Affine variety codes are better than their reputation

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Affine variety codes

$$I \subseteq \mathbb{F}_q[X_1, \ldots, X_m] \qquad I_q = I + \langle X_1^q - X_1, \ldots, X_m^q - X_m \rangle.$$

$$\{P_1, \ldots, P_n\} = \mathbb{V}_{\mathbb{F}_q}(I_q),$$

$\{N_1 + I_q, \ldots, N_n + I_q\}$ a basis for $\mathbb{F}_q[X_1, \ldots, X_m]/I_q$.

We get a basis for $\mathbb{F}_q^n$:
$\{\vec{b}_1 = \big(N_1(P_1), \ldots, N_1(P_n)\big), \ldots, \vec{b}_n = \big(N_n(P_1), \ldots, N_n(P_n)\big)\}$

Definition

Consider $L \subseteq \{1, \ldots, n\}$. $C(I, L) = \mathrm{Span}_{\mathbb{F}_q}\{\vec{b}_i \mid i \in L\}$
$C^\perp(I, L) = \big(C(I, L)\big)^\perp$.

Theorem

C is a linear code $\Leftrightarrow$ C is an affine variety code.

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One-point AG codes

Theorem

If Q is a rational place then $\cup_{s=0}^{\infty}\mathcal{L}(sQ) \simeq \mathbb{F}_q[X_1,\ldots,X_m]/I$ where I satisfies the order domain conditions.

Theorem

A map $h : \mathbb{F}_q[X_1,\ldots,X_m]/I \to \mathbb{F}_q^n$ such that

- *h is $\mathbb{F}_q$-linear,*
- *$h(f) = (c_1,\ldots,c_n)$ and $h(g) = (d_1,\ldots,d_n)$*
  $$\Rightarrow h(fg) = (c_1d_1,\ldots,c_nd_n)$$

is of the form $h(f = F + I) = (F(P_1),\ldots,F(P_n))$, where $P_1,\ldots,P_n$ are affine points.

Our work

- Most known affine variety codes are one-point AG codes in disguise.
- We introduce a much broader class of affine variety codes.
- We
  - generalise the Feng-Rao-bound/order-bound for dual codes (also simply known as the Feng-Rao-bound/order-bound). Our method builds on work by Salazar et al.
  - generalise the Feng-Rao-bound/order-bound for primary codes (sometimes called the Andersen–G bound),

  We treat affine variety codes and general linear codes. We treat minimum distance and generalised Hamming weights.

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The footprint bound

Definition

Given an ideal $J \subseteq k[X_1, \ldots, X_m]$ and a monomial ordering $\prec$ then $\Delta_\prec(J) = \{M \text{ is a monomial } \mid M \notin \mathrm{lm}(J)\}$

Theorem

(The footprint bound:) If $J \subseteq k[X_1, \ldots, X_m]$ is radical and zero-dimensional and if k is a perfect field then $\#\mathbb{V}(J) = \#\Delta_\prec(J)$.

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- Our new bound for primary codes relies on the footprint bound.
- Our new bound for dual codes uses Feng-Rao arguments, and the connection to the primary bound is not completely clear.

The footprint bound and other bounds

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The rest of this talk

- Our bound for dual codes is powerful, but too technical for this talk.
- Our bound for primary codes can easily be explained for affine variety codes.

Agenda:

- We start by studying the order domain conditions and primary codes.
- Then we throw away half of the order domain conditions and consider primary codes.
- We present numerical data for both primary and dual codes.

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Hermitian code

$$I = \langle X^2 + X - Y^3 \rangle \subseteq \mathbb{F}_4[X, Y],\ I_q = I + \langle X^4 - X, Y^4 - Y \rangle.$$

A weighted degree lexicographic ordering

From the weight function $w(X^i Y^j) = 3i + 2j$ we define the monomial ordering $\prec_w$ by $N \prec_w M$ if

- either $w(N) < w(M)$,
- or $w(N) = w(M)$ but $\deg_X(N) < \deg_X(M)$.

$$\{P_1, \ldots, P_8\} = \mathbb{V}(I_q).$$

Consider $\vec{c} = (F(P_1), \ldots, F(P_8))$.

$$\begin{aligned} w_H(\vec{c}) &= 8 - \#\text{ common zeros between } F \text{ and } I_q \\ &= \#\big(\Delta_{\prec_w}(I_q) \backslash \Delta_{\prec_w}(I_q + \langle F \rangle)\big) \\ &= \#\{M \in \Delta_{\prec_w}(I_q) \mid M \in \mathrm{lm}(I_q + \langle F \rangle)\}. \end{aligned}$$

Hermitian code - cont.

Consider $\vec{c} = (F(P_1), \ldots, F(P_8))$, say $F = a_1 + a_2 Y + X$

$w_H(\vec{c}) = \#\{M \in \Delta_{\prec_w}(I_q) \mid M \in \mathrm{Im}(I_q + \langle F \rangle)\}$.

Y^3	XY^3	6	9
Y^2	XY^2	4	7
Y	XY	2	5
1	X	0	3

$X = \mathrm{Im}(F)$, $XY = \mathrm{Im}(YF)$,
$XY^2 = \mathrm{Im}(Y^2F)$,
$XY^3 = \mathrm{Im}(Y^3F)$,
$Y^3 = \mathrm{Im}(XF - (X^2 + X - Y^3))$

In conclusion, $w_H(\vec{c}) \geq 5$.

We could also have counted the numbers in $\{0, 2, 3, 4, 5, 6, 7, 9\}$ which are being hit by $w(\mathrm{Im}(F)) = 3$.

This is due to $X^2 + X - Y^3$ having two monomials of the highest weight and all monomials in $\Delta_{\prec_w}(I)$ being of different weight.

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The order domain conditions

Definition

Consider an ideal $J \subseteq k[X_1, \ldots, X_m]$ where k is a field. Let a weighted degree ordering $\prec_w$ be given. Assume that J possesses a Gröbner basis $\mathcal{F}$ with respect to $\prec_w$ such that:

(C1) Any $F \in \mathcal{F}$ has exactly two monomials of highest weight.

(C2) No two monomials in $\Delta_{\prec_w}(J)$ are of the same weight.

Then we say that J and $\prec_w$ satisfy the order domain conditions.

The Feng-Rao bounds do not work well when the order domain conditions are not satisfied.

We throw away condition (C2) and introduce a method that works well for the corresponding codes.

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An affine variety code over $\mathbb{F}_8$.

$$I = \langle (X^4 + X^2 + X) - (Y^6 + Y^5 + Y^3) \rangle \subseteq \mathbb{F}_8[X, Y].$$
$$I_q = I + \langle X^8 - X, Y^8 - Y \rangle.$$

Define $\prec_w$ on the basis of $w(X^iY^j) = 3i + 2j$.

Y^7	XY^7	X^2Y^7	X^3Y^7
Y^6	XY^6	X^2Y^6	X^3Y^6
Y^5	XY^5	X^2Y^5	X^3Y^5
Y^4	XY^4	X^2Y^4	X^3Y^4
Y^3	XY^3	X^2Y^3	X^3Y^3
Y^2	XY^2	X^2Y^2	X^3Y^2
Y	XY	X^2Y	X^3Y
1	X	X^2	X^3

$\Delta_{\prec_w}(I_q)$

14	17	20	23
12	15	18	21
10	13	16	19
8	11	14	17
6	9	12	15
4	7	10	13
2	5	8	11
0	3	6	9

Corresponding weights

An affine variety code over $\mathbb{F}_8$ - cont.

14	17	20	23
12	15	18	21
10	13	16	19
8	11	14	17
6	9	12	15
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2	5	8	11
0	3	6	9

$$\mathbb{V}(I_q) = \{P_1, \ldots, P_{32}\}$$

$$\vec{c} = (F(P_1), \ldots, F(P_{32}))$$

where

$$F = a_1 + a_2Y + a_3X + a_4Y^2 + a_5XY + a_6Y^3 + a_7X^2 + a_8XY^2 + a_9Y^4 + a_{10}X^2Y + a_{11}XY^3 + X^3$$

Observe that $w(XY^3) = w(X^3) = 9$. Hence, we must be careful.

An affine variety code over $\mathbb{F}_8$ - cont.

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8	11	14	17
6	9	12	15
4	7	10	13
2	5	8	11
0	3	6	9

Case 1: $a_{11} = 0$

$\text{Im}\left(XF - ((X^4 + X^2 + X) - (Y^6 + Y^5 + Y^3))\right) = Y^6$ and therefore we find not only X^3, X^3Y, X^3Y^2, X^3Y^3, X^3Y^4, X^3Y^5, X^3Y^6, X^3Y^7 but also Y^6, XY^6, X^2Y^6, Y^7, XY^7, X^2Y^7 as leading monomials.

Remember: $w_H(\vec{c}) = \#\{M \in \Delta_{\prec_w}(I_q) \mid M \in \text{Im}(I_q + \langle F \rangle)\}$.

An affine variety code over $\mathbb{F}_8$ - cont.

$F = a_1 + a_2Y + a_3X + a_4Y^2 + a_5XY + a_6Y^3 + a_7X^2 + a_8XY^2 + a_9Y^4 + a_{10}X^2Y + a_{11}XY^3 + X^3.$

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Case 2: $a_{11} \neq 0$

$\mathrm{lm}\left(XF - ((X^4+X^2+X) - (Y^6+Y^5+Y^3))\right) = X^2Y^3$ and therefore we find not only X^3, X^3Y, X^3Y^2, X^3Y^3, X^3Y^4, X^3Y^5, X^3Y^6, X^3Y^7 but also X^2Y^3, X^2Y^4, X^2Y^5, X^2Y^6, X^2Y^7 as leading monomials.

Case 1 gave $w_H(\vec{c}) \geq 14$ and Case 2 gave $w_H(\vec{c}) \geq 13$.

Hence, $w_H(\vec{c}) \geq 13$. (The Feng-Rao bound gives $w_H(\vec{c}) \geq 8$)

An affine variety code over $\mathbb{F}_8$ - cont.

$F = a_1 + a_2Y + a_3X + a_4Y^2 + a_5XY + a_6Y^3 + a_7X^2 + a_8XY^2 + a_9Y^4 + a_{10}X^2Y + a_{11}XY^3 + X^3$.

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Case 2: $a_{11} \neq 0$

$\text{lm}\left(XF - ((X^4+X^2+X)-(Y^6+Y^5+Y^3))\right) = X^2Y^3$ and therefore we find not only X^3, X^3Y, X^3Y^2, X^3Y^3, X^3Y^4, X^3Y^5, X^3Y^6, X^3Y^7 but also X^2Y^3, X^2Y^4, X^2Y^5, X^2Y^6, X^2Y^7 as leading monomials.

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Terminology for general linear code

Feng-Rao introduced the concept of well-behaving pairs (WB),

Miura the concept of weakly well-behaving pairs (WWB),

G–Thommesen the concept of one-way well-behaving pairs (OWB).

OWB $\Leftarrow$ WWB $\Leftarrow$ WB
Therefore OWB gives the strongest bounds.

OWB becomes crucial when we skip the second order domain condition.

Results for dual codes

$$I = \langle (X^9 + X^3 + X) - (Y^{12} + Y^{10} + Y^4) \rangle \subseteq \mathbb{F}_{27}[X, Y].$$

Code length $n = 243$.

	Feng-Rao WB	Feng-Rao WWB	Feng-Rao OWB	"Advisory bound"	Our bound
$d_1(C(75))$	15	15	21	29	33
$d_2(C(75))$	16	16	24	34	38
$d_1(C(76))$	15	15	21	33	36
$d_2(C(76))$	16	16	24	38	39
$d_1(C(83))$	16	16	24	34	38
$d_2(C(83))$	17	17	27	39	41

A method for constructing many examples

Definition

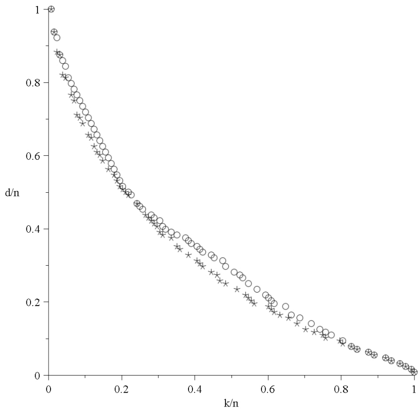
An $(\mathbb{F}_{q^t}, \mathbb{F}_q)$-polynomial is a polynomial $F(T) \in \mathbb{F}_{q^t}[T]$ such that $F(\gamma) \in \mathbb{F}_q$ holds for all $\gamma \in \mathbb{F}_{q^t}$.

Theorem

Consider the cyclotomic coset C_i modulo $q^t - 1$. Then $F(T) = \sum_{s \in C_i} X^s$ is an $(\mathbb{F}_{q^t}, \mathbb{F}_q)$-polynomial.

Corollary

Let $F(T)$ be a polynomial as in the above theorem and different from the trace-polynomial. Then $Tr_{\mathbb{F}_{q^t}/\mathbb{F}_q}(X) - F(Y)$ has exactly q^{2t-1} zeros.



Improved codes over $\mathbb{F}_{16}$ of length $n = 128$.
Using the trace-polynomial and the polynomial corresponding to the cyclotomic coset C_{10} we get $w(X) = 5$ and $w(Y) = 4$. These are the $\circ$s.
Using the trace-polynomial and the norm-polynomial we get the $*$s.