Bounds for the number of Rational points on curves over finite fields

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Classical Bounds

Let $\mathcal{X}$ be a projective, irreducible, non-singular curve of genus g, defined over $\mathbb{F}_q$. If N is the number of $\mathbb{F}_q$-rational points of $\mathcal{X}$ then

- Hasse-Weil-Serre:

$$|N-(q+1)| \leq g\lfloor 2q^{1/2}\rfloor.$$

- "Zeta":

$$N_2 \leq q^2+1+2gq-\frac{(N_1-q-1)^2}{g}$$

where N_r is the number of $\mathbb{F}_{q^r}$-rational points of $\mathcal{X}$.

- **Stöhr-Voloch** (baby version): If $\mathcal{X}$ has a plane model of degree d, and a finite number of inflection points, then

$$N \leq g-1+d(q+2)/2.$$

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Morphisms vs. Linear Series

Let $\mathcal{X}$ be a proj. irred. smooth curve of genus g defined over $\mathbb{F}_q$. Associted to a non-degenerated morphism $\phi = (f_0 : ... : f_n) : \mathcal{X} \longrightarrow \mathbb{P}^n(\mathbb{K})$, there exists a base-point-free linear series, of dimension n and degree d, given by

$$\mathcal{D} = \left\{ div \left(\sum_{i=0}^{n} a_i f_i \right) + E \mid a_0, ..., a_n \in \mathbb{K} \right\},$$

where

$$E := \sum_{P \in \mathcal{X}} e_P P, \quad \text{with } e_P = -min\{v_P(f_0), ..., v_P(f_n)\}$$

and $d = \deg E$

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Order sequence

For each point $P \in \mathcal{X}$, we have

$$\phi(P) = ((t^{e_P} f_0)(P) : ... : (t^{e_P} f_n)(P)),$$

where $t \in \mathbb{K}(\mathcal{X})$ is a local parameter at P.

For each point $P \in \mathcal{X}$, we define a sequence of non-negative integers

$$(j_0(P), ..., j_n(P))$$

where $j_0(P) < ... < j_n(P)$, are called $(\mathcal{D}, P)$ orders.This can be obtained from

$$\{j_0(P), \cdots, j_n(P)\} := \{v_P(D) : D \in \mathcal{D}\}.$$

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Order sequence

- We define $L_i(P)$ to be the intersection of all hyperplanes H of $\mathbb{P}^n(\mathbb{K})$ such that $v_P(\phi^*(H)) \geq j_{i+1}(P)$. Therefore, we have

$$L_0(P) \subset L_1(P) \subset \cdots \subset L_{n-1}(P).$$

- $L_i(P)$ is called i-th osculating space at P.
- Note that $L_0 = \{P\}$, $L_1(P)$ is the tangent line at P, etc.
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Via Wronskianos

Theorem

Let t be a local parameter at a point $P \in \mathcal{X}$. Suppose that each coordinate f_i of the morphism $\phi = (f_0 : ... : f_n)$ is regular at P. If $j_0, ..., j_{s-1}$ are the s first $(\mathcal{D}, P)$-orders of P, then j_s is the smallest integer such that the points

$$((D_t^{(j_s)} f_0)(P) : ... : (D_t^{(j_s)} f_n)(P)),$$

where $i = 0, ..., s$ are linearly independent over $\mathbb{K}$. Moreover, $L_i(P)$, the i-th osculating space at P is generated by these points.

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Via Wronskians

The order sequence $(j_0(P), ..., j_n(P))$ is the same for all but finitely many points $P \in \mathcal{X}$. This sequence is called the order sequence of $\mathcal{X}$ with respect to $\mathcal{D}$, and it is denoted by

$$(\epsilon_0, ..., \epsilon_n).$$

This sequence is also obtained as the minimal sequence (in lexicographic order), for which

$$\det(D_t^{(\epsilon_i)} f_j)_{0 \le i,j \le n} \neq 0,$$

where $t \in \mathbb{K}(\mathcal{X})$ is a separating variable.A curve $\mathcal{X}$ is called **classical** w.r.t. ϕ (or $\mathcal{D}$) if $(\epsilon_0, \epsilon_1, ..., \epsilon_n) = (0, 1, ..., n)$. Otherwise, $\mathcal{X}$ is called **non-classical**.

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Frobenius orders

Suppose ϕ is defined over $\mathbb{F}_q$, i.e., $f_i \in \mathbb{F}_q(\mathcal{X})$ for all $i = 0, ..., n$. The sequence of non-negative integers $(\nu_0, ..., \nu_{n-1})$, chosen minimally (lex order) such that

$$\det \begin{pmatrix} f_0^q & \dots & f_n^q \\ D_t^{(\nu_0)} f_0 & \dots & D_t^{(\nu_0)} f_n \\ \vdots & \dots & \vdots \\ D_t^{(\nu_{n-1})} f_0 & \dots & D_t^{(\nu_{n-1})} f_n \end{pmatrix} \neq 0,$$

where t is a separating variable of $\mathbb{F}_q(\mathcal{X})$, is called $\mathbb{F}_q$-order sequence of $\mathcal{X}$ with respect to ϕ.

Frobenius order

It is known that

$$\{\nu_0, ..., \nu_{n-1}\} = \{\epsilon_0, ..., \epsilon_n\} \backslash \{\epsilon_I\},$$

for some $I \in \{1, ..., n\}$. The ν_i's are called $\mathbb{F}_q$-Frobenius orders. If

$$(\nu_0, ..., \nu_{n-1}) = (0, ..., n-1),$$

then the curve $\mathcal{X}$ is called $\mathbb{F}_q$**-Frobenius classical** w.r.t. ϕ. Otherwise, $\mathcal{X}$ is called $\mathbb{F}_q$**-Frobenius non-classical**.

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Stöhr-Voloch Theorem

Theorem

Let $\mathcal{X}$ be a projective, irreducible smooth curve of genus g, defined over $\mathbb{F}_q$. If $\phi : \mathcal{X} \longrightarrow \mathbb{P}^n(\mathbb{K})$ is a non-degenerated morphism defined over $\mathbb{F}_q$, with $\mathbb{F}_q$-Frobenius orders $(\nu_0, ..., \nu_{n-1})$, then

$$N_1 \leq \frac{(\nu_1 + ... + \nu_{n-1})(2g-2) + (q+n)d}{n}, \quad (1)$$

where d is the degree of $\mathcal{D}$ associated to ϕ.

remark. Over the last twenty years, the Stöhr-Voloch Theory has been used as a key ingredient for many results related to points on curves over finite fields.

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A variation of the Stöhr-Voloch approach

Fix positive integers u and m, with $m > u$ and $mdc(u, m) = 1$.
The ideia is to estimate the number of points $P \in \mathcal{X}$ such that the line defined by $\Phi_{q^u}(\phi(P))$ and $\Phi_{q^m}(\phi(P))$, intersects the $(n-2)$-th osculating space of $\phi(\mathcal{X})$ at P. Let $\mathcal{D}$ be the linear series associated to ϕ and t be a local parameter at P. We know that the $(n-2)$-th osculating hyperplane at P is generated by

$$((D_t^{(j_i)} f_0)(P) : ... : (D_t^{(j_i)} f_n)(P)), \qquad i = 0, ..., n-2,$$

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It is easy to see that P satisfies the geometric properties above if and only if

$$\det \begin{pmatrix} f_0(P)^{q^m} & f_1(P)^{q^m} & \dots & f_n(P)^{q^m} \\ f_0(P)^{q^u} & f_1(P)^{q^u} & \dots & f_n(P)^{q^u} \\ (D_t^{(j_0)} f_0)(P) & (D_t^{(j_0)} f_1)(P) & \dots & (D_t^{(j_0)} f_n)(P) \\ \vdots & \vdots & \dots & \vdots \\ (D_t^{(j_{n-2})} f_0)(P) & (D_t^{(j_{n-2})} f_1)(P) & \dots & (D_t^{(j_{n-2})} f_n)(P) \end{pmatrix} = 0.$$

A variation of the Stöhr-Voloch approach

This leads us to study the following functions

$$\mathcal{A}_t{}^{\rho_0,\dots,\rho_{n-2}} := \det \begin{pmatrix} f_0^{q^m} & f_1^{q^m} & \dots & f_n^{q^m} \\ f_0^{q^u} & f_1^{q^u} & \dots & f_n^{q^u} \\ D_t^{(\rho_0)} f_0 & D_t^{(\rho_0)} f_1 & \dots & D_t^{(\rho_0)} f_n \\ \vdots & \vdots & \dots & \vdots \\ D_t^{(\rho_{n-2})} f_0 & D_t^{(\rho_{n-2})} f_1 & \dots & D_t^{(\rho_{n-2})} f_n \end{pmatrix} \tag{2}$$

in $\mathbb{F}_q(\mathcal{X})$, where $t \in \mathbb{F}_q(\mathcal{X})$ is a separating variable, and $\rho_0, \rho_1, \cdots, \rho_{n-2}$ are non-negative integers. It can be shown that there exist non-zero function in $\mathbb{F}_q(\mathcal{X})$ of the above type.

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Let $0 \leq \kappa_0 < ... < \kappa_{n-2}$ be the smallest sequnce (lex order) such that ${\mathcal{A}_t}^{\rho_0,\ldots,\rho_{n-2}} \neq 0$. The κ_i's will be called (q^u, q^m)-Frobenius orders of $\mathcal{X}$ w.r.t. ϕ. If $\kappa_i = i$ for $i = 0, 1, ..., n-2$, we say that the curve is (q^u, q^m)**-Frobenius classical**. Otherwise, $\mathcal{X}$ is called (q^u, q^m)**-Frobenius non-classical**.

A variation of the Stöhr-Voloch approach

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Proposition

There exist integers I and J such that $\{\kappa_0, ..., \kappa_{n-2}\} = \{\nu_0, ..., \nu_{n-1}\} \backslash \{\nu_I\} = \{\mu_0, ..., \mu_{n-1}\} \backslash \{\mu_J\}$.

Invariants

Based on the previus proposition, one can see that the sequence $(\kappa_0, ..., \kappa_{n-2})$ depends only on the morphism.

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Definition

The (q^u, q^m)-Frobenius divisor de of $\mathcal{D}$ is defined by $T_{u,m} =$

$$div(\mathcal{A}_t^{\kappa_0,\dots,\kappa_{n-2}}(f_i's)) + (\kappa_0 + \kappa_1 + ... + \kappa_{n-2})div(dt) + (q^m + q^u + n - 1)E,$$

where t is a separating variable of $\mathbb{F}_q(\mathcal{X})$, $E = \sum_{P \in \mathcal{X}} e_P P$ and $e_P = -min\{v_P(f_0), ..., v_P(f_n)\}$.

Invariants

The following can be checked

- The divisor $T_{u,m}$ is effective.
- All the points $P \in \mathcal{X}(\mathbb{F}_{q^r})$, for $r = u, m, m-u$ are in the support of $T_{u,m}$.

Now the idea is to estimate the weights of the points

$$P \in \mathcal{X}(\mathbb{F}_{q^u}) \cup \mathcal{X}(\mathbb{F}_{q^m}) \cup \mathcal{X}(\mathbb{F}_{q^{m-u}})$$

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Estimating the weights of the points on $T_{u,m}$

Proposition

Let $P \in \mathcal{X}(\mathbb{F}_q)$ *with* $(\mathcal{D}, P)$*-orders* $j_0, j_1, \dots, j_n$*. Then*

$$v_P(T_{u,m}) \geq q^u j_1 + \sum_{i=0}^{n-2} (j_{i+2} - \kappa_i),$$

and equality holds if and only if

$$\det\left(\binom{j_i}{\kappa_s}\right)_{2 \leq i \leq n, 0 \leq s \leq n-2} \not\equiv 0 \mod p.$$

Estimating the weights

Proposition

Let $P \in \mathcal{X}$ be an arbitrary point with $(\mathcal{D}, P)$-orders $j_0, j_1, ..., j_n$. Then

$$v_P(T_{u,m}) \geq \sum_{i=0}^{n-2} (j_i - \kappa_i),$$

and if

$$\det\left(\binom{j_i}{\kappa_s}\right)_{0 \leq i,s \leq n-2} \equiv 0 \mod p,$$

strict inequality holds.

Estimating the weights

Proposition

Let $P \in \mathcal{X}$ be a point $\mathbb{F}_{q^r}$-rational, for $r = u, m$, with $(\mathcal{D}, P)$-orders $j_0, j_1, ..., j_n$. Then

$$v_P(T_{u,m}) \geq max\left\{\sum_{i=1}^{n-1}(j_i - \kappa_{i-1}), 1\right\}.$$

Moreover, if

$$\det\left(\binom{j_i}{\kappa_s}\right)_{1\leq i\leq n-1, 0\leq s\leq n-2} \equiv 0 \mod p \quad \text{and} \quad \sum_{i=1}^{n-1}(j_i - \kappa_{i-1}) \geq 1$$

then the strict inequality holds.

Estimating the weights

Proposition

Let $P \in \mathcal{X}$ be a $\mathbb{F}_{q^{(m-u)}}$-rational point. Then

$$v_P(T_{u,m}) \geq q^u.$$

The main result

Theorem

Let $\mathcal{X}$ be a projective, irreducible, smooth curve of genus g, defined over $\mathbb{F}_q$, and let N_r be its number of $\mathbb{F}_{q^r}$ rational points, for $r = 1, u, m, m-u$. If $\phi : \mathcal{X} \longrightarrow \mathbb{P}^n(\mathbb{K})$ is a non-degenerated morphism, defined over $\mathbb{F}_q$, with (q^u, q^m)-Frobenius orders $(\kappa_0, \kappa_1, ..., \kappa_{n-2})$, then

$$\begin{aligned} & (c_1 - c_u - c_m - c_{m-u})\mathbf{N_1} + c_u\mathbf{N_u} + c_m\mathbf{N_m} + c_{m-u}\mathbf{N_{m-u}} \\ \leq \; & (\kappa_1 + ... + \kappa_{n-2})(2g-2) + (q^m + q^u + n - 1)d, \end{aligned} \tag{3}$$

where d is the degree of the linear series $\mathcal{D}$ associated to ϕ.

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Theorem

and c_r *are the lower bound for the weights of* $P \in \mathcal{X}(\mathbb{F}_{q^r})$ *on the divisor* $T_{u,m}$*, for* $r = 1, u, m, m-u$*. Moreover,* $c_{m-u} \geq q^u$ *e* $c_1 \geq q^u + 2(n-1)$.

Some Consequences

Corollary

Let $\mathcal{X}$ be a projective, irreducible, smooth curve of genus g, defined over $\mathbb{F}_q$, and let N_r be its number of $\mathbb{F}_{q^r}$ rational points, for $r = 1, u, m, m-u$. If $\mathcal{X}$ is (q^u, q^m)-Frobenius classical w.r.t. a non-degenerated morphism $\phi : \mathcal{X} \longrightarrow \mathbb{P}^n(\mathbb{K})$ defined over $\mathbb{F}_q$, then

$$(n-1)\mathbf{N_u} + (n-1)\mathbf{N_m} + q^u\mathbf{N_{m-u}} \leq (n-1)(n-2)(g-1)$$

$$+(q^m + q^u + n - 1)d,$$

where d is the degree of the linear series $\mathcal{D}$ associated to ϕ.

Remark. $p < d$ is sufficient condition for $\mathcal{X}$ to be r (q^u, q^m)-Frobenius classical.

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Some comparisons

Let $\mathcal{X}$ be a plane curve of genus g and degree d given by $f(x,y)=0$, where $f(x,y)\in\mathbb{F}_q[x,y]$. For $s\in\{1,...,d-3\}$, consider the Veronese morphism.

$$\phi_s=(1:x:y:x^2:...:x^iy^j:...:y^s):\mathcal{X}\longrightarrow\mathbb{P}^M(\mathbb{K}),$$

where $i+j\leq s$.

We know that the linear series $\mathcal{D}_s$ associated to ϕ_s is base-point-free, of degree sd and dimension

$$M=\binom{s+2}{2}-1=(s^2+3s)/2.$$

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Examples

If $\mathcal{X}$ is (q^u, q^m)-Frobenius classical for $\mathcal{D}_s$, then the new result gives us

$$(M-1)\mathbf{N_u} + (M-1)\mathbf{N_m} + q^u\mathbf{N_{m-u}} \leq (M-1)(M-2)(g-1)$$
$$+sd(q^m + q^u + M - 1).$$

If we have (q^u, q^m)-Frobenius classicality for $\mathcal{D}_2$, then the result yields

$$4N_u + 4N_m + q^u N_{m-u} \leq 12(g-1) + 2d(q^m + q^u + 4). \qquad (4)$$

Examples

Example

Let $\mathcal{X}$ *be a curve of degree* 6 *over* $\mathbb{F}_3$ *given by*

$$\sum_{r+s+k=6} x^r y^s z^k = 0.$$

We wil estimate N_3*, the number of* $\mathbb{F}_{27}$*-rationail points of* $\mathcal{X}$*. We use the new bound for* $m = 3$ *e* $u = 1$*. It is known that* $N_1 = 0$ *and* $N_2 = d(d + q^2 - 1)/2 = 42$*. We have*

Bound	$N_3 \leq$
Hasse-Weil	131
Störh-Voloch	96
New bound	60

(5)

Examples

Example

For $p = 7$ *and* $q = p^3 = 343$ *consider the a curva de Fermat*

$$\mathcal{X} : x^{57} + y^{57} = z^{57}$$

over $\mathbb{F}_{343}$. *It is known that* $N_1 = 16416$, *and it can be checked that the curve is* (q, q^2)*-Frobenius classical for* $\mathcal{D}_2$. *Thus we have*

Bound	$N_2 \leq$
Hasse-Weil	1154882
Zeta	1006356
Garcia-Stöhr-Voloch	957233
new bound	152874

. (6)

Using computer, one can check that 152874 *is the actual value*

The end

Thanks!!