Superspecial rank of supersingular curves

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Abstract: A curve X of genus g over a finite field is *supersingular* if the Newton polygon of its L-polynomial is a line segment of slope $1/2$.

Equivalently, X is supersingular if and only if the Jacobian $\mathrm{Jac}(X)$ is isogenous to a product of supersingular elliptic curves.

Only in rare cases is $\mathrm{Jac}(X)$ isomorphic to a product of supersingular elliptic curves, in which case X is called *superspecial*.

I will define the *superspecial rank*, which is an invariant of the Dieudonné module or Ekedahl-Oort type of a p.p. abelian variety.

If X is a supersingular curve, then the superspecial rank determines the number of elliptic factors in the decomposition of $\mathrm{Jac}(X)$ up to isomorphism.

As examples, we compute the superspecial rank of Hermitian curves and Suzuki curves. I will describe results about the superspecial rank of curves in characteristic 2.

Overview

An elliptic curve $E/\overline{\mathbb{F}}_p$ can be ordinary or supersingular.
How do you generalize supersingular property?
Study: abelian varieties of dimension $g > 1$ and curves of genus $g > 1$.

A. p-rank $f = 0$
B. supersingular
C. superspecial

Today: define superspecial rank (invariant of Dieudonneé module) differentiating B and C for curves

Motivation - supersingular elliptic rank of supersingular Jacobians
Examples (with Elkin, Weir, Malmskog)

Another day: Newton polygon results - differentiating A and B for curves

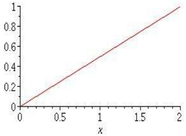
Supersingular elliptic curves

Let E be a smooth elliptic curve over $k = \overline{k}$, with $\mathrm{char}(k) = p$.
Let $E[p]$ be the kernel of the inseparable multiplication-by-p morphism.

E is **supersingular** if it satisfies the following equivalent conditions:

A. The only p-torsion point is the identity: $E[p](k) = \{\mathrm{id}\}$.

B. The Newton polygon of E is a line segment of slope $\frac{1}{2}$.



C. The group scheme $E[p]$ contains 1 copy of α_p, the kernel of Frobenius on $\mathbb{G}_a$.

For all p, there exists a supersingular elliptic curve E over $\mathbb{F}_{p^2}$ (Igusa).

These properties are not all the same when $g > 1$

Let A be a p.p. abelian variety of dimension g over $k = \overline{k}$, $\mathrm{char}(k) = p$.
Let $A[p]$ be the kernel of the inseparable multiplication-by-p morphism.

The following conditions are all different for $g \geq 3$.

A. p-rank 0 - The only p-torsion point is the identity: $A[p](k) = \{\mathrm{id}\}$.

B. supersingular - The Newton polygon of A is a line of slope $\frac{1}{2}$.

C. superspecial - The group scheme $A[p]$ contains g copies of α_p, the kernel of Frobenius on $\mathbb{G}_a$.

Then $C \Rightarrow B \Rightarrow A$.

Goal: study $A \not\Rightarrow B \not\Rightarrow C$ for Jacobians of curves of genus $g \geq 3$.

B. Definition of Newton polygon

Let X be a smooth projective curve defined over $\mathbb{F}_q$.
Zeta function of X is $Z(X/\mathbb{F}_q, t) = L(X/\mathbb{F}_q, t)/(1-t)(1-qt)$

where $L(X/\mathbb{F}_q, t) = \prod_{i=1}^{2g}(1 - w_i t) \in \mathbb{Z}[t]$ and $|w_i| = \sqrt{q}$.

The Newton polygon of X is the NP of the L-polynomial $L(t)$.
Find p-adic valuation v_i of coefficient of t^i in $L(t)$.
Draw lower convex hull of $(i, v_i/a)$ where $q = p^a$.

Facts: The NP goes from $(0,0)$ to $(2g, g)$.
NP line segments break at points with integer coefficients;
If slope λ occurs with length m_λ, so does slope $1-\lambda$.

Definition

$X/\mathbb{F}_q$ is *supersingular* if the Newton polygon of $L(X/\mathbb{F}_q, t)$ is a line segment of slope $1/2$.

There is a partial ordering on NPs; the supersingular NP is 'smallest'.

B. Definition of supersingular abelian variety

Let A be a p.p. abelian variety of dimension g over k.

Manin: for c, d relatively prime s.t. $\lambda = \frac{c}{d} \in \mathbb{Q} \cap [0,1]$,
define a p-divisible group $G_{c,d}$ of dimension c and height d.

The Dieudonné module D_λ for $G_{c,d}$ is a $W(k)$-module.
Over $\mathrm{Frac}(W(k))$, there is a basis $x_1, \ldots, x_d$ for D_λ s.t. $F^d x_i = p^c x_i$.

There is an isogeny of p-divisible groups $A[p^\infty] \sim \oplus_\lambda G_{c,d}^{m_\lambda}$.

Newton polygon:
lower convex hull - line segments of slope λ and length m_λ.

Definition: *A supersingular* iff $\lambda = \frac{1}{2}$ is the only slope.

Existence of supersingular objects

Abelian varieties:
For all p and g, there exists a supersingular p.p. abelian variety of dimension g, namely E^g.

Let $\mathcal{A}_g$ be the moduli space of p.p. abelian varieties of dimension g. The supersingular locus of $\mathcal{A}_g$ has dimension $\lfloor \frac{g^2}{4} \rfloor$.

Smooth Curves: Many experts on supersingular curves are here.

Van der Geer/Van der Vlugt:

If $p = 2$, there exists a supersingular curve of every genus.

Open problem

For $p \geq 3$, it is unknown if there exists a supersingular curve of every genus.

B. Example: Hermitian curves are supersingular

Let $q = p^n$. The *Hermitian curve* X_q has affine equation $y^q + y = x^{q+1}$.

It has genus $g = q(q-1)/2$.
It is maximal over $\mathbb{F}_{q^2}$ because $\#X_q\left(\mathbb{F}_{q^2}\right) = q^3 + 1$.

Ruck/Stichtenoth: X_q is unique curve of genus g maximal over $\mathbb{F}_{q^2}$.

Hansen: X_q is the Deligne-Lusztig variety for $\mathrm{Aut}(X_q) = \mathrm{PGU}(3,q)$.

The zeta function of X_q is $Z(X_q/\mathbb{F}_q, t) = \frac{(1+qt^2)^g}{(1-t)(1-qt)}$.
The only slope of the Newton polygon of $L(t) = (1+qt^2)^g$ is $1/2$.

Thus $\mathrm{Jac}(X_q)$ is supersingular.

C. Supersingular elliptic curve - revisited

C. The group scheme $E[p]$ contains 1 copy of α_p, the kernel of Frobenius on $\mathbb{G}_a$.

As a k-scheme, $\alpha_p \simeq \mathrm{Spec}(k[x]/x^p)$ with co-multiplication $m^*(x) = x \otimes 1 + 1 \otimes x$ and co-inverse $\mathrm{inv}^*(x) = -x$.

$E[p]$ is a group scheme of rank p^2, fitting in a non-split exact sequence

$$0 \to \alpha_p \to E[p] \to \alpha_p \to 0.$$

The image of α_p is the kernel of F (Frobenius) and V (Verschiebung).

What is a good generalization of this condition?

First approach - the a-number

Let α_p denote the kernel of Frobenius on $\mathbb{G}_a$.

Definition

The a-number of A is $a(A) = \mathrm{dim}_k \mathrm{Hom}(\alpha_p, A[p])$.

Computation: Let X be a curve of genus g.
Let r be the rank of the Cartier operator on $H^0(X, \Omega^1)$.
Then the a-number of $A = \mathrm{Jac}(X)$ is $a = g - r$.

Example - the Hermitian curve

Let $q = p^n$. Recall that $X_q : y^q + y = x^{q+1}$ has genus $g = q(q-1)/2$.

If $n = 1$, then $a = g$. If $n = 2$, then $a = g/2$.

Gross: $a = p^n(p^{n-1} + 1)(p - 1)/4$.

Computation of *a*-number of Hermitian curve

The Cartier operator C acts on $H^0(X_q, \Omega^1)$.

Let $\Delta = \{(i,j) \mid i,j \in \mathbb{Z},\ i,j \geq 0,\ i+j \leq q-2\}$.
A basis for $H^0(X_q, \Omega^1)$ is $B = \{\omega_{i,j} := x^i y^j dx \mid (i,j) \in \Delta\}$.

Write $i = i_0 + p i_n^T$ and $j = j_0 + p j_n^T$ with $0 \leq i_0, j_0 \leq p-1$.

$$\begin{aligned} C(x^i y^j dx) &= x^{i_n^T} y^{j_n^T} C\left(x^{i_0}(x^{q+1} - y^q)^{j_0} dx\right) \\ &= x^{i_n^T} y^{j_n^T} \sum_{l=0}^{j_0} \binom{j_0}{l} (-1)^l x^{p^{n-1}(j_0 - l)} y^{p^{n-1} l} C\left(x^{i_0 + j_0 - l} dx\right). \end{aligned}$$

$C(x^k dx) \neq 0$ iff $k \equiv -1 \bmod p$. Need $i_0 + j_0 - \ell \equiv -1 \bmod p$.

If $i_0 + j_0 < p-1$, then $C(\omega_{i,j}) = 0$.
If $i_0 + j_0 \geq p-1$, then $C(\omega_{i,j}) = \omega_{p^{n-1}(p-1-i_0)+i_n^T,\, p^{n-1}(i_0+j_0-(p-1))+j_n^T}$.

Let A be a p.p. abelian variety of dimension g over $k = \overline{k}$, $\mathrm{char}(k) = p$.

Recall $a(A) = \dim_k \mathrm{Hom}(\alpha_p, A[p])$, with α_p kernel of Frobenius on $\mathbb{G}_a$.

Def: An abelian variety A is *superspecial* if $a(A) = g$.

Let E be a supersingular elliptic curve over k.

Oort

A is supersingular iff $A \sim E^g$ is ISOGENOUS to a product of supersingular elliptic curves.

A is superspecial iff $A \simeq \times_{i=1}^{g} E_i$ is ISOMORPHIC to a product of supersingular elliptic curves.

Existence of superspecial objects

Abelian varieties: The number of superspecial p.p. abelian varieties of dim g is finite and non-zero; (it is a class number).

The superspecial locus of $\mathcal{A}_g$ has dimension 0.

Smooth curves: Problem: there are not many superspecial curves.

Example: The Hermitian curve $X_q : y^q + y = x^{q+1}$ is supersingular for all $q = p^n$, but superspecial iff $n = 1$.

Ekedahl

If $X/\overline{\mathbb{F}}_p$ is a superspecial curve of genus g, then $g \leq p(p-1)/2$.

Upper bound realized by Hermitian curve $X_p : y^p + y = x^{p+1}$.

Another approach - Dieudonné module

The Dieudonné module $D(A[p])$ of the group scheme $A[p]$ is an $\mathbb{E}$-module.

Here $\mathbb{E} = k[F, V]$ is the non-commutative ring generated by semi-linear operators F and V with relations $FV = VF = 0$ and $F\lambda = \lambda^p F$ and $\lambda V = V\lambda^p$ for all $\lambda \in k$.

Let $(R) = \mathbb{E}R$ be the left ideal of $\mathbb{E}$ generated by R.

Let E be a supersingular elliptic curve.
Let $I_{1,1}$ be the isomorphism class of the rank p^2 group scheme $E[p]$.
The Dieudonné module of $I_{1,1}$ is $\mathbb{E}/(F + V)$

Fact:

A is superspecial if and only if $A[p] \simeq (I_{1,1})^g$.
A is superspecial if and only if $D(A[p]) = (\mathbb{E}/(F + V))^g$

Example - Hermitian curve

Let $q = p^n$. Recall that $X_q : y^q + y = x^{q+1}$ has genus $g = q(q-1)/2$.
Let $\mathbb{D}(X_q)$ denote the Dieudonné module of $\mathrm{Jac}(X_q)[p]$.

Theorem -Pries/Weir

We determine the Dieudonné module $D(X_q)$ for all $q = p^n$, complementing earlier work of Dummigan. Its distinct indecomposable factors are in bijection with orbits of $\mathbb{Z}/(2^n+1) - \{0\}$ under $\times 2$.

Examples:

$\mathbb{D}(X_p) = (\mathbb{E}/(F+V))^g$.
$\mathbb{D}(X_{p^2}) = (\mathbb{E}/(F^2+V^2))^{g/2}$.
$\mathbb{D}(X_{p^3}) = (\mathbb{E}/\mathbb{E}(F^3+V^3))^{r_{3,2}} \oplus (\mathbb{E}/\mathbb{E}(F+V))^{g-3r_{3,2}}$,
where $r_{3,2} = p^3(p+1)^2(p-1)/2^3$.

Goal

Let X be a supersingular curve of genus g.

Recall that $\mathrm{Jac}(X) \sim E^g$ with E supersingular elliptic curve.

But, almost always, $\mathrm{Jac}(X) \not\simeq \times_{i=1}^{g} E_i$.

Find an invariant that measures the extent to which $\mathrm{Jac}(X)$ decomposes up to isomorphism.

The a-number is not good.
Could have large a-number even when $\mathrm{Jac}(X)$ indecomposable.

The Dieudonné module is too complicated.
There are 2^g options for the DM, 2^{g-1} when the p-rank is $f = 0$.
The DM could have many factors even when $\mathrm{Jac}(X)$ indecomposable.

Superspecial rank

Let A be a p.p. abelian variety of dimension g over k.

Definition

The *superspecial rank* $s(A)$ is the multiplicity of $\mathbb{E}/(F+V)$ in the Dieudonné module of $A[p]$.

Fact: $s(A)$ is the dimension of $\mathrm{Ker}(F+V)$ on $H^1_{\mathrm{dR}}(A)$.

If A ordinary, then $s(A)=0$.

In fact, $0 \leq s(A) \leq a(A) \leq g - f(A)$

because each factor of $\mathbb{E}/(F+V)$ contributes to the a-number and because $f(A)$ is the multiplicity of $\mathbb{Z}/p \oplus \mu_p$ in $A[p]$.

Also $s(A) = g$ iff A is superspecial.

Example - dimension $g = 1$:

Name	cod	f	a	ν	μ	Dieudonné module	s(A)
L	0	1	0	[1]	$\emptyset$	$\mathbb{E}/(F, 1-V) \oplus \mathbb{E}/(V, 1-F)$	0
$I_{1,1}$	1	0	1	[0]	$\{1\}$	$\mathbb{E}/(F+V)$	1

The isomorphism type of the p-torsion of an elliptic curve E is:
$L = \mathbb{Z}/p \oplus \mu_p$ if E is ordinary
$I_{1,1}$ if E is supersingular

Here $\mathbb{E}$ is the non-commutative ring generated by semi-linear operators F and V with the relations $FV = VF = 0$ and $F\lambda = \lambda^p F$ and $\lambda V = V\lambda^p$ for all $\lambda \in k$; and (ρ) denotes the left ideal of $\mathbb{E}$ generated by ρ.

Example - dimension $g = 2$:

Name	cod	f	a	ν	μ	Dieudonné module	s(A)
L^2	0	2	0	$[1,2]$	$\emptyset$	$D(L)^2$	0
$L \oplus I_{1,1}$	1	1	1	$[1,1]$	$\{1\}$	$D(L) \oplus \mathbb{E}/(F+V))$	1
$I_{2,1}$	2	0	1	$[0,1]$	$\{2\}$	$\mathbb{E}/(F^2+V^2)$	0
$I_{1,1}^2$	3	0	2	$[0,0]$	$\{2,1\}$	$(\mathbb{E}/(F+V))^2$	2

The supersingular locus contains both types $(I_{1,1})^2$ and $I_{2,1}$.

Dimension $g = 3$:

Name	cod	f	a	ν	Dieudonné module	$s(A)$
L^3	0	3	0	$[1,2,3]$	$D(L)^3$	0
$L^2 \oplus I_{1,1}$	1	2	1	$[1,2,2]$	$D(L)^2 \oplus \mathbb{E}/(F+V)$	1
$L \oplus I_{2,1}$	2	1	1	$[1,1,2]$	$D(L) \oplus \mathbb{E}/(F^2+V^2)$	0
$L \oplus I_{1,1}^2$	3	1	2	$[1,1,1]$	$D(L) \oplus (\mathbb{E}/(F+V))^2$	2
$I_{3,1}$	3	0	1	$[0,1,2]$	$\mathbb{E}/(F^3+V^3)$	0
$I_{3,2}$	4	0	2	$[0,1,1]$	$\mathbb{E}/(F^2-V) \oplus \mathbb{E}/(V^2-F)$	0
$I_{1,1} \oplus I_{2,1}$	5	0	2	$[0,0,1]$	$\mathbb{E}/(F+V) \oplus \mathbb{E}/(F^2+V^2)$	1
$I_{1,1}^3$	6	0	3	$[0,0,0]$	$(\mathbb{E}/(F+V))^3$	3

The group scheme $A[p]$ does not determine the NP of A when $g \geq 3$.

If $A[p] \simeq I_{3,1}$, then the NP of A is usually $G_{1,2} + G_{2,1}$ (three slopes of $1/3$ and $2/3$) but it can also be $3G_{1,1}$ (supersingular).

The Ekedahl-Oort type

The p-torsion $A[p]$ is a group scheme of rank p^{2g}.

The isomorphism class of $A[p]$ is determined by its Ekedahl-Oort type.

Find *final filtration* $N_1 \subset \cdots \subset N_{2g}$ of $D(A[p])$ as k-vector space, stable under the action of V and F^{-1} such that $i = \dim(N_i)$.

The *Ekedahl-Oort type* is $\nu = [\nu_1, \ldots, \nu_g]$ where $\nu_i = \dim(V(N_i))$.

There are nec/suff conditions $\nu_i \leq \nu_{i+1} \leq \nu_i + 1$ on ν.

There are 2^g possibilities for the isomorphism class of $A[p]$.

Superspecial rank zero

Lemma

For all p and $g \geq 2$, a generic supersingular p.p. abelian variety has superspecial rank 0.

Proof: A generic supersingular p.p. abelian variety has p-rank 0 and a-number 1 Li/Oort.

This forces DM to be $\mathbb{E}/(F^g + V^g)$, thus superspecial rank 0 for $g \geq 2$.

Remark:
Most abelian varieties with $\mathrm{DM} = \mathbb{E}/(F^g + V^g)$ are not supersingular.
An EO strata is fully contained in supersingular locus iff smaller than $[0, 0, \ldots, 0, 1, 2, \ldots, \lfloor \frac{g}{2} \rfloor]$.

Supersingular ranks of supersingular abelian varieties

Classify all the supersingular ranks which occur for supersingular abelian varieties

Proposition

For all p and g, there exists a supersingular abelian variety of dimension g over $\overline{\mathbb{F}}_p$ with superspecial rank s if and only if $0 \leq s \leq g-2$ or $s = g$.

Proof sketch:

There exists a supersingular abelian variety A_1 of dim $g-s$ with $a = 1$.

The DM is $\mathbb{E}/(F^{g-s} + V^{g-s})$ so $s(A_1) = 0$ as long as $s \neq g-1$.

Let $A = E^s \times A_1$ where E is a supersingular elliptic curve.

Application of superspecial rank to Selmer groups

Let K be the function field of a curve over k and v place of K.
Let $f : A \to A'$ be an isogeny of abelian varieties over K.

Motivation - superspecial rank

Let A be a constant supersingular elliptic curve over K and $f = [p]$.
Ulmer: The rank of $\mathrm{Sel}(K, [p])$ is the superspecial rank.

Tate-Shafarevich group: $Ш(K, A)_f = \mathrm{Ker}(\, Ш(K, A) \to \, Ш(K, A'))$
where $Ш(K, A) = \mathrm{Ker}(H^1(K, A) \to \prod_v H^1(K_v, A))$.

Selmer group: $\mathrm{Sel}(K, f)$ is subset of $H^1(K, \mathrm{Ker}(f))$ s.t. restriction is in image of $\mathrm{Sel}(K_v, f) = \mathrm{Im}(A'(K_v) \to H^1(K_v, \mathrm{Ker}(f)))$ for all v.

Exact sequence $0 \to A'(K)/f(A(K)) \to \mathrm{Sel}(K, f) \to \, Ш(K, A)_f \to 0$.

(supersingular)-Elliptic rank

Let A be an abelian variety of dimension g over k.

Definition

The *supersingular elliptic rank* $e(A)$ of A is

$$e(A) := \max\{e \mid \iota : A \xrightarrow{\simeq} B \times (\times_{i=1}^{e} E_i)\},$$

E_i supersingular elliptic curves, B abelian variety of dimension $g-e$, and ι an isomorphism of abelian varieties.

Note: $0 \leq e(A) \leq s(A)$ because each E_i contributes a factor of $\mathbb{E}/(F+V)$ to the Dieudonné module

If A absolutely simple, then $e(A) = 0$.

Lenstra/Oort: If η non-supersingular symmetric formal isogeny type, there exists a simple abelian variety A with isogeny type η.
There are many A with $s(A) > 0$ and $e(A) = 0$.

Motivation for superspecial rank

An observation of Oort

If A is supersingular, then the supersingular elliptic rank $e(A)$ of A equals the superspecial rank $s(A)$ of A.

Proof sketch: Know $e \leq s$. Let E be supersingular elliptic curve.

Write $A[p^\infty] \simeq G_{1,1}^s \times Z$ with $s(Z) = 0$. Since A supersingular,

there exists a finite group scheme $N \hookrightarrow G_{1,1}^{g-s}$ s.t. $Z \simeq G_{1,1}^{g-s}/N$.

Now E^{g-s} and $G_{1,1}^{g-s}$ have the same finite subgroup schemes.

So $N \hookrightarrow E^{g-s}$; it follows that $A \simeq E^s \times (E^b/N)$ and $e(A) \geq s$.

Supersingular, but not superspecial

Proposition - revised

For all p and $g \geq 2$ and $0 \leq s \leq g-2$,
there exists a supersingular abelian variety of dimension g over $\overline{\mathbb{F}}_p$ with supersingular elliptic rank s.

Thus $A \sim E^g$ with E ss,
and $A \simeq B \times (\times_{i=1}^{s} E_i)$ but $A \not\simeq C \times (\times_{i=1}^{s+1} E_i)$.

Question

Given p and $g \geq 2$ and $0 \leq s \leq g-2$,
does there exist a supersingular curve X over $\overline{\mathbb{F}}_p$ with genus g whose Jacobian has supersingular elliptic rank s?

Supersingular elliptic rank of Hermitian curves

The Hermitian curve $X_q : y^q + y = x^{q+1}$ with $q = p^n$ is supersingular with genus $g = q(q-1)/2$.

P/Weir: The distinct indecomposable factors of Dieudonné module $D(X_q)$ are in bijection with orbits of $\mathbb{Z}/(2^n+1) - \{0\}$ under $\times 2$.
Example: $D(X_{p^2}) = (\mathbb{E}/(F^2+V^2))^{g/2}$.

Check: if n even, then $\mathbb{E}/(F+V)$ is not a factor of $D(X_{p^n})$.

Combinatorial reason: n odd iff there is an element of order three in $\mathbb{Z}/(2^n+1)$ iff there is an orbit of length 2 in $\mathbb{Z}/(2^n+1)$ under $\times 2$.

Application: P/Weir

If n is even, the supersingular elliptic rank of $\mathrm{Jac}(X_{p^n})$ equals 0.
If n is odd, the supersingular elliptic rank of $\mathrm{Jac}(X_{p^n})$ equals $(\frac{p(p-1)}{2})^n$.

Computing the superspecial rank

Let X be a smooth projective curve over k.
Let $D(X)$ be the Dieudonné module of the p-torsion of $\mathrm{Jac}(X)$.

Oda: isomorphism of $\mathbb{E}$-modules between $D(X)$ and $H^1_{\mathrm{dR}}(X)$.

Exact sequence:

$$0 \to H^0(X,\Omega^1) \to H^1_{\mathrm{dR}}(X) \to H^1(X,\mathcal{O}) \to 0.$$

Compute the structure of $H^1_{\mathrm{dR}}(X)$ under F and V.
Write $(f,\omega) \in H^1_{\mathrm{dR}}(X)$ where $d\phi = \omega_1 - \omega_2$.
Then $F(f,\omega) := (f^p,0)$ and $V(f,\omega) := (0,C(\omega))$,
where C is the Cartier operator on Ω^1.

ss-Elliptic rank of hyperelliptic curves when $p = 2$

Let Y be a hyperelliptic curve of genus g over $k = \overline{\mathbb{F}}_2$.
Then $Y : y^2 + y = h(x)$ for some $h(x) \in k(x)$.

Theorem - Elkin/P

We determine structure of Dieudonné module of $\mathrm{Jac}(Y)$.
It depends only on the orders of the poles of $h(x)$.

E.g., p-rank 0 iff $h(x) \in k[x]$ iff the EO type is $[0, 1, 1, 2, 2, \ldots, \lfloor \frac{g}{2} \rfloor]$.

Application: Elkin/P

Let Y be a supersingular hyperelliptic curve of genus g over $\overline{\mathbb{F}}_2$.
Let e be the supersingular elliptic rank of $\mathrm{Jac}(Y)$.
Then $e = 1$ when $g \equiv 1 \bmod 3$ and $e = 0$ otherwise.

Such curves exist: if $h(x) = xR(x)$ for an additive polynomial $R(x)$ of degree 2^s, then Y is supersingular of genus 2^{s-1} VdG/VdV.

Supersingular elliptic rank of Suzuki curves

Let $p = 2$, let $q_0 = 2^m$ and let $q = 2^{2m+1}$.

The Suzuki curve S_q has equation $y^q + y = x^{q_0}(x^q + x)$.
It is supersingular with genus $q_0(q-1)$.

Furlmann/Torres: S_q is the unique $\mathbb{F}_q$-optimal curve of genus g .

Hansen92: S_q is the Deligne-Lusztig curve for group $Sz(q) = {}^2B_2(q)$.

Application: Malmskog/P/Weir

The ss-elliptic rank of $\mathrm{Jac}(S_q)$ is 1 when m even and is 0 when m odd.

Proof - supersingular elliptic rank of S_q

The Suzuki curve S_q has genus $q_0(q-1)$ where $q_0 = 2^m$, $q = 2^{2m+1}$.

Application: Malmskog/P/Weir

The ss-elliptic rank of $\mathrm{Jac}(S_q)$ is 1 when m even and is 0 when m odd.

Proof: Consider tame action of $T \simeq \mathbb{Z}/(q-1)$ on S_q.

Giulietti, Korchmaros, Torres: The quotient S_q/T is a supersingular hyperelliptic curve of genus q_0. (must also have p-rank 0).

Applying Elkin/P to S_q/T, the *old ss-elliptic rank* is 1 when $q_0 \equiv 1 \bmod 3$ (even m) and is 0 when $q_0 \equiv 2 \bmod 3$ (odd m).

The *new elliptic rank* is zero: the orbits of action of F on non-trivial eigenspaces of $H^1_{\mathrm{dR}}(S_q)$ are too long.

Work in progress - 1

Supersingular curves with ss-elliptic rank 0 when $p = 2$

P: Let $p = 2$. Let $r, s \in \mathbb{N}$ with s even. Let $q = 2^r$.

There exists a supersingular curve Y of genus $g = 2^{s-1}(q-1)$ with ss-elliptic rank 0.

If s odd, let $\alpha = \mathrm{val}_3(q-1)$ and $\beta = \mathrm{val}_3(2^s+1)$.
Then ss-elliptic rank of Y is bounded by $3^\alpha(3^\beta - 1)/2$.

Proof: Use VdG/VdV to construct supersingular curve of genus g, with automorphism of large prime-to-p order.

Old elliptic rank bounded by that of quotient curve.
New elliptic rank zero because orbits of action of F on non-trivial eigenspaces of $H^1_{\mathrm{dR}}(Y)$ too long.

Work in progress - 2

With Malmskog and Weir, computing the DM of the Suzuki curves.

Examples If $m = 1$, then $\mathrm{DM} = \mathbb{E}/(F^2 + V^2) \oplus (\mathbb{E}/(F^3 + V^3))^4$.

If $m = 2$, then $\mathrm{DM} = \mathbb{E}/(F + V) \oplus \mathbb{E}/(F^3 + V^3) \oplus D^4 \oplus (\mathbb{E}/(F^5 + V^5))^{16}$

where D has 3 generators (given by word $F^{-4}V^3F^{-3}V^4F^{-3}V^3$).

Strategy: find structure in general using representation theory.

Conclusion

The superspecial rank is an invariant of the DM of abelian variety A.

If A supersingular, then it equals the number of superspecial elliptic factors in the decomposition of A up to isomorphism.

Superspecial rank is generically 0 for supersingular abelian varieties.

Expect same is true for supersingular Jacobians of curves

Questions

1. The supersingular locus S_g of $\mathcal{A}_g$ can be stratified by superspecial rank into subspaces $S_{g,s}$.
   For $0 \leq s \leq g-2$, what is the geometry of $S_{g,s}$?
2. What superspecial ranks occur for supersingular curves?
3. If A supersingular, understand connection between superspecial rank and properties of the isogeny $A \sim E^g$.

Thanks!