Weierstrass semigroups at several points, total inflection points on curves and coding theory

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Algebraic curves over finite fields
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Let $X$ be a smooth complete irreducible curve of genus $g \geq 1$ defined over a field $\mathbb{F}$, assumed to be the full field of constants of $\mathbb{F}(X)$.

Let $P_1, \ldots, P_m$ be distinct rational points of $X$.

**Definition** The Weierstrass semigroup at $P_1, \ldots, P_m$ is defined as

$$H = H(P_1, \ldots, P_m) := \{(\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m \mid \exists f \in \mathbb{F}(X) \text{ with} \quad \text{div}_\infty(f) = \alpha_1 P_1 + \cdots + \alpha_m P_m\}$$

Its systematic study was initiated by S. J. Kim and M. Homma in mid 90’s. They studied specially the case $m = 2$; investigated properties of $H$ and its relationship with the theory of algebraic geometry (Goppa) codes.

In a joint work with F. Torres we extended their results for any value of $m$, and also applied the results to obtain better lower bounds for the minimum distance of certain algebraic geometry codes.

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From now on we assume that \( \#(F) \geq m \).

Properties of \( H \):

- For all \( i = 1, \ldots, m \) we get that \( a \in H(P_i) \) if and only if \( a.e_i \in H \).
- Let \((n_1, \ldots, n_m), (p_1, \ldots, p_m) \in H\) and set \( q_i := \max\{n_i, p_i\} \), \( i = 1, \ldots, m \). Then \((q_1, \ldots, q_m) \in H\).

Define \((n_1, \ldots, n_m) \preceq (p_1, \ldots, p_m)\) if \( n_i \leq p_i \ \forall i = 1, \ldots, m \). Then \( \preceq \) is a partial order in \( \mathbb{N}_0^m \).

Let \( i \in \{1, \ldots, m\} \), let \( n_i \in \mathbb{N}_0 \) and let \( n = (n_1, \ldots, n_m) \) be a minimal element (w.r.t. \( \preceq \)) of the set \( \{(p_1, \ldots, p_m) \in H \mid p_i = n_i\} \). If \( n_i > 0 \) and \( n_j > 0 \) for some \( j \in \{1, \ldots, m\} \), \( j \neq i \), then:

(i) \( n_i e_i \notin H \) (hence \( n_i \notin H(P_i) \));

(ii) \( n \) is a minimal element of the set \( \{(p_1, \ldots, p_m) \in H \mid p_j = n_j\} \), so \( n_j \notin H(P_j) \).
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Cícero Carvalho (UFU)
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Weierstrass semigroup and AG codes

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Denote the set of pure gaps by \( \mathbf{G}_0 \).

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Weierstrass semigroup and AG codes

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Assume that $\mathbb{F}$ is a finite field, let $D := Q_1 + \cdots + Q_n$, where $Q_1, \ldots, Q_n$ are distinct rational points of $X$, all distinct from $P_1, \ldots, P_m$, and let $G$ be a divisor with support on $P_1, \ldots, P_m$.

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Weierstrass semigroup and AG codes

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Weierstrass semigroup and AG codes

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Weierstrass semigroup and AG codes

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Weierstrass semigroup and AG codes

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Assume that $\mathbb{F}$ is a finite field, let $D := Q_1 + \cdots + Q_n$, where $Q_1, \ldots, Q_n$ are distinct rational points of $X$, all distinct from $P_1, \ldots, P_m$, and let $G$ be a divisor with support on $P_1, \ldots, P_m$.

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Weierstrass semigroup and AG codes

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Weierstrass semigroup and AG codes

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Theorem. The $r$-th generalized Hamming distance of an AG code of length $n$ defined over $X$ satisfies

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A similar improvement can be made to bounds for the generalized Hamming distance of AG codes. Let $r$ be a positive integer, and $C \subset \mathbb{F}^m$ a linear code. Let $U$ be a subcode of $C$, the support of $U$ is defined as
$$\text{supp}(U) := \{i \mid c_i \neq 0 \text{ for some } (c_1, \ldots, c_m) \in U\}.$$ The $r$-th generalized Hamming distance of $C$ is defined as
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Assume that $X$ is a smooth plane, projective curve, of degree $r > 4$. We say that $P \in X$ is a total inflection point if the tangent line at $P$ intersects $X$ only at $P$. In a work with T. Kato, we proved the following.

Theorem. Let $P_1, P_2$ and $P_3$ be rational, total inflection points of $X$ which do not lie in a line. Then $((r - 4)r, 1, 1), (1, (r - 4)r, 1)$ and $(1, 1, (r - 4)r)$ are pure gaps of $H(P_1, P_2, P_3)$.

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We say that $P \in X$ is a total inflection point if the tangent line at $P$ intersects $X$ only at $P$. In a work with T. Kato, we proved the following.

Theorem. Let $P_1, P_2$ and $P_3$ be rational, total inflection points of $X$ which do not lie in a line. Then $((r - 4)r, 1, 1), (1, (r - 4)r, 1)$ and $(1, 1, (r - 4)r)$ are pure gaps of $H(P_1, P_2, P_3)$.

Theorem. Let $P_1, \ldots, P_m$ be total inflection points on $X$. Then $(s_1 r + \alpha_1, \ldots, s_m r + \alpha_m)$ is a pure gap of $H(P_1, \ldots, P_m)$, whenever $s_i, \alpha_i$ are integers such that $s_i \geq 0$, $1 \leq \alpha_i \leq r - 1 - i - \sum_{j=1}^{m} s_j$, for all $i = 1, \ldots, m$, and $\sum_{j=1}^{m} s_j \leq r - 2 - m$.

Theorem. Let $P, P_1, \ldots, P_m \in X$ be rational points, with $P$ a total inflection point. Let $0 \leq i < r - 3$ and $\alpha_1, \ldots, \alpha_m$ be positive integers such that $\sum_{j=1}^{m} \alpha_j \leq r - i - 3$. Then $(ir + 1, \alpha_1, \ldots, \alpha_m)$ is a pure gap of $H(P, P_1, \ldots, P_m)$. 
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Application. Take $X \subset \mathbb{P}^2(K)$ the Hermitian curve of degree $q + 1$ defined over $\mathbb{F} = GF(q^2)$. Let $s$ and $m$ be positive integers such that $s + m \leq q - 1$; let $P_1, \ldots, P_m$ be distinct rational points of $X$.

Take $s_1 = s$, $s_2 = \cdots = s_m = 0$, from the above theorem we get that

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Let $G = (2sr + q - 1 - s)P_1 + \sum_{i=2}^m (q - i - s)P_i$ and let $D$ be the sum of the other $q^3 + 1 - m$ rational points of $X$. From the work together with F. Torres we know that $C_{\Omega}(D, G)$ is an $[q^3 + 1 - m, k, d]$ code with

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So \( d \geq 2s(q + 1) + m(2q - 2s - 1) - m^2 - (q + 1)(q - 2) \)

and if we take \( s \geq (q - 1)/2 \) then \( \deg(G) > 2g - 2 \) and

\[ k = g + q^3 - 2s(q + 1) - m(q - s) + m(m - 1)/2. \]

We compared \( C_\Omega(D, G) \) with codes on the Hermitian curve supported on

one point and having the same dimension \( k \), finding many situations where

\( C_\Omega(D, G) \) has better parameters.

For example, assume that \( q \) is odd and \( q \geq 5 \), take \( m = s = (q - 1)/2 \).

Then \( C_\Omega(D, G) \) is an \([q^3 + 1 - (q - 1)/2, k, d]-\)code with

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Taking \( F = (q^3 - (q^2/8 + 3q/2 - 5/8))P \), where \( P \) is a rational point of

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We compared \( C_\Omega(D, G) \) with codes on the Hermitian curve supported on one point and having the same dimension \( k \), finding many situations where \( C_\Omega(D, G) \) has better parameters.

For example, assume that \( q \) is odd and \( q \geq 5 \), take \( m = s = (q - 1)/2 \).

Then \( C_\Omega(D, G) \) is an \([q^3 + 1 - (q - 1)/2, k, d]\)-code with

\[ k = q^3 - (5q + 13)(q - 1)/8 \quad \text{and} \quad d \geq q^2/4 + q + 3/4. \]

Taking \( F = (q^3 - (q^2/8 + 3q/2 - 5/8))P \), where \( P \) is a rational point of \( X \) and \( E \) is the sum of the other rational points, we get that \( C_\Omega(F, E) \) is an \([q^3, k, d']\) code, where \( d' = q^2/8 + 3q/2 - 5/8 \) (from works by Stichtenoth, Yang and Kummar) so that \( d - d' \geq (q(q - 4) + 11)/8 \).
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T H A N K  Y O U!