Garden of curves with many automorphisms

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joint work with Massimo Giulietti

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Outline

X := (projective, non-singular, geometrically irreducible,) algebraic curve of genus $g \geq 2$, defined over an algebraically closed filed $K$ of characteristic $p > 0$.

$\text{Aut}(X)$ := the $K$-automorphism group of $X$.

Upper bounds on $|\text{Aut}(X)|$ depending on $g$, a survey.

What are the possibilities for $\text{Aut}(X)$ when $X$ has zero $p$-rank?

A classification in even characteristic $p$-subgroups of $\text{Aut}(X)$ of curves with positive $p$-rank.

Remark

The general study of $\text{Aut}(X)$ relies on the fundamental group of the curve, see R. Pries and K. Stevenson, A survey of Galois theory of curves in characteristic $p$, Amer. Math. Soc., (2011)

For further developments in specific questions and for effective constructions we need the potential of Finite Group Theory.

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Curves with many automorphisms
\( \mathcal{X} := \) (projective, non-singular, geometrically irreducible,) algebraic curve of genus \( g \geq 2 \), defined over an algebraically closed field \( \mathbb{K} \) of characteristic \( p > 0 \). \( \text{Aut}(\mathcal{X}) := \) the \( \mathbb{K} \)-automorphism group of \( \mathcal{X} \).
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The classical Hurwitz bound

If $G$ is tame then $|G| \leq 84(g - 1)$.
(Hurwitz bound)

$|\text{Aut}(X)| < 16g^4$; up to one exception, the Hermitian curve, [Stichtenoth (1973)].

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- $|\text{Aut}(\mathcal{X})| < 8g^3$; up to four exceptions. [Henn (1976)]
Four infinite families of curves $\mathcal{X}$ with $|\text{Aut}(\mathcal{X})| \geq 8g^3$

(I) $v(Y^2 + Y + X^2 k + 1)$, $p = 2$, a hyperelliptic curve of genus $g = 2k - 1$ with $|\text{Aut}(\mathcal{X})| = 2^{2k+1} (2k + 1)$.

(II) The Roquette curve: $v(Y^2 - (X q - X))$ with $p > 2$, a hyperelliptic curve of genus $g = 1/2(q - 1)$ with $\text{Aut}(\mathcal{X})/\mathcal{M} \sim \text{PSL}(2, q)$ or $\text{Aut}(\mathcal{X})/\mathcal{M} \sim \text{PGL}(2, q)$, where $q = p^r$ and $|\mathcal{M}| = 2$.

(III) The Hermitian curve: $v(Y^n + Y - X^n + 1)$ with $n = p^r$, genus $1/2n(n - 1)$, $\text{Aut}(\mathcal{X}) \sim \text{PGU}(3, n)$, $n$ a power of 2. $|\text{Aut}(\mathcal{X})| = (n^3 + 1)n^3(n^2 - 1)$.

(IV) The DLS curve (Deligne-Lusztig curve of Suzuki type): $v(X^n_0(X^n + X) + Y^n + Y)$, with $p = 2$, $n_0 = 2r \geq 2$, $n = 2n_0^2$, $g = n_0(n - 1)$, $\text{Aut}(\mathcal{X}) \sim \text{Sz}(n)$ where $\text{Sz}(n)$ is the Suzuki group, $|\text{Aut}(\mathcal{X})| = (n^2 + 1)n^2(n - 1)$.
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Two more infinite families of curves $\mathcal{X}$ with large $\text{Aut}(\mathcal{X})$.

The DLR curve (the Deligne-Lusztig curve arising from the Ree group):

$$v(Y^n_2 - [1 + (X^n - X)]^{n-1}Y^{n-1} + (X^n - X)^n - X^n(X^n - X)^3 + 3^n),$$
with $p = 3$, $n_0 = 3^r$, $n = 3^{n_2^0}$;

$g = 3^{\frac{2}{n_0}(n - 1)(n + n_0 + 1)}$; $\text{Aut}(\mathcal{X}) \cong \text{Ree}(n)$ where $\text{Ree}(n)$ is the Ree group,

$|\text{Aut}(\mathcal{X})| = (n^3 + 1)n^3(n - 1)$.

The G.K curve:

$$v(Y^{3n+1} + (X^n + X)(\sum_{i=0}^{n} (-1)^{i+1}X^i(n-1)^i)n^{n+1}),$$
a curve of genus $g = \frac{1}{2}(n^3 + 1)(n^2 - 2) + 1$ with $\text{Aut}(\mathcal{X})$ containing a subgroup isomorphic to $\text{SU}(3, n)$, $n = p^r$.

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(V) The DLR curve (the Deligne-Lusztig curve arising from the Ree group):
\[ v(Y^{n^2} - [1 + (X^n - X)^{n-1}]Y^n + (X^n - X)^{n-1}Y - X^n(X^n - X)^{n+3n_0}) \], with \( p = 3 \), \( n_0 = 3^r \), \( n = 3n_0^2 \);
\[ g = \frac{3}{2}n_0(n - 1)(n + n_0 + 1); \text{ Aut}(\mathcal{X}) \cong \text{Ree}(n) \text{ where } \text{Ree}(n) \text{ is the Ree group, } |\text{Aut}(\mathcal{X})| = (n^3 + 1)n^3(n - 1). \]
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Problems on curves with large automorphism groups, $\gamma = 0$

Remark
All the above examples have zero $p$-rank.

Problem 1: Find a function $f(g)$ such that if $|\text{Aut}(X)| > f(g)$ then $\gamma = 0$.

Problem 2: Determine the structure of large automorphism groups of curves with $\gamma = 0$. This includes the study of large automorphism groups of maximal curves over a finite field.

Problem 3: $\exists$ simple or almost simple groups, other than those in the examples (II), . . . (VI), occurring as an automorphism group of a maximal curve?
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Problem 4: “Big action problem” (Lehr-Matignon): What about zero $p$-rank curves with very large $p$-group $S$ of automorphisms?
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If $\text{Aut}(\mathcal{X})$ fixes no point and $|S| > pg/(p - 1)$ then $\mathcal{X}$ is one of the curves (II) . . . (VI). (Giulietti-K. 2010).
Large $p$-subgroups of automorphisms of zero $p$-rank curves

Lemma [Bridge lemma]

Let $X$ be a zero $p$-rank curve, i.e. $\gamma = 0$. Let $S \leq \text{Aut}(X)$ with $|S| = p^h$. Then $S$ fixes a point of $P$ of $X$, and no non-trivial element in $S$ fixes a point distinct from $P$.

Definition

A Sylow $p$-subgroup $S_p$ of a finite group $G$ is a trivial intersection set if $S_p$ meets any other Sylow $p$-subgroup of $G$ trivially. If this is the case, $G$ has the TI-condition with respect to the prime $p$.

Theorem (Giulietti-K. 2005)

Let $X$ be a curve with $\gamma = 0$. Then every wild subgroup $G$ of $\text{Aut}(X)$ satisfies the TI-condition for its $p$-subgroups of Sylow.

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Finite groups satisfying TI-condition for some prime $p$

Theorem (Burnside-Gow, 1976)

Let $G$ be a finite solvable group satisfying the TI-condition for $p$. Then a Sylow $p$-subgroup $S_p$ is either normal or cyclic, or $p = 2$ and $S_2$ is a generalized quaternion group.

Remark
Non-solvable groups satisfying the TI-condition are also exist. The known examples include the simple groups involved in the examples (II) ... (VI). Their complete classification is not done yet, important partial classifications (under further conditions) were given by Hering, Herzog, Aschbacher, and more recently by Guralnick-Pries-Stevenson.
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Let $p = 2$ and $X$ a zero $2$-rank algebraic curve of genus $g \geq 2$. Let $G \leq \text{Aut}(X)$ with $2 \mid |G|$. Then one of the following cases holds.

(a) $G$ fixes no point of $X$ and the subgroup $N$ of $G$ generated by all its $2$-elements is isomorphic to one of the groups $\text{PSL}(2, n)$, $\text{PSU}(3, n)$, $\text{SU}(3, n)$, $\text{Sz}(n)$ with $n = 2^r \geq 4$; Here $N$ coincides with the commutator subgroup $G'$ of $G$.

(b) $G$ fixes no point of $X$ and it has a non-trivial normal subgroup of odd order. A Sylow $2$-subgroup $S_2$ of $G$ is either a cyclic group or a generalized quaternion group. Furthermore, either $G = \text{O}(G) \rtimes S_2$, or $G / \text{O}(G) \cong \text{SL}(2, 3)$, or $G / \text{O}(G) \cong \text{GL}(2, 3)$, or $G / \text{O}(G) \cong G_{48}$.

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Corollary

Let $\mathcal{X}$ be a zero 2-rank curve such that the subgroup $G$ of $\text{Aut}(\mathcal{X})$ fixes no point of $\mathcal{X}$.

If $G$ is solvable, then the Hurwitz bound holds for $G$; more precisely $|G| \leq 72(g - 1)$.

If $G$ is not solvable, then $G$ is known and the possible genera of $\mathcal{X}$ are computed from the order of its commutator subgroup $G'$ provided that $G$ is large enough, namely whenever $|G| \geq 24g(g - 1)$. 

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Corollary

Let $X$ be a zero 2-rank curve such that the subgroup $G$ of $\text{Aut}(X)$ fixes no point of $X$.

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Problem 5: Find some more examples of zero 2-rank curves of genus $g$ with $|\text{Aut}(X)| \geq 24g(g - 1)$.

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Maximal curves with few orbits on rational points

Remark
For the Hermitian curve, \( \text{Aut}(X) \) is transitive on \( X(\mathbb{F}_{q^2}) \).
For other two classical maximal curves, \( \text{Aut}(X) \) has two orbits on the set of rational points.

Theorem (Giulietti-K. 2009)
Let \( p = 2 \).
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Problem 8: Prove a similar characterization theorem for the other "classical" maximal curves.
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Curves with large $p$-groups of automorphisms, case $\gamma > 0$

$x :=$ curve with genus $g$ and $p$-rank $\gamma > 0$.

$S := p$-subgroup of $\text{Aut}(x)$;

Nakajima's bound (1987):

$$|S| \leq \begin{cases} 4(\gamma - 1) & \text{for } p = 2, \gamma > 1 \\ p^\gamma - 2(\gamma - 1) & \text{for } p \neq 2, \gamma > 1 \\ g - 1 & \text{for } \gamma = 1 \end{cases}.$$

Problem 9: Determine the possibilities for the structures of $S$ when $x$ extremal w.r. Nakajima's bound, or $|S|$ is closed to it.

Hypothesis (I): $|S| > 2(g - 1)$ (and $|S| \geq 8$), $\Rightarrow p = 2, 3$.

If $S$ fixes a point then $|S| \leq p^g / (p - 1)$.

Hypothesis (II): $S$ fixes no point on $x$. 

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Curves with many automorphisms
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Hypothesis (I): \(|S| > 2(g - 1)\) (and \(|S| ≥ 8\)), \(⇒ p = 2, 3\).

If \( S \) fixes a point then \(|S| ≤ pg / (p - 1)\).

Hypothesis (II): \( S \) fixes no point on \( \mathcal{X} \).
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Curves with large $p$-groups of automorphisms, case $\gamma > 0$

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Theorem (Giulietti-K. 2013)

Let $p = 3$. If $|S| > 2(g-1)$ and $S$ fixes no point on $X$, then $g = \gamma$; $|S| = 3(\gamma-1)$ (Extremal curve w.r. Nakajima's bound); $S$ is generated by two elements, $S$ is abelian only for $|S| = 3, 9$; $S$ has two short orbits on $X$ each of size $|S|/3$; $S$ has a normal subgroup $M$ such that $S = M \rtimes \langle \varepsilon \rangle$ with $\varepsilon^3 = 1$ and $M$ semiregular on $X$; $X$ is an unramified Galois extension of an ordinary genus $2$ curve $\bar{X}$ with $\text{Gal}(X|\bar{X}) = M$; if $M$ is abelian then $|Z(S)| = 3$ and $S$ has maximal (nilpotency) class. $\Rightarrow$ the structure of $S$ is known.

Problem 10: Find examples where $S$ has not maximal class.
Case $p = 3$

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- $g = \gamma$;
- $|S| = 3(\gamma - 1)$ (*Extremal curve w.r. Nakajima’s bound*);
- *S is generated by two elements, S is abelian only for $|S| = 3, 9$*;
- *S has two short orbits on X each of size $|S|/3$*. 

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**Theorem (Giulietti-K. 2013)**

Let $p = 3$. If $|S| > 2(g - 1)$ and $S$ fixes no point on $\mathcal{X}$, then

- $g = \gamma$;
- $|S| = 3(\gamma - 1)$ (Extremal curve w.r. Nakajima’s bound);
- $S$ is generated by two elements, $S$ is abelian only for $|S| = 3, 9$;
- $S$ has two short orbits on $\mathcal{X}$ each of size $|S|/3$;
- $S$ has a normal subgroup $M$ such that $S = M \rtimes \langle \varepsilon \rangle$ with $\varepsilon^3 = 1$ and $M$ semiregular on $\mathcal{X}$;
- $\mathcal{X}$ is an unramified Galois extension of an ordinary genus 2 curve $\overline{\mathcal{X}}$ with $\text{Gal}(\mathcal{X}|\overline{\mathcal{X}}) = M$; $M = \langle \alpha, \beta \rangle$;
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Case $p = 3$

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Problem 10: Find examples where $S$ has not maximal class.
Case $p = 3$, Examples for small genera

If $|S| = 3$ then $X = v \left( (X Y^3 - X) - X^2 + c \right)$ with $c \in K^*$.  

If $|S| = 9$ then $S = C_3 \times C_3$ and $X = v \left( (X^3 - X)(Y^3 - Y) + c \right)$ with $c \in K^*$, $g(X) = 4$.  

If $|S| = 27$ then $S = \text{UT}(3, 3)$ and $X = v \left( (X^3 - X)(Y^3 - Y) + c, U^3 - U - X \right)$ with $c \in K^*$, $g(X) = 10$.  

For $|S| = 81$ an explicit example: $S \sim Syl_3(\text{Sym} \ 9)$, $X = v \left( (X^3 - X)(Y^3 - Y) + c, U^3 - U - X \right)$ with $c \in K^*$, $g(X) = 28$.  

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Curves with many automorphisms
Case $p = 3$, Examples for small genera

- If $|S| = 3$ then $\mathcal{X} = v((X(Y^3 - Y) - X^2 + c)$ with $c \in \mathbb{K}^*$. 
Case \( p = 3 \), Examples for small genera

- If \( |S| = 3 \) then \( \mathcal{X} = v((X(Y^3 - Y) - X^2 + c) \) with \( c \in \mathbb{K}^* \).
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Case \( p = 3 \), Examples for small genera

- If \(|S| = 3\) then \( \mathcal{X} = v((X(Y^3 - Y) - X^2 + c) \) with \( c \in \mathbb{K}^* \).
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- If \(|S| = 27\) then \( S = UT(3, 3) \) and 
  \( \mathcal{X} = v((X^3 - X)(Y^3 - Y) + c, Z^3 - Z - X^3 Y + YX^3) \) with 
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Case $p = 3$, Examples for small genera

- If $|S| = 3$ then $\mathcal{X} = v((X(Y^3 - Y) - X^2 + c)$ with $c \in \mathbb{K}^*$.
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- If $|S| = 27$ then $S = UT(3, 3)$ and $\mathcal{X} = v((X^3 - X)(Y^3 - Y) + c, Z^3 - Z - X^3 Y + YX^3)$ with $c \in \mathbb{K}^*$, $g(\mathcal{X}) = 10$.
- For $|S| = 81$ an explicit example: $S \cong Syl_3(\text{Sym}_9)$, $\mathcal{X} = v((X^3 - X)(Y^3 - Y) + c, U^3 - U - X, (U - Y)(W^3 - W) - 1, (U - (Y + 1))(T^3 - T) - 1)$ with $c \in \mathbb{K}^*$, $g(\mathcal{X}) = 28$. 
Case $p = 3$, infinite families of examples

$$F := K(x, y^3 - y^3 - x^2 + c), \quad c \in K^*;$$

$$g(F) = \gamma(F) = 2.$$ 

$$\phi := (x, y) \mapsto (x, y + 1), \quad \phi \in \text{Aut}(F).$$ 

$F_N$ denotes the largest unramified abelian extension of $F$ of exponent $N$ with two generators, 

(i) $F_N \mid F$ is an unramified Galois extension of degree $3^2 N$,

(ii) $F_N$ is generated by all function fields which are cyclic unramified extensions of $F$ of degree $p^2 N$,

(iii) $\text{Gal}(F_N \mid F) = C_3^N \times C_3^N$ and $u^3 N = 1$ for every element $u \in \text{Gal}(F_N \mid F)$.

$M$ denotes the Galois closure of $F_N \mid K$.

Lemma $\text{Gal}(M \mid K(x))$ preserves $F$.

$\Rightarrow \text{Gal}(M \mid K(x)) \leq \text{Aut}(F_N)$.

Corollary $F_N$ is an extremal function field w.r. Nakajima's bound.
Case $p = 3$, infinite families of examples

- $F:=\mathbb{K}(x, y), \ x(y^3 - y) - x^2 + c = 0, \ c \in \mathbb{K}^*$;
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*Curves with many automorphisms*
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  (iii) $\text{Gal}(F_N|F) = C_{3^N} \times C_{3^N}$ and $u^{3^N} = 1$ for every element $u \in \text{Gal}(F_N|F)$. 

$M :=$ Galois closure of $F_N|\mathbb{K}$.

Lemma $\text{Gal}(M|\mathbb{K}(x))$ preserves $F$.

$\Rightarrow$ $\text{Gal}(M|\mathbb{K}) \leq \text{Aut}(F_N)$.

Corollary $F_N$ is an extremal function field w.r. Nakajima's bound.
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- $F := \mathbb{K}(x, y), \ x(y^3 - y) - x^2 + c = 0, \ c \in \mathbb{K}^*$;
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  \begin{enumerate}
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**Lemma**

$\text{Gal}(M|\mathbb{K}(x))$ preserves $F$.  

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Curves with many automorphisms
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**Lemma**

\[ \text{Gal}(M|\mathbb{K}(x)) \text{ preserves } F \implies \text{Gal}(M|\mathbb{K}(x)) \leq \text{Aut}(F_N). \]
Case $p = 3$, infinite families of examples

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**Lemma**

$\text{Gal}(M|\mathbb{K}(x))$ preserves $F$. $\Rightarrow$ $\text{Gal}(M|\mathbb{K}(x)) \leq \text{Aut}(F_N)$.

**Corollary**

$F_N$ is an extremal function field w.r. Nakajima’s bound.
Remark

If \( N \sqsubseteq S \) and \( |S:N| \geq 9 \) then \( X/N \) is also an extremal curve w.r.t. Nakajima's bound.

\[
F := K(x, y) \left( x(y^3 - y) - x^2 + c = 0 \right), \quad c \in K^*.
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Let \( F \) be the set of all unramified Galois extensions \( K \) of \( F \) such that \( K \) is extremal w.r.t. Nakajima's bound.

For every \( K \in F \) take the (unique) maximal 3-subgroup of \( \text{Aut}(K) \) together with all surjections of index \( \geq 9 \).

These groups and surjections form an inverse system.

Problem 10: What about the arising profinite group (limit of this inverse system)?
Remark

If $N \triangleleft S$ and $[S : N] \geq 9$ then $\mathcal{X}/N$ is also an extremal curve w.r. Nakajima's bound.
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If $N \trianglelefteq S$ and $[S : N] \geq 9$ then $\mathcal{X}/N$ is also an extremal curve w.r. Nakajima's bound.

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Let $F := \mathbb{K}(x, y), \quad x(y^3 - y) - x^2 + c = 0, \quad c \in \mathbb{K}^*$, set $F$ be the set of all unramified Galois extensions $K$ of $F$ such that $K$ is extremal w.r. Nakajima’s bound. For every $K \in F$ take the (unique) maximal 3-subgroup of $\text{Aut}(K)$ together with all surjections of index $\geq 9$. 
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**Problem 10:** What about the arising profinite group (limit of the this inverse system)?
Case $p = 2$

**Theorem (Giulietti-K. 2012)**

Let $p = 2$. If $|S| > 2(g - 1)$, $|S| \geq 8$ and $S$ fixes no point on $X$, then one of the following cases occurs

- $|S| = 4(g - 1)$, $X$ is an ordinary bielliptic curve.
- Either (ia) $S$ is dihedral, or (ib) $S = (E \times \langle u \rangle) \rtimes \langle w \rangle$ where $E$ is cyclic group of order $g - 1$ and $u$ and $w$ are involutions.

- $|S| = 2g + 2$, and $S = A \rtimes B$, $A$ is an elementary abelian subgroup of index 2 and $B = 2$; Every central involution of $S$ is inductive.

Involution $u \in Z(S)$ is inductive: $S/\langle u \rangle$, viewed as a subgroup of $\text{Aut}(\bar{X})$ of the quotient curve $X = X/\langle u \rangle$ satisfies the hypotheses of the theorem.
Case \( p = 2 \)

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**Theorem (Giulietti-K. 2012)**

Let $p = 2$. If $|S| > 2(g - 1)$, $|S| \geq 8$ and $S$ fixes no point on $X'$, then one of the following cases occurs:

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Case $p = 2$, examples

For every $2h$, $\exists$ a curve of type (ia): (extremal curve w.r. Nakajima’s bound with dihedral 2-group of automorphisms).

$\exists$ a sporadic example of type (ib) with $g = 9$ and $S = D_8 \times C_2$.

For $q = 2h$, the hyperelliptic curve $X := v(Y^2 + Y + X)(X^q + X) + \sum_{\alpha \in F_q} X^q + X + \alpha$ has genus $g = q - 1$ and an elementary abelian automorphism group of order $2^q$. Examples involving inductive involutions are also known.

Problem 11: Construct infinite family of curves of type (ib).
For every $2^h$, there exists a curve of type (ia): (extremal curve w.r. Nakajima’s bound with dihedral 2-group of automorphisms).
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$$\mathcal{X} := v((Y^2 + Y + X)(X^q + X) + \sum_{\alpha \in \mathbb{F}_q} \frac{X^q + X}{X + \alpha})$$

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Examples involving inductive involutions are also known.
Case $p = 2$, examples

- For every $2^h$, $\exists$ a curve of type (ia): (extremal curve w.r. Nakajima’s bound with dihedral 2-group of automorphisms).
- $\exists$ a sporadic example of type (ib) with $g = 9$ and $S = D_8 \times C_2$.
- For $q = 2^h$, the hyperelliptic curve

$$\mathcal{X} := v((Y^2 + Y + X)(X^q + X) + \sum_{\alpha \in \mathbb{F}_q} \frac{X^q + X}{X + \alpha})$$

has genus $g = q - 1$ and an elementary abelian automorphism group of order $2q$.
- Examples involving inductive involutions are also known.

Problem 11: Construct infinite family of curves of type (ib).