Discrepancy estimates for sequences: New results and open problems

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Some recent results

Some concrete open problems

- Lower bounds for the discrepancy of certain classes of sequences
- Discrepancy of Halton-Kronecker sequences
- Best lower bounds for star-discrepancy of sequences in $[0, 1)$

I. Lower bounds for the discrepancy of certain classes of sequences

big open problem:

find best lower general bound for discrepancy of sequences (finite point sets) in $[0,1)^s$.

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Conjectures:

- $\forall\, x_1, \ldots, x_N \in [0,1)^s$

$$D_N \geq c_s \cdot \frac{(\log N)^{s-1}}{N}$$

- $\forall\, x_1, x_2, \ldots \in [0,1)^s$

$$D_N \geq c_s \cdot \frac{(\log N)^{s}}{N}$$

for infinitely many N.

most important contributions:

- K.F.Roth 1954:
  $\forall\, x_1, x_2, \ldots$ in $[0,1)^s$

$$D_N \geq c_s \cdot \frac{(\log N)^{s/2}}{N}$$

  for infinitely many N.

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  $\forall\, x_1, x_2, \ldots$ in $[0,1)^s$

$$D_N \geq c_s \cdot \frac{(\log N)^{s/2}}{N}$$

  for infinitely many N.
- W.M.Schmidt 1972:
  Conjecture for sequences is true for $s = 1$

most important contributions:

- J.Beck 1989:
  First improvement of result of Roth by some $\log\log N$ factor

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- J.Beck 1989:
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- D.Bilyk, M.Lacey and A. Vagharshakyan 2008:
  $\forall\, x_1, x_2, \ldots$ in $[0,1)^s$

$$D_N \geq c_s \cdot \frac{(\log N)^{s/2+\delta(s)}}{N}$$

  for some small $\delta(s) > 0$

Why believe that always

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for infinitely many N?

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$\Rightarrow$ Know many examples of concrete sequences (low-discrepancy sequences) with

$$D_N \leq c_s' \cdot \frac{(\log N)^s}{N}$$

for all N.

None better! (of course)

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for $q \geq 2$

$$n = n_0 + n_1 q + \ldots + n_r q^r$$
$$\phi_q(n) := \frac{n_0}{q} + \frac{n_1}{q^2} + \ldots + \frac{n_r}{q^{r+1}}$$

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$$D_N = O\left(\frac{(\log N)^2}{N}\right)$$

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no lower bound better than general lower bound of Bilyk and Lacey

$$D_N = \Omega\left(\frac{(\log N)^{1+\delta(2)}}{N}\right)$$

Open Problem 1

Give improved discrepancy bounds for the two-dimensional Halton sequence, i.e., improve

$$D_N = \Omega\left(\frac{(\log N)^{1+\delta(2)}}{N}\right)$$

$$\left(\text{or} \quad D_N = O\left(\frac{(\log N)^2}{N}\right)\right).$$

Or

2-dimensional Kronecker sequence

$$(\{n\alpha\}, \{n\beta\})_{n \geq 0}$$

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$$(\{n\alpha\}, \{n\beta\})_{n \geq 0}$$

$$D_N = \Omega\left(\frac{(\log N)^2}{N}\right) \quad \text{for all} \quad (\alpha, \beta)?$$

$$D_N = O\left(\frac{(\log N)^2}{N}\right) \quad \text{for some} \quad (\alpha, \beta)?$$

Metric behaviour of discrepancy of Kronecker sequences
Beck, 1994 (Annals of Mathematics)

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For almost all $(\alpha_1, \ldots, \alpha_s) \in [0,1)^s$ for discrepancy of $(\{n\alpha_1\}, \ldots, \{n\alpha_s\})_{n \geq 0}$:

$$D_N = O\left(\frac{(\log N)^s \cdot g(\log \log N)}{N}\right)$$

iff $\sum_{n=1}^{\infty} \frac{1}{g(n)}$ converges e.g.: For almost all $(\alpha_1, \ldots, \alpha_s)$:

$$D_N = O\left(\frac{(\log N)^s \cdot (\log\log N)^{1+\epsilon}}{N}\right)$$

and

$$D_N = \Omega\left(\frac{(\log N)^s \cdot \log\log N}{N}\right)$$

Third main source for low-discrepancy sequences:
$\rightarrow$ digital sequences in sense of Niederreiter

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$\rightarrow$ digital sequences in sense of Niederreiter

- generate $\boldsymbol{x}_n = (x_n^{(1)}, \ldots, x_n^{(s)}) \in [0,1)^s$
- need $\mathbb{N} \times \mathbb{N}$ matrices $C_1, \ldots, C_s$ over $\mathbb{Z}_q, q$ prime

- $n = n_0 + n_1 q + \ldots n_r q^r$

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$\rightarrow$ $x_n^{(i)} := \frac{y_0^{(i)}}{q} + \frac{y_1^{(i)}}{q^2} + \dots \in [0,1)$

$\boldsymbol{x}_n = (x_0^{(1)}, \dots, x_n^{(s)}) \in [0,1)^s$

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$\rightarrow$ many methods to find such $C_1, \ldots, C_s$

Sobol, Faure, Niederreiter, Niederreiter-Xing,
Niederreiter-Hofer, ...
$\rightarrow$ small O-constant

L., 1998 metric discrepancy bound

For almost all $C_1, \ldots, C_s$

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lower bound?

Theorem 1 (L., Pillichshammer, 2013)

For almost all choices of $C_1, \ldots, C_s$

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method:

- analogous to method of Beck
- but adapted to Walsh series
- and auxiliary results from non-archimedean diophantine approximation

$\rightarrow$ Consider finite versions of

- Kronecker sequences $\rightarrow$ good lattice-point sets
- digital sequences $\rightarrow$ digital (t, m, s)-nets

(good) lattice -point set:

$$x_n = \left(\left\{n \cdot \frac{a_1}{N}\right\}, \ldots, \left\{n \cdot \frac{a_s}{N}\right\}\right); n = 0, 1, \ldots, N-1$$

($a_1, \ldots, a_s$ given integers)

Hlawka, Korobov, Niederreiter:

For "almost all" choices of $a_1, \dots, a_s$

$$D_N = O\left(\frac{(\log N)^s}{N}\right)$$

(Note: For **finite** point sets $D_N = O\left(\frac{(\log N)^{s-1}}{N}\right)$ probably best possible order)

Improved for $s = 2$, L., 1986:

For "almost all" choices of a_1, a_2

$$D_N \leq c \cdot \frac{\log N \cdot \log \log N}{N}$$

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Generalised to arbitrary dimension, Bykovskii, 2012

For "almost all" choices of $a_1, a_2, \ldots, a_s$

$$D_N \leq c \cdot \frac{(\log N)^{s-1} \cdot \log\log N}{N}$$

Remarks

2-dimensional result equivalent to:

For every $N \in \mathbb{N}$, for "almost all" $a \in \{1, \ldots, N-1\}$ with $(a, N) = 1$, $\frac{a}{N} = [0; a_1, a_2, \ldots, a_r]$:

$$\sum_{k=1}^{r} a_k \leq c \cdot \log N \cdot \log \log N$$

$\rightarrow$ connections to conjecture of Zaremba:

Conjecture of Zaremba:

$\exists c$, such that for all $N \in \mathbb{N}$ there **exists** $a \in \{1, \ldots, N-1\}$ with $(a, N) = 1$, $\frac{a}{N} = [0; a_1, \ldots, a_r]$:

$$a_k \leq c \quad \text{for all } k.$$

$\rightarrow$ recent essential progress by Bourgain and Kontorovich

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$$\Rightarrow \qquad \sum_{k=1}^{r} a_k \leq c \cdot \log N$$

$$\Rightarrow \qquad D_N \leq c' \cdot \frac{\log N}{N}$$

for 2-dim. lattice point set

General result of Bykovskii:

For “almost all” choices of $a_1, a_2, \ldots, a_s$

$$D_N \leq c \cdot \frac{(\log N)^{s-1} \cdot \log \log N}{N}$$

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Best possible?
lower bound?
adaption of method of Beck used for Kronecker sequences?

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Best possible?
lower bound?
adaption of method of Beck used for Kronecker sequences?

In principle yes!
Seems much more difficult to give necessary auxiliary results now from “discrete diophantine approximation”

Open Problem 2

Give a "metric" (average type) lower bound for the discrepancy of good lattice point sets, thereby completing the result of Bykovskii.

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Open Problem 3

Give a "metric" (average type) lower bound for the discrepancy of digital (t, m, s)-nets, thereby completing a result (upper "metric" bound) of L. from 1996

II. Discrepancy of Halton-Kronecker sequences

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Halton sequence:

$$q \quad \text{prime}$$

$$n = n_0 + n_1 q + \ldots + n_r \cdot q^r$$

Then

$$\phi_q(n) := \frac{n_0}{q} + \frac{n_1}{q^2} + \ldots + \frac{n_r}{q^{r+1}}$$

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Then

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$q_1, \ldots, q_s$ different primes

$$x_n := (\phi_{q_1}(n), \ldots, \phi_{q_s}(n)) \in [0,1)^s$$

$(x_n)_{n \geq 0}$ Halton sequence in $[0,1)^s$

Kronecker sequence:

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_t) \in \mathbb{R}^t$$

Then

$$y_n := (\{n\alpha_1\}, \ldots, \{n\alpha_t\}) \in [0,1)^t$$

$(y_n)_{n \geq 0}$ Kronecker sequence in $[0,1)^t$

Consequently Halton-Kronecker sequence:

$$z_n := (x_n, y_n) \in [0,1)^{s+t}$$

$\nearrow$ Halton in $[0,1)^s$ $\quad$ $\nwarrow$ Kronecker in $[0,1)^t$

$(z_n)_{n\geq 0}$ Halton-Kronecker sequence in $[0,1)^{s+t}$

- Halton sequence:
  - uniformly distributed in $[0,1)^s$
  - low-discrepancy sequence, i.e., $D_N = O\left(\frac{(\log N)^s}{N}\right)$

- Kronecker sequence:
  - uniformly distributed in $[0,1)^t$ iff $1, \alpha_1, \dots, \alpha_t$ linearly independent over $\mathbb{Z}$
  - For almost all $\boldsymbol{\alpha}$ we have $D_N = O\left(\frac{(\log N)^{t+\epsilon}}{N}\right)$ for all $\epsilon > 0$.

J.Beck: *Probabilistic diophantine approximation, I. Kronecker-sequences*. Annals of Mathematics 140, 451–502 (1994)

(earlier result of W.M.Schmidt: $O\left(\frac{(\log N)^{t+1+\epsilon}}{N}\right)$)

Halton-Kronecker sequences

$$z_n = (\underbrace{x_n}_{\substack{s\text{-dim.}\\ \text{Halton}}}, \underbrace{y_n}_{\substack{t\text{-dim.}\\ \text{Kronecker}}}); \quad n = 0, 1, 2, \dots$$

- uniform distribution?
- discrepancy?

Niederreiter:

- Halton-Kronecker sequence $z_n = (x_n, y_n)$ uniformly distributed iff the Kronecker part (y_n) is uniformly distributed
- discrepancy estimates for z_n in dependence of simultaneous approximation properties of $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_t)$

Metrical discrepancy estimates?

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Not at all clear that combination of two **low-discrepancy** sequences gives a **low-discrepancy** sequence e.g.:

combination of

Halton sequence	and	low-discrepancy **digital sequence** in sense of Niederreiter

usually **not** low-discrepancy ($D_N \sim \frac{1}{N^\gamma}$ with $\gamma < 1$)

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usually **not** low-discrepancy ($D_N \sim \frac{1}{N^\gamma}$ with $\gamma < 1$)

follows from general version of **phenomenon of Newman**:

$$\# \{0 \leq n < N \mid S_2(3n) \text{ even} \} > \frac{N}{2} + c \cdot N^{\frac{\log 3}{\log 4}}$$

for some $c > 0$ and all $N > N_0$.

Metrical discrepancy estimates:

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Theorem 2 (L., 2013)

For almost all $\alpha \in [0,1)^t$ *we have for the discrepancy* D_N *of the* $(s+t)$*-dimensional Halton-Kronecker sequence*

$$D_N = O\left(\frac{(\log N)^{s+t+\epsilon}}{N}\right)$$

for every $\epsilon > 0$.

Metrical discrepancy estimates:

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for every $\epsilon > 0$.

- Most probably essentially best possible order
- $s = 0 \Rightarrow$ result of Beck for pure Kronecker sequence

Method: partly by using, adapting, generalising method of Beck in probabilistic diophantine approximation

For example:

- Beck:
  For almost all $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_t)$ we have

$$\#\{\boldsymbol{n} = (n_1, \ldots, n_t) | V \leq \boldsymbol{n} \leq W, \{n_1\alpha_1 + \ldots + n_t\alpha_t\} \leq g(\boldsymbol{n})\} =$$
$$= \sum_{V \leq \boldsymbol{n} \leq W} g(\boldsymbol{n}) + R$$

  with "small error R".

- For Halton-Kronecker sequences:
  For almost all $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_t)$ we have

$$\sum_{\boldsymbol{j}=(j_1,\ldots,j_s)\in J} \#\{\boldsymbol{n} \mid V_{\boldsymbol{j}} \le \boldsymbol{n} \le W_j,$$

$$\{q_1^{j_1} \cdot \ldots \cdot q_s^{j_s}(n_1\alpha_1 + \ldots + n_s\alpha_s)\} \le g_{\boldsymbol{j}}(\boldsymbol{n})\} =$$

$$= \sum_{\boldsymbol{j}\in J} \sum_{V_{\boldsymbol{j}} \le \boldsymbol{n} \le W_{\boldsymbol{j}}} g_{\boldsymbol{j}}(\boldsymbol{n}) + R$$

  with "small error R".

Of course, like for Kronecker sequences

- For $t \geq 2$ no **concrete** α is known such that

$$D_N = O\left(\frac{(\log N)^{s+t+\epsilon}}{N}\right)$$

- For $t \geq 2$ it is not known if there exist α such that

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Even for $\boldsymbol{t = 1, s = 1}$ this is **not known** in the case of Halton-Kronecker sequences!!

$$z_n = (\underbrace{\phi_2(n)}_{\substack{\text{van der} \\ \text{Corput} \\ \text{sequence}}}, \underbrace{\{n\alpha\}}_{\substack{1\text{-dim.} \\ \text{Kronecker} \\ \text{sequence}}})$$

candidates: α with bounded continued fraction coefficients

Open Problem 4

Does the discrepancy of

$$z_n = (\phi_2(n), \{n \cdot \sqrt{2}\}); \quad n = 1, 2, \ldots$$

satisfy

$$D_N = O\left(\frac{(\log N)^{2+\epsilon}}{N}\right)?$$

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satisfy

$$D_N = O\left(\frac{(\log N)^{2+\epsilon}}{N}\right)?$$

Needs: Investigation of behaviour of continued fractions of $2^K \cdot \sqrt{2}$ for $K = 1, 2, 3, \ldots$

Is $$\sum_{K=1}^{\log N} \sum_{e=1}^{\log \frac{N}{2^K}} a_e(2^K \cdot \sqrt{2}) = O\left((\log N)^{2+\epsilon}\right)?$$

III. Best lower bounds for star-discrepancy of sequences in $[0,1)$

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W.M.Schmidt, 1972

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$$D_N^* \geq \frac{1}{100} \cdot \frac{\log N}{N}$$

for infinitely many N.

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For all sequences $(x_n)_{n \geq 0}$ in $[0,1)$

$$D_N^* \geq \frac{1}{100} \cdot \frac{\log N}{N}$$

for infinitely many N.

order $\frac{\log N}{N}$ best possible

Open question:

c^* maximal such that for all $(x_n)_{n \geq 0}$:

$$D_N^* \geq c^* \cdot \frac{\log N}{N}$$

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Best known bounds:

$0.06015\ldots$	$\leq c^* \leq$	$0.222\ldots$
Bejian, 1982		Ostromoukhov, 2009 (Faure)

Proof of Bejian:

- rather technical
- old

goals:

→ simplify

→ improve

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use and refine a method of Liardet, 1979; Tijdeman, Wagner, 1980

Method:

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- $x_1, x_2, \ldots, x_N \in [0,1)$
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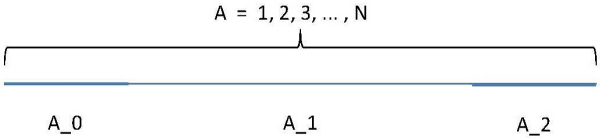
Method:

- $x_1, x_2, \ldots, x_N \in [0,1)$
- $N = a^t$, some $a > 2, a \in \mathbb{R}; t \in \mathbb{N}$
- $D_n(x) := \#\{i \leq n \,|x_i < x\} - n \cdot x$
- Divide $A = 1, 2, \ldots, N$ in 3 parts

$$A_0 = 1, 2, \ldots, a^{t-1}$$
$$A_1 = a^{t-1} + 1, \ldots, a^t - a^{t-1}$$
$$A_2 = a^t - a^{t-1} + 1, \ldots, a^t$$



Consider

$$P(t) := \int_0^1 \left(\max_{n \in A} D_n(x) - \min_{n \in A} D_n(x) \right) dx$$

Use simple general inequality:

- A any set
- $A_0, A_2 \subseteq A$
- $f : A \to \mathbb{R}$

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$$
\begin{aligned}
&\Rightarrow \max_{y \in A} f(y) - \min_{y \in A} f(y) \geq \\
&\geq \frac{1}{2}\left[\left(\max_{y \in A_2} f(y) - \min_{y \in A_2} f(y)\right) + \left(\max_{y \in A_0} f(y) - \min_{y \in A_0} f(y)\right)\right. \\
&\left. + \left|\max_{y \in A_2} f(y) - \max_{y \in A_0} f(y)\right| + \left|\min_{y \in A_2} f(y) - \min_{y \in A_0} f(y)\right|\right]
\end{aligned}
$$

Apply to

$$\begin{aligned} f : & A \to \mathbb{R} \\ & n \to f(n) := D_n(x) \end{aligned}$$

$\Rightarrow$ essentially

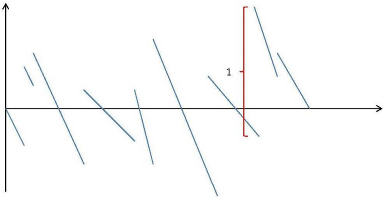
$$P(t) \geq P(t-1) + \int_0^1 \Bigg| \underbrace{\max_{n \in A_2} D_n(x) - \max_{n \in A_0} D_n(x)}_{=: g_t(x)} \Bigg| dx$$

Discuss function $g_t(x)$ on $[0, 1]$:

$$P(t) \geq P(t-1) + \int_0^1 |g_t(x)|\, dx$$

$g_t(x)$:

- piecewise linear with negative slopes
- $\leq a^t$ discontinuities
- $-a^t \leq$ slope $\leq -a^{t-1}(a-2)$ always
- $a^{t-1}(a-2)$ jumps of height 1
- $g_t(0) = g_t(1) = 0$



something like that

$$P(t) \geq P(t-1) + \int_0^1 |g_t(x)|\, dx$$

elementary calculus:
determine g_t^* such that

$$\min_{g_t} \int_0^t |g_t(x)|\, dx = \int_0^t |g_t^*(x)|\, dx =: m_t$$

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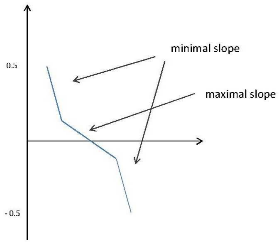
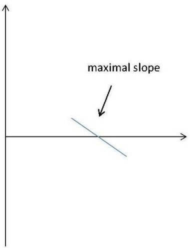
$$\min_{g_t} \int_0^t |g_t(x)|\, dx = \int_0^t |g_t^*(x)|\, dx =: m_t$$

combination of parts



and



minimal and maximal slopes

$$\Rightarrow P(t) \geq P(t-1) + m_t$$

$$\Rightarrow P(t) \geq m_t + m_{t-1} + \ldots + m_2 + m_1 = \chi_a \cdot t$$

$$\underset{\substack{a \\ a \sim 3.81}}{\overrightarrow{\max}} \quad \chi \cdot t$$

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$$\Rightarrow P(t) \geq m_t + m_{t-1} + \ldots + m_2 + m_1 = \chi_a \cdot t$$

$$\underset{\substack{a \\ a\sim 3.81}}{\overrightarrow{\max}} \quad \chi \cdot t$$

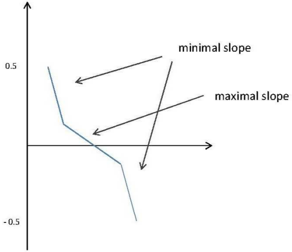
i.e.:

$$\int_0^1 \left(\max_{n=1,\ldots,N} D_n(x) - \min_{n=1,\ldots,N} D_n(x) \right) dx \geq \chi \cdot \frac{\log N}{\log a}$$

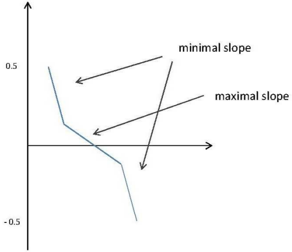
$\Rightarrow \quad \exists\, n \leq N, x \in [0,1)$ with

$$|D_n(x)| \geq \frac{\chi}{2 \cdot \log a} \cdot \log N \geq \underbrace{0.06015 \cdot \log n}_{\text{Bejian}}$$

Further discussion of $g_t(x)$ shows, that $g_t(x)$ **cannot** contain parts of form



Further discussion of $g_t(x)$ shows, that $g_t(x)$ **cannot** contain parts of form



$\Rightarrow$ further discussion of g_t^* under additional constraint leads to (not yet published)

$$c^* \geq 0.06458\ldots$$

(instead of Bejian's $0.06015\ldots$)

More discussion of $g_t(x)$ **and** $g_{t-1}(x)$

$$\begin{aligned} P(t) \geq & P(t-1) + \int_0^1 |g_t(x)|\, dx \geq \\ & P(t-2) + \int_0^1 |g_t(x)|\, dx + \int_0^1 |g_{t-1}(x)|\, dx \end{aligned}$$

More discussion of $g_t(x)$ **and** $g_{t-1}(x)$

$$P(t) \geq P(t-1) + \int_0^1 |g_t(x)|\, dx \geq$$
$$P(t-2) + \int_0^1 |g_t(x)|\, dx + \int_0^1 |g_{t-1}(x)|\, dx$$

$\Rightarrow$ it is not possible that

$$g_t = g_t^* \quad \textbf{and} \quad g_{t-1} = g_{t-1}^*$$

More discussion of $g_t(x)$ **and** $g_{t-1}(x)$

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$\Rightarrow$ it is not possible that

$$g_t = g_t^* \quad \textbf{and} \quad g_{t-1} = g_{t-1}^*$$

additional condition of form:

"if slope of g_{t-1} is in $[\alpha, \beta]$ then slope of g_t is in $[\gamma, \delta]$"

$\rightarrow$ result $c^* \geq 0.06458\ldots$ can be further improved

→ result $c^* \geq 0.06458\ldots$ can be further improved

→ best way:
find $\widehat{g}^*_{t-1}, \widehat{g}^*_t$ **admissible** such that
$\int_0^1 |\widehat{g}^*_t(x)|\, dx + \int_0^1 \left|\widehat{g}^*_{t-1}(x)\right| dx$
is minimal

$\rightarrow$ result $c^* \geq 0.06458\ldots$ can be further improved

$\rightarrow$ best way:
find $\widehat{g}^*_{t-1}, \widehat{g}^*_t$ **admissible** such that
$\int_0^1 |\widehat{g}^*_t(x)|\,dx + \int_0^1 \left|\widehat{g}^*_{t-1}(x)\right|\,dx$
is minimal

$\Rightarrow$ should be possible by elementary calculation, but cumbersome

$\Rightarrow$ should give $c^* \geq 0.07\ldots$

Open Problem 5

Improve the above estimate for c^* for example by the method indicated above