{24})2^n +$ $+(\frac{1}{6}n^3 - n^2 - \frac{25}{6}n - 3)$ |
| $a(4,n) =$ | $\frac{1}{24}(2^n)^4 + (\frac{1}{6}n - \frac{1}{4})(2^n)^3 +$ $+(\frac{1}{4}n^2 - \frac{3}{4}n - \frac{61}{24})(2^n)^2 + (\frac{1}{6}n^3 - \frac{3}{4}n^2 - \frac{61}{12}n - \frac{21}{4})2^n +$ $+(\frac{1}{24}n^4 - \frac{1}{4}n^3 - \frac{61}{24}n^2 - \frac{21}{4}n + 61)$ |
| $a(5,n) =$ | $\frac{1}{120}(2^n)^5 + (\frac{1}{24}n - \frac{1}{24})(2^n)^4 + (\frac{1}{12}n^2 - \frac{1}{6}n - \frac{23}{24})(2^n)^3 +$ $+(\frac{1}{12}n^3 - \frac{1}{4}n^2 - \frac{23}{8}n - \frac{95}{24})(2^n)^2 +$ $(\frac{1}{24}n^4 - \frac{1}{6}n^3 - \frac{23}{8}n^2 - \frac{95}{12}n + \frac{1159}{20})2^n +$ $+(\frac{1}{120}n^5 - \frac{1}{24}n^4 - \frac{23}{24}n^3 - \frac{95}{24}n^2 + \frac{1159}{20}n - 163)$ |

# coefficients of $L$-polynomials

| | |
|---|---|
| $a(1,n) =$ | $2^n + (n-3)$ |
| $a(2,n) =$ | $\frac{1}{2}(2^n)^2 + (n-3)$ |
| $a(3,n) =$ | $\frac{1}{6}(2^n)^3 + (\frac{n}{2} - 1)(2^n)^2 + (\frac{1}{2}n^2 - \frac{3}{4}n - \frac{61}{24})2^n + {} + (\frac{1}{6}n^3 - n^2 - \frac{25}{6}n - 3)$ |
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## Observation-III

$a(m,n) = \sum_{i=0}^{m}$(a polynomial in $\mathbf{n}$ of degree $\mathbf{i}$ over $\mathbb{Q}$) $\cdot(2^{\mathbf{n}})^{\mathbf{m-i}}$

# Optimal towers and basic inequality

## Basic inequality

$$\sum_{n\geq 1} \mu_n q^{n/2} \leq 1$$

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If $\mathcal{T}/\mathbb{F}_{q^2}$ is optimal then it implies that

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and

$$Z_{\mathcal{T}}(t) = \frac{1}{(1-t)^{\sqrt{q}-1}}$$

# Optimal towers and basic inequality

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and

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## Question

Does the equality $\sum_{n \geq 1} \mu_n q^{n/2} = 1$ imply $\mathcal{T}/\mathbb{F}_{q^2}$ is optimal?

# Zeta functions of Galois closure of Garcia-Stichtenoth tower

**Theorem**

*Let* $\tilde{\mathcal{T}} = (\tilde{T}_n)_n$ *is a Galois closure of the Garcia-Stichtenoth tower over* $\mathbb{F}_{p^2}$ $(p > 2)$*. Then for each* $m$ *there exists* $M(m)$ *such that if* $n \geq M(m)$ *then*

$$N_m(\tilde{T}_n) = p^{3n-4} - p^{3n-5} + p^{2n-5} + p^{2n-6}$$

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$$N_m(\tilde{T}_n) = p^{3n-4} - p^{3n-5} + p^{2n-5} + p^{2n-6}$$

Hence for each $n$ there exists $N(n)$ such that

$$\log Z_{\tilde{T}_n} = \sum_{m \geq 1}^{N(n)} \frac{p^{3n-4} - p^{3n-5} + p^{2n-5} + p^{2n-6}}{m} x^m + \sum_{m \geq N(n)} \frac{N_m(\tilde{T}_n)}{m} x^m$$

and $N(n) \to \infty$ as $n \to \infty$

Thank your for your attention!