Tractability for Analytic Multivariate Problems

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Based on Papers

- **J. Dick, G. Larcher, F. Pillichshammer, H.W.**

  "Exponential convergence and tractability of multivariate integration for Korobov spaces", Math. Comp. 2011.

- **J. Dick, P. Kritzer, F. Pillichshammer, H.W.**

  "Approximation of analytic functions in Korobov spaces", to appear in J. Complexity.

- **P. Kritzer, F. Pillichshammer, H.W.**

  "Multivariate integration of infinitely many times differentiable functions in weighted Korobov spaces", to appear in Math. Comp.

- **P. Kritzer, F. Pillichshammer, H.W.**

  "Tractability of multivariate analytic problems", to appear in Radon Series.

- **P. Kritzer, F. Pillichshammer, H.W.**

  L_∞-approximation of analytic functions in Korobov spaces, in progress.

Setting

$$S_s \; : \; F_s \; \to \; G_s$$

s	**number of variables**
ε	**error demand**
$n(\varepsilon, S_s)$	**minimal number of function values or linear functionals**

Main Question: How does $n(\varepsilon, S_s)$ depend on ε^{-1} and s?

Standard Tractability Notions

- **WT-weak tractability:**

$$\lim_{s+\varepsilon^{-1}\to\infty} \frac{\log n(\varepsilon, S_s)}{s+\varepsilon^{-1}} = 0$$

- **PT-polynomial tractability:** $\exists\ C, q, p \ge 0$

$$n(\varepsilon, S_s) \le C\, s^{\,q}\, \varepsilon^{-p} \qquad \forall\ \varepsilon \in (0,1),\ \ s \in \mathbb{N}.$$

- **SPT- strong polynomial tractability:** $\exists\ C, p \ge 0$

$$n(\varepsilon, S_s) \le C\, \varepsilon^{-p} \qquad \forall\ \varepsilon \in (0,1),\ \ s \in \mathbb{N}.$$

See

Tractability of Multivariate Problems

E. Novak + H.W., Volumes I, II, III, EMS, 2008, 2010, 2012.

Standard Tractability Notions

- **Appropriate for problems of finite smoothness,**
  e.g., for $s = 1$ we have $n(\varepsilon, S_1) = \Theta(\varepsilon^{-p})$
- **Polynomial rates of convergence,**
  e.g., for $s = 1$ the minimal error is $\Theta(n^{-1/p})$ with n function values/linear functionals
- **Many negative results for unweighted spaces,**
  i.e., when all variables are equally important
  (often the curse of dimensionality !!!)
- **Many positive results for weighted spaces,**
  i.e., when successive variables are of diminishing importance

New Tractability Notions: Motivation

- **What about analytic problems or problems of infinite smoothness?**
- **For** $s = 1$ **we then may have** $n(\varepsilon, S_1) = \Theta((1 + \log\, \varepsilon^{-1})^p)$
- **This corresponds to exponential convergence, i.e., the minimal error for** $s = 1$ **is** $\Theta(q^n)$ **for some** $q \in (0, 1)$
- **Should we replace**

$$(\varepsilon^{-1}, s) \qquad \textbf{by} \qquad (1 + \log\, \varepsilon^{-1}, s) \quad ?$$

New Tractability Notions

- **EC-WT- exponential convergence and weak tractability:**

$$\lim_{s+\log\,\varepsilon^{-1}\to\infty} \frac{\log\,n(\varepsilon,S_s)}{s+\log\,\varepsilon^{-1}}=0$$

- **EC-PT-exponential convergence and polynomial tractability:** $\exists\ C,q,p\ge 0$

$$n(\varepsilon,S_s)\le C\,s^{\,q}\,(1+\log\,\varepsilon^{-1})^p \qquad \forall\ \varepsilon\in(0,1),\ \ s\in\mathbb{N}.$$

- **EC-SPT- exponential convergence and strong polynomial tractability:** $\exists\ C,p\ge 0$

$$n(\varepsilon,S_s)\le C\,(1+\log\,\varepsilon^{-1})^p \qquad \forall\ \varepsilon\in(0,1),\ \ s\in\mathbb{N}.$$

Challenge

Which problems $S = \{S_d\}$ satisfy the new tractability notions?

Korobov Space

$$F_s = H(K_s) \quad \textbf{a RKHS}$$

$$K_s(x,y) = \sum_{h\in\mathbb{Z}^s} \omega_h \, \exp\left(2\pi \mathrm{i} \sum_{j=1}^{s} h_j(x_j - y_j)\right) \qquad \forall\, x,y \in [0,1]^s$$

$$\omega_h = \omega^{\sum_{j=1}^{s} a_j |h_j|^{b_j}}$$

where

$$\omega \in (0,1), \qquad 0 < a_1 \le a_2 \le \cdots, \qquad \inf_j b_j > 0$$

Korobov Space

- F_s **is a space of analytic functions**
- **the norm in terms of Fourier coefficients,**

$$\|f\|_{F_s} = \left(\sum_{h \in \mathbb{Z}^s} |\hat{f}(h)|^2 \, \omega_h^{-1} \right)^{1/2} < \infty$$

- **Let** $e_h(x) = \omega_h^{1/2} \exp\left(2\pi \mathrm{i} \sum_{j=1}^s h_j x_j\right)$**.**

  Then $\{e_h\}_{h \in \mathbb{Z}^s}$ **is an orthonormal basis of** F_s

Three Problems $S_s : F_s \to G_s$

- **Integration: INT**

$$S_s(f) = \int_{[0,1]^s} f(x)\,\mathrm{d}x \quad \textbf{with} \quad G_s = \mathbb{R}$$

- **L_2-approximation: APP_2**

$$S_s(f) = f \quad \textbf{with} \quad G_s = L_2([0,1]^s)$$

- **L_∞-approximation: APP_∞**

$$S_s(f) = f \quad \textbf{with} \quad G_s = L_\infty([0,1]^s)$$

Complexity

For $f \in F_s$ we want to approximate $S_s(f) \quad \approx \quad A_n(f)$

- **Algorithms:**

$$\mathbf{\Lambda}^{\rm std}: \quad A_n(f) = \phi_n(f(x_1), f(x_2), \dots, f(x_n)) \ \text{ with } \ x_j \in [0,1]^s$$

or

$$\mathbf{\Lambda}^{\rm all}: \quad A_n(f) = \phi_n(L_1(f), L_2(f), \dots, L_n(f)) \ \text{ with } \ L_j \in F_s^*$$

- **Minimal Worst Case Error:**

$$e(n,s) = \inf_{A_n} \sup_{\|f\|_{F_s} \le 1} \|S_s(f) - A_n(f)\|_{G_s}$$

- **Information Worst Case Complexity:**

$$n(\varepsilon, s) \ = \ \min\{\, n \,|\ \ e(n,s) \ \le \ \varepsilon \,\}$$

Relations between Three Problems

For both $\Lambda^{\rm std}$ and $\Lambda^{\rm all}$:

$$\mathbf{INT} \le \mathbf{APP}_2 \le \mathbf{APP}_\infty$$

Of course, INT is trivial for $\Lambda^{\rm all}$

Results for Integration (obviously now for $\Lambda^{\rm std}$)

$$\omega_h = \omega^{\sum_{j=1}^s a_j |h_j|^{b_j}}$$

- **EXP:** $\exists\, q \in (0,1)$ and functions $C, C_1, p : \mathbb{N} \to (0,\infty)$ such that

$$e(n,s) \le C(s)\, q^{(n/C_1(s))^{p(s)}} \qquad p^*(s) = \sup p(s)$$

- **UEXP:** if $p(s) = p$ for all $s \in \mathbb{N}$ and $p^* = \sup p$

Then

- **EXP always holds and** $p^*(s) = 1/\left(\sum_{j=1}^s b_j^{-1}\right)$
- **UEXP holds iff** $B := \sum_{j=1}^\infty b_j^{-1} < \infty$**. If so** $p^* = 1/B$**.**

Results for Integration

For standard notions of tractability (only sufficient conditions):

$$A := \lim_{j \to \infty} \frac{a_j}{\log\, j}$$

- **SPT, i.e.,** $n(\varepsilon, S_s) \le C\,\varepsilon^{-p}$**,**

$$A > \frac{1}{\log\, \omega^{-1}} \qquad \Longrightarrow \qquad \text{SPT}$$

- **PT, i.e.,** $n(\varepsilon, S_s) \le C\, s^{\,q}\, \varepsilon^{-p}$**,**

$$\frac{a_j}{\log\, j} \ge \frac{1}{\log\, \omega^{-1}} \quad \text{for large } j \qquad \Longrightarrow \qquad \text{PT}$$

- **WT, i.e.,** $\lim_{s+\varepsilon^{-1}\to\infty} (s+\varepsilon^{-1})^{-1} \log\, n(\varepsilon, S_s) = 0$**,**

$$\lim_{j \to \infty} a_j = \infty \qquad \Longrightarrow \qquad \text{WT}$$

Results for Integration

For new notions of tractability:

- **EC-SPT, i.e.,** $n(\varepsilon, S_s) \le C\,(1+\log\,\varepsilon^{-1})^p$,

$$\textbf{EC-SPT} \quad \textbf{iff} \quad B := \sum_{j=1}^{\infty} b_j^{-1} < \infty \quad \textbf{and} \quad \alpha^* = \liminf_{j\to\infty} \frac{\log\, a_j}{j} > 0$$

If so then $p \in \left[B, B + \min\left(B, \frac{\log\, 3}{\alpha^*}\right)\right]$

- **PT, i.e.,** $n(\varepsilon, S_s) \le C\,s^{\,q}\,(1+\log\,\varepsilon^{-1})^p$,

$$\textbf{EC-PT} \quad \textbf{iff} \quad \textbf{EC-SPT}$$

- **EC-WT, i.e.,** $\lim_{s+\log\,\varepsilon^{-1}\to\infty}(s+\log\,\varepsilon^{-1})^{-1}\ln\, n(\varepsilon, S_s) = 0$,

$$\textbf{EC-WT} \quad \textbf{iff} \quad \lim_{j\to\infty} a_j = \infty$$

Results for L_2-approximation

For both $\Lambda^{\rm std}$ and $\Lambda^{\rm all}$:

EXP and UEXP as for integration, i.e.,

- **EXP always holds and** $p^*(s) = 1/\left(\sum_{j=1}^{s} b_j^{-1}\right)$
- **UEXP holds iff** $B := \sum_{j=1}^{\infty} b_j^{-1} < \infty$ **and then** $p^* = 1/B$**.**

Results for L_2-approximation

For standard notions of tractability and Λ^{all}:

Let

$$A := \lim_{j \to \infty} \frac{a_j}{\log\, j}$$

- **SPT iff** $A > 0$. **If so then** $n(\varepsilon, S_s) \le C\, \varepsilon^{-p}$ **with**

$$p = \frac{2}{A\, \log\, \omega^{-1}}$$

- **PT iff SPT**
- **WT always holds**

Results for L_2-approximation

For standard notions of tractability and Λ^{std} (only sufficient conditions): Let

$$A := \lim_{j\to\infty} \frac{a_j}{\log\, j}$$

- $A > \frac{1}{\log\, \omega^{-1}} \quad \Longrightarrow \quad$ **SPT.** **If so then** $n(\varepsilon, S_s) \le C\,\varepsilon^{-p}$ **with**

$$p \le \frac{2}{A\, \log\, \omega^{-1}} \left(1 + \frac{1}{A\, \log\, \omega^{-1}}\right)$$

- $\frac{a_j}{\log\, a_j} \ge \frac{1}{\log\, \omega^{-1}}$ **for large** $j \quad \Longrightarrow \quad$ **PT**
- $\lim_{j\to\infty} a_j = \infty \quad \Longrightarrow \quad$ **WT**

Results for L_2-approximation

For new notions of tractability and both $\Lambda^{\rm std}$ and $\Lambda^{\rm all}$:

$$\mathbf{APP}_2 \quad \equiv \quad \mathbf{INT},\ \textit{i.e.},$$

- **EC-SPT iff** $B := \sum_{j=1}^\infty b_j^{-1} < \infty$ **and** $\alpha^* = \liminf_{j\to\infty} \frac{\log\, a_j}{j} > 0$.

  If so then $n(\varepsilon, S_s) \le C\,(1+\varepsilon^{-1})^p$ **with** $p \in \left[B, B + \min\left(B, \frac{\log\, 3}{\alpha^*}\right)\right]$
- **EC-PT iff EC-SPT**
- **EC-WT iff** $\lim_{j\to\infty} a_j = \infty$

Results for L_∞-approximation

... work in progress

For new notions of tractability we have

$$\mathbf{APP}_\infty \quad \equiv \quad \mathbf{APP}_2$$

for both Λ^{std} and Λ^{all}.

Why?

Let $\{\lambda_j\}$ be the ordered eigenvalues of $W_s = S_s^* S_s$ for $S_s = \mathrm{APP}_2$.

Then the nth **minimal error for $\mathbf{APP}_2$ is related to λ_{n+1} whereas for $\mathbf{APP}_\infty$ to $\sum_{j=n+1}^\infty \lambda_j$, Due to exponential convergence**

$$\lambda_{n+1} \sim \sum_{j=n+1}^\infty \lambda_j$$