The hybrid spectral test

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Context: MC and QMC Methods

The underlying space
$\mathbb{R}^s/\mathbb{Z}^s$ ... s-dimensional torus, identified with $[0,1)^s$, $s \geq 1$
λ_s ... Haar measure on $[0,1)^s$ (s-dimensional Lebesgue measure)

The object of interest
$\omega = (\mathbf{x}_n)_{n\geq 0}$ a given sequence in $[0,1)^s$,
and $f: [0,1)^s \to \mathbb{C}$ Riemann-integrable.

We are interested in the difference between the sample means

$$S_N(f,\omega) = \frac{1}{N}\sum_{n=0}^{N-1} f(\mathbf{x}_n), \quad N \in \mathbb{N},$$

and the integral

$$I(f) = \int_{[0,1)^s} f \, d\lambda_s.$$

Notation

Intervals in $[0,1)^s$
Arbitrary (aligned) intervals and intervals anchored at the origin:

$$\mathcal{J} = \{\prod_{i=1}^{s}[u_i, v_i[: \ 0 \leq u_i < v_i \leq 1\},$$
$$\mathcal{J}^* = \{\prod_{i=1}^{s}[0, v_i[: \ 0 < v_i \leq 1\}.$$

For $J \in \mathcal{J}$, write

$$\mathbf{1}_J \ \ldots \text{the indicator function of } J$$

Remark

$$\begin{aligned} S_N(\mathbf{1}_J, \omega) - \lambda_s(J) &= \frac{1}{N}(\text{number of hits in } J) - \text{volume}(J) \\ &= S_N(\mathbf{1}_J - \lambda_s(J), \omega). \end{aligned}$$

Uniform distribution of sequences

Uniform distribution in $[0,1)^s$
$\omega = (\mathbf{x}_n)_{n\geq 0}$ uniformly distributed in $[0,1)^s$ (u.d.), if

$$\forall J \in \mathcal{J}: \qquad \lim_{N\to\infty} S_N(\mathbf{1}_J - \lambda_s(J), \omega) \; = \; 0.$$

Discrepancy
Extreme discrepancy of the first N points of ω:

$$D_N(\omega) \; = \; \sup_{J\in\mathcal{J}} \left| S_N(\mathbf{1}_J - \lambda_s(J), \omega) \right|,$$

Star discrepancy of the first N points of ω:

$$D_N^*(\omega) \; = \; \sup_{J\in\mathcal{J}^*} \left| S_N(\mathbf{1}_J - \lambda_s(J), \omega) \right|.$$

Examples of sequences

Example: Kronecker sequences and GLPs
$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_s) \in \mathbb{R}^s$
$\omega = (\{n\boldsymbol{\alpha}\})_{n \geq 0}$ ... sequence of fractional parts of the vector $n\boldsymbol{\alpha}$
If $\boldsymbol{\alpha} = \mathbf{g}/N$, $\mathbf{g} \in \mathbb{Z}^s$: good lattice points and lattice rules.

Digital sequences in base b
$b \geq 2$ integer
$n \in \mathbb{N}_0$: $\vec{n}$ denotes digit vector of n in base b
$\omega = ((\varphi_b(C_1 \vec{n}^T), \ldots, \varphi_b(C_s \vec{n}^T)))_{n \geq 0}$, with suitably chosen matrices $C_1, \ldots, C_s$ and some output function φ_b with values in $[0, 1)$.

Van der Corput sequence in base b
$\omega = (\sum_{j \geq 0} n_j b^{-j-1})_{n \geq 0}$, where $n = \sum_{j \geq 0} n_j b^j$ b-adic expansion

Halton sequence in base $\mathbf{b} = (b_1, \ldots, b_s)$
Van der Corput sequence in base b_i in every coordinate.

Methodology: a duality principle

Questions

- How to prove u.d. of a given sequence ω?
- How to measure the u.d. of a given sequence ω?

A duality principle
If we employ the compact group G to construct the sequence $\omega = (\mathbf{x}_n)_{n\geq 0}$ in $[0,1)^s$, the dual group $\hat{G}$ provides a suitable function system $\mathcal{F} = \{\xi_{\mathbf{k}}\}$ on $[0,1)^s$ for the analysis of ω.

G	Type of addition	$\mathcal{F}$
$[0,1)^s$	add. mod 1	trigonometric functions
$\mathbb{F}_p^m$	add. without carry	Walsh functions in base p
$\mathbb{Z}_b$	add. with carry	b-adic function system

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The b-adic integers

The b-adic representation
Let $b \geq 2$, $b \in \mathbb{Z}$.
For $k \in \mathbb{N}_0$, let

$$k = \sum_{j \geq 0} k_j b^j, \quad k_j \in \{0, 1, \ldots, b-1\}.$$

For $x \in [0, 1)$, let

$$x = \sum_{j \geq 0} x_j b^{-j-1}, \quad x_j \in \{0, 1, \ldots, b-1\}.$$

The representation of x is unique if $x_j \neq b-1$ for infinitely many j.

The b-adic integers $\mathbb{Z}_b$

$$\mathbb{Z}_b = \left\{ z = \sum_{j \geq 0} z_j b^j, \quad \text{with digits } z_j \in \{0, 1, \ldots, b-1\} \right\}.$$

Remark
$\mathbb{Z}$ is embedded in $\mathbb{Z}_b$.
For example, $-1 = (b-1) + (b-1)b + (b-1)b^2 + \ldots$

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Monna's map

Monna's map φ_b in base b

$$\varphi_b : \quad \mathbb{Z}_b \to [0,1),$$
$$\varphi_b\Big(\sum_{j\geq 0} z_j\, b^j\Big) \;=\; \sum_{j\geq 0} z_j\, b^{-j-1} \pmod 1.$$

φ_b is continuous and surjective, but not injective.

Notation

Let

$$\mathrm{e}(y) \;=\; \mathrm{e}^{2\pi i y}, \quad y \in \mathbb{R}.$$

Dual group $\hat{\mathbb{Z}}_b$ enumerated

Enumerate the characters as follows:

$$\hat{\mathbb{Z}}_b \;=\; \{z \mapsto \underbrace{e\Big(\varphi_b(k)(z_0 + z_1 b + z_2 b^2 + \cdots)\Big)}_{\chi_k(z)} : \; k \in \mathbb{N}_0\},$$

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Pseudo-inverse of Monna map

Pseudo-inverse φ_b^+ of Monna's map

$$\varphi_b^+ : \quad [0,1) \to \mathbb{Z}_b,$$
$$\varphi_b^+(x) = \sum_{j \geq 0} x_j\, b^j, \quad \left(x = \sum_{j\geq 0} x_j\, b^{-j-1}, \text{ with } x_j \neq b-1 \text{ infinitely often}\right)$$

The b-adic function system

$k \in \mathbb{N}_0$

$$\gamma_k : [0,1) \to \{c \in \mathbb{C} : |c| = 1\},$$
$$\gamma_k(x) = \chi_k(\varphi_b^+(x)).$$

$\Gamma_b = \{\gamma_k : k \in \mathbb{N}_0\}$ ... the b-adic function system on $[0,1)$

$\mathbf{b} = (b_1, \ldots, b_s)$, with not necessarily distinct integers $b_i \geq 2$, $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$,

$$\gamma_{\mathbf{k}} : [0,1)^s \to \{c \in \mathbb{C} : |c| = 1\},$$
$$\gamma_{\mathbf{k}}(\mathbf{x}) = \prod_{i=1}^{s} \gamma_{k_i}(x_i), \quad \gamma_{k_i} \in \Gamma_{b_i}.$$

$\Gamma_{\mathbf{b}}^{(s)} = \{\gamma_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}_0^s\}$ ... the $\mathbf{b}$-adic function system on $[0,1)^s$

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$\Gamma_{\mathbf{b}}^{(s)} = \{\gamma_{\mathbf{k}} : \; \mathbf{k} \in \mathbb{N}_0^s\}$ ... the **b**-adic function system on $[0,1)^s$

b-adic Weyl criterion

ONB property
$\Gamma_{\mathbf{b}}^{(s)}$ is an ONB of $L^2([0,1)^s)$.

b-adic Weyl Criterion
$\mathbf{b} = (b_1, \ldots, b_s)$ a vector of s not necessarily distinct integers $b_i \geq 2$
Then ω u.d. in $[0,1)^s$ if and only if

$$\forall \mathbf{k} \neq \mathbf{0} : \quad \lim_{N\to\infty} \underbrace{\frac{1}{N} \sum_{n=0}^{N-1} \gamma_{\mathbf{k}}(\mathbf{x}_n)}_{S_N(\gamma_{\mathbf{k}},\omega)} \;=\; 0.$$

Remark
The Halton sequence in base **b**, with coprime b_i, $1 \leq i \leq s$, has the form

$$\omega = (\varphi_{\mathbf{b}}(n))_{n\geq 0}, \quad \varphi_{\mathbf{b}}(n) = (\varphi_{b_1}(n), \ldots, \varphi_{b_s}(n)).$$

Therefore,

$$S_N(\gamma_{\mathbf{k}},\omega) = \frac{1}{N}\frac{C^N - 1}{C - 1}, \quad C = C(\mathbf{k}) = \mathrm{e}(\varphi_{b_1}(k_1) + \cdots + \varphi_{b_s}(k_s)).$$

Corollary
The Halton sequence in base **b** is uniformly distributed modulo one.

b-adic Inequality of Erdös-Turán-Koksma

Theorem

ω a sequence in $[0,1)^s$,
$\mathbf{b} = (b_1, \ldots, b_s) \in \mathbb{N}^s$, $b_i \geq 2$, not necessarily distinct,
$\mathbf{g} = (g_1, \ldots, g_s) \in \mathbb{N}^s$, arbitrary.
Then

$$D_N(\omega) \ \leq \ \epsilon_{\mathbf{b}}(\mathbf{g}) + \sum_{\mathbf{k} \in \Delta^*_{\mathbf{b}}(\mathbf{g})} \rho_{\mathbf{b}}(\mathbf{k}) \left| S_N(\gamma_{\mathbf{k}}, \omega) \right| \ ,$$

where, for $b \geq 2$ integer:

$$\rho_b(k) \ = \ \begin{cases} 1 & \text{if} \quad k = 0, \\ \dfrac{2}{b^t \sin(\pi k_{t-1}/b)} & \text{if} \quad b^{t-1} \leq k < b^t, \ t \in \mathbb{N}, \end{cases}$$

$$\rho_{\mathbf{b}}(\mathbf{k}) \ = \ \prod_{i=1}^{s} \rho_{b_i}(k_i), \quad \mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s,$$

$$\Delta_{\mathbf{b}}(\mathbf{g}) \ = \ \left\{ \mathbf{k} = (k_1, \ldots, k_s) : \ 0 \leq k_i < b_i^{g_i}, 1 \leq i \leq s \right\},$$

$$\Delta^*_{\mathbf{b}}(\mathbf{g}) \ = \ \Delta_{\mathbf{b}}(\mathbf{g}) \setminus \{\mathbf{0}\},$$

$$\epsilon_{\mathbf{b}}(\mathbf{g}) \ = \ 1 - \prod_{i=1}^{s} (1 - 2b_i^{-g_i}).$$

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$$D_N(\omega) \;\leq\; \epsilon_{\mathbf{b}}(\mathbf{g}) + \sum_{\mathbf{k} \in \Delta^*_{\mathbf{b}}(\mathbf{g})} \rho_{\mathbf{b}}(\mathbf{k}) \left| S_N(\gamma_{\mathbf{k}}, \omega) \right| \;,$$

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Main lemma

Question
Where do the weights $\rho_{\mathbf{b}}(\mathbf{k})$ come from?

Lemma
Choose $\mathbf{g} = (g_1, \dots, g_s) \in \mathbb{N}_0^s$ and consider $f(\mathbf{x}) = \mathbf{1}_I(\mathbf{x}) - \lambda_s(I)$,
where $I = \prod_{i=1}^{s} \left[a_i b_i^{-g_i}, d_i b_i^{-g_i} \right[, \quad 0 \le a_i < d_i \le b_i^{g_i},\ g_i \ge 1.$

Then

1. $\forall\, \mathbf{k} \in \mathbb{N}_0^s \setminus \Delta_{\mathbf{b}}^*(\mathbf{g})$,
$$\hat{f}(\mathbf{k}) = 0,$$
2. $\forall\, \mathbf{k} \in \Delta_{\mathbf{b}}^*(\mathbf{g})$,
$$|\hat{\mathbf{1}}_I(\mathbf{k})| \le \rho_{\mathbf{b}}(\mathbf{k}),$$
3. for all $\mathbf{x} \in [0,1)^s$:
$$f(\mathbf{x}) \;=\; \sum_{\mathbf{k} \in \Delta_{\mathbf{b}}^*(\mathbf{g})} \hat{\mathbf{1}}_I(\mathbf{k})\, \gamma_{\mathbf{k}}(\mathbf{x}).$$

Relating u.d. in $[0,1)$, $\mathbb{Z}$, and $\mathbb{Z}_b$

Theorem (H. and Niederreiter, 2011)
$f_i : \mathbb{N}_0 \to \mathbb{N}_0$, $1 \le i \le s$,
$\boldsymbol{f} = (f_1, \ldots, f_s)$, $\boldsymbol{f} : \mathbb{N}_0^s \mapsto \mathbb{N}_0^s$,
$\mathbf{b} = (b_1, \ldots, b_s)$, with not necessarily distinct integers $b_i \ge 2$,

$$\mathbf{x}_n = (\varphi_{b_1}(f_1(n)), \ldots, \varphi_{b_s}(f_s(n))) \in [0,1)^s, \quad n = 0, 1, 2, \ldots$$

Then, the following properties are equivalent:

1. the sequence $(\mathbf{x}_n)_{n\ge 0}$ is uniformly distributed in $[0,1)^s$;
2. the sequence $(\boldsymbol{f}(n))_{n\ge 0}$ of elements of $\mathbb{Z}^s$ is uniformly distributed mod $(b_1^{g_1}, \ldots, b_s^{g_s})$ for all $g_1, \ldots, g_s \in \mathbb{N}_0$ that are not all 0;
3. the sequence $(\boldsymbol{f}(n))_{n\ge 0}$ is uniformly distributed mod $(b_1^{g}, \ldots, b_s^{g})$ for all $g \in \mathbb{N}$;
4. the sequence $(\boldsymbol{f}(n))_{n\ge 0}$ of elements of $\mathbb{Z}_{\mathbf{b}}$ is uniformly distributed in $\mathbb{Z}_{\mathbf{b}}$.

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4. the sequence $(\boldsymbol{f}(n))_{n\ge0}$ of elements of $\mathbb{Z}_\mathbf{b}$ is uniformly distributed in $\mathbb{Z}_\mathbf{b}$.

Examples

Example I

$\alpha_1, \ldots, \alpha_s$ positive real numbers: $1, \alpha_1, \ldots, \alpha_s$ linearly independent over $\mathbb{Q}$

$f_i(n) = \lfloor n\alpha_i \rfloor$, $1 \leq i \leq s$.
Then $(\mathbf{x}_n)_{n \geq 0}$ is uniformly distributed in $[0, 1)^s$.

Example II

$\alpha_1, \ldots, \alpha_s$ positive real numbers: $\alpha_1 = 1/d$ with some $d \in \mathbb{N}$,
$1, \alpha_2, \ldots, \alpha_s$ linearly independent over $\mathbb{Q}$

$f_i(n) = \lfloor n\alpha_i \rfloor$, $1 \leq i \leq s$.
Then $(\mathbf{x}_n)_{n \geq 0}$ is uniformly distributed in $[0, 1)^s$.

Example III

$d_1, \ldots, d_s$ distinct positive integers,
$\alpha_1, \ldots, \alpha_s$ positive irrational numbers,

$f_i(n) = \lfloor \alpha_i n^{d_i} \rfloor$, $1 \leq i \leq s$.
Then $(\mathbf{x}_n)_{n \geq 0}$ is uniformly distributed in $[0, 1)^s$.

Walsh functions on $[0,1)^s$

Definition

$k \in \mathbb{N}_0$, $k = \sum_{j\geq 0} k_j b^j$, $\quad x \in [0,1)$, $x = \sum_{j\geq 0} x_j b^{-j-1}$:

k-th Walsh function in base b:

$$w_k(x) \;=\; e\left((\sum_{j\geq 0} k_j x_j)/b \right).$$

$\mathbf{b} = (b_1, \ldots, b_s)$ a vector of not necessarily distinct integers $b_i \geq 2$,
$\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$, $\quad \mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s$:

k-th Walsh function in base **b**:

$$w_{\mathbf{k}}(\mathbf{x}) \;=\; \prod_{i=1}^{s} w_{k_i}(x_i).$$

Write $\mathcal{W}_{\mathbf{b}}^{(s)} = \{w_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}_0^s\}$ . . . the Walsh system in base **b**

Properties of $\mathcal{W}_{\mathbf{b}}^{(s)}$

- $\mathcal{W}_{\mathbf{b}}^{(s)}$ is an ONB of $L^2([0,1)^s)$.
- A Weyl Criterion holds.
- We have an Erdös-Turán-Koksma inequality for discrepancy.

Hybrid sequences

Hybrid sequences

Hybrid sequence $\omega = (\mathbf{x}_n)_{n\geq 0}$ in $[0,1)^s$:
mix at least two lower-dimensional sequences of different types s.t. certain coordinates of $\mathbf{x}_n$ stem from the first sequence, certain of the remaining coordinates stem from the second sequence, and so on.

Concept due to Spanier (1997); recent developments due to Niederreiter (2009 –)

Situation

We have different function systems on $[0,1)^s$ at our disposition:

$$\mathcal{T}^{(s)}, \mathcal{W}_{\mathbf{b}}^{(s)}, \Gamma_{\mathbf{b}}^{(s)}$$

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Hybrid function systems

Notation

$s \geq 1$: $s = s_1 + s_2 + s_3,\ s_i \in \mathbb{N}_0$

$\mathbf{y} = (y_1, \ldots, y_s) \in \mathbb{R}^s :\ \mathbf{y} = (\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \mathbf{y}^{(3)})$,
with $\mathbf{y}^{(1)} = (y_1, \ldots, y_{s_1})$, $\mathbf{y}^{(2)} = (y_{s_1+1}, \ldots, y_{s_1+s_2})$, and so on.

Hybrid function systems

$\mathbf{b} = (b_1, \ldots, b_{s_1}, b_{s_1+1}, \ldots, b_{s_1+s_2})$, with $s_1 + s_2$ not necessarily distinct integers $b_i \geq 2$,

Hybrid function system:

$$\mathcal{F} = \mathcal{W}_{\mathbf{b}^{(1)}}^{(s_1)} \otimes \Gamma_{\mathbf{b}^{(2)}}^{(s_2)} \otimes \mathcal{T}^{(s_3)} = \{\xi_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}_0^{s_1} \times \mathbb{N}_0^{s_2} \times \mathbb{Z}^{s_3}\}$$

$$\xi_{\mathbf{k}}(\mathbf{x}) = w_{\mathbf{k}^{(1)}}(\mathbf{x}^{(1)})\gamma_{\mathbf{k}^{(2)}}(\mathbf{x}^{(2)})e_{\mathbf{k}^{(1)}}(\mathbf{x}^{(3)})$$

Properties

- $\mathcal{F}$ is an ONB of $L^2([0,1)^s)$.
- For $\mathcal{F}$, the Weyl criterion holds.

Hybrid function systems

Notation

$s \geq 1$: $s = s_1 + s_2 + s_3,\ s_i \in \mathbb{N}_0$

$\mathbf{y} = (y_1, \ldots, y_s) \in \mathbb{R}^s:\ \mathbf{y} = (\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \mathbf{y}^{(3)})$,
with $\mathbf{y}^{(1)} = (y_1, \ldots, y_{s_1})$, $\mathbf{y}^{(2)} = (y_{s_1+1}, \ldots, y_{s_1+s_2})$, and so on.

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$\mathbf{b} = (b_1, \ldots, b_{s_1}, b_{s_1+1}, \ldots, b_{s_1+s_2})$, with $s_1 + s_2$ not necessarily distinct integers $b_i \geq 2$,

Hybrid function system:

$$\mathcal{F} = \mathcal{W}_{\mathbf{b}^{(1)}}^{(s_1)} \otimes \Gamma_{\mathbf{b}^{(2)}}^{(s_2)} \otimes \mathcal{T}^{(s_3)} = \{\xi_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}_0^{s_1} \times \mathbb{N}_0^{s_2} \times \mathbb{Z}^{s_3}\}$$

$$\xi_{\mathbf{k}}(\mathbf{x}) = w_{\mathbf{k}^{(1)}}(\mathbf{x}^{(1)})\gamma_{\mathbf{k}^{(2)}}(\mathbf{x}^{(2)})e_{\mathbf{k}^{(1)}}(\mathbf{x}^{(3)})$$

Properties

- $\mathcal{F}$ is an ONB of $L^2([0,1)^s)$.
- For $\mathcal{F}$, the Weyl criterion holds.

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A hybrid inequality of ETK

Theorem
$s = s_1 + s_2$, with $s_1, s_2 \in \mathbb{N}_0$, not both equal to 0,
$\mathbf{b} = (b_1, \ldots, b_s)$ not necessarily distinct integers $b_i \geq 2$,
$\mathcal{W}_{\mathbf{b}^{(1)}}^{(s_1)}$ the Walsh system in base $\mathbf{b}^{(1)}$ and $\Gamma_{\mathbf{b}^{(2)}}^{(s_2)}$ the $\mathbf{b}^{(2)}$-adic system in base $\mathbf{b}^{(2)}$,
$\mathbf{b}^{(1)} = (b_1, \ldots, b_{s_1})$, $\mathbf{b}^{(2)} = (b_{s_1+1}, \ldots, b_s)$.
Consider the hybrid function system $\mathcal{W}_{\mathbf{b}^{(1)}}^{(s_1)} \otimes \Gamma_{\mathbf{b}^{(2)}}^{(s_2)} = \{\xi_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}_0^s\}$.
Then, for all $\mathbf{g} \in \mathbb{N}^s$,

$$D_N(\omega) \;\leq\; \epsilon_{\mathbf{b}}(\mathbf{g}) + \sum_{\mathbf{k} \in \Delta_{\mathbf{b}}^*(\mathbf{g})} \rho_{\mathbf{b}}(\mathbf{k}) \, |S_N(\xi_{\mathbf{k}}, \omega)| \;,$$

and

$$D_N^*(\omega) \;\leq\; \epsilon_{\mathbf{b}}^*(\mathbf{g}) + \sum_{\mathbf{k} \in \Delta_{\mathbf{b}}^*(\mathbf{g})} \rho_{\mathbf{b}}^*(\mathbf{k}) \, |S_N(\xi_{\mathbf{k}}, \omega)| \;,$$

with error terms $\epsilon_{\mathbf{b}}(\mathbf{g})$ and $\epsilon_{\mathbf{b}}^*(\mathbf{g})$,

$$\epsilon_{\mathbf{b}}(\mathbf{g}) = 1 - \prod_{i=1}^{s}(1 - 2b_i^{-g_i}), \quad \epsilon_{\mathbf{b}}^*(\mathbf{g}) = 1 - \prod_{i=1}^{s}(1 - b_i^{-g_i}).$$

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Further results: hybrid sequences

Theorem(Kritzer and H., 2012)
$\omega^{(1)} = (\varphi_{\mathbf{p}}(n))_{n=0}^{\infty}$, an s_1-dimensional Halton sequence to the base $\mathbf{p} = (p_1, \ldots, p_{s_1})$, $p_1, \ldots, p_{s_1}$ distinct primes;
$\omega^{(2)} = (\{n\frac{\mathbf{g}}{N}\})_{n=0}^{N-1}$,
where N is a prime different from $p_1, \ldots, p_{s_1}$, and $\mathbf{g} = (g, g^2, \ldots, g^{s_2}) \in \mathbb{Z}^{s_2}$.
Then there exists a lattice point $\mathbf{g} = (g, g^2, \ldots, g^{s_2})$ with $g \in \{1, \ldots, N-1\}$ such that for $\omega = (\varphi_{\mathbf{p}}(n), \{\frac{n\mathbf{g}}{N}\})_{n=0}^{N-1}$,

$$F_N(\omega) \le c\frac{(\log N)^{s_1+s_2+1}}{N},$$

c a constant independent of N.
$F_N(\omega)$: the diaphony with respect to $\mathcal{F} = \Gamma_{\mathbf{b}}^{(s_1)} \otimes \mathcal{T}^{(s_2)}$,

$$F_N(\omega) := \left(\frac{1}{\sigma - 1} \sum_{\substack{\mathbf{k}^{(1)} \in \mathbb{N}_0^{s_1} \\ \mathbf{k}^{(2)} \in \mathbb{Z}^{s_2} \\ (\mathbf{k}^{(1)}, \mathbf{k}^{(2)}) \neq \mathbf{0}}} \rho(\mathbf{k}^{(1)}, \mathbf{k}^{(2)}) \left| \frac{1}{N} \sum_{n=0}^{N-1} \gamma_{\mathbf{k}^{(1)}}(\mathbf{x}_n^{(1)}) e_{\mathbf{k}^{(2)}}(\mathbf{x}_n^{(2)}) \right|^2 \right)^{1/2}.$$

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U.d.d. function systems

Index sets

s-dimensional index set Λ: $(\mathbb{Z}^s, +)$, or $(\mathbb{N}_0^s, +)$, or $(\mathbb{N}^s, +)$, or finite direct products of these semigroups s.t. the dimensions of the factors add up to s, with the possible exclusion of the zero vector $\mathbf{0}$.

Examples

$\mathbb{N}_0^{s_1} \times \mathbb{N}_0^{s_2} \times \mathbb{Z}^{s_3}$, where $s = s_1 + s_2 + s_3$, with $s_1, s_2, s_3 \in \mathbb{N}_0$, not all s_i equal to 0.

U.d.d. function systems

A uniform distribution determining (u.d.d.) function system on $[0,1)^s$:
a subclass $\mathcal{F} = \{\xi_{\mathbf{k}} : \mathbf{k} \in \Lambda\}$ of the Riemann integrable functions on $[0,1)^s$ s.t. for all $\mathbf{k} \in \Lambda$, $||\xi_{\mathbf{k}}||_\infty = \sup\{|\xi_{\mathbf{k}}(\mathbf{x})| : \mathbf{x} \in [0,1)^s\} \leq 1$ and $\int_{[0,1)^s} \xi_{\mathbf{k}} d\lambda_s = 0$, and such that, for any sequence ω in $[0,1)^s$,

$$\forall \mathbf{k} \in \Lambda : \lim_{N \to \infty} S_N(\xi_{\mathbf{k}}, \omega) = 0$$

implies the uniform distribution of ω in $[0,1)^s$.

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Weight functions

Examples of u.d.d. function systems
The systems $\mathcal{W}_{\mathbf{b}}^{(s)}$, $\Gamma_{\mathbf{b}}^{(s)}$, $\mathcal{T}^{(s)}$, and the hybrid function systems $\mathcal{W}_{\mathbf{b}^{(1)}}^{(s_1)} \otimes \Gamma_{\mathbf{b}^{(2)}}^{(s_2)} \otimes \mathcal{T}^{(s_3)}$.

Weight functions on Λ
Λ an s-dimensional index set and $||\cdot||$ an arbitrary norm on $\mathbb{R}^s$.
A weight function on Λ: a real-valued function ρ with $\rho(\mathbf{k}) > 0$ for all $\mathbf{k} \in \Lambda$, and, for all $\epsilon > 0$, there exists a positive real number $K_0 = K_0(\epsilon)$ such that $\rho(\mathbf{k}) < \epsilon$ for all $\mathbf{k} \in \Lambda$ with $||\mathbf{k}|| > K_0$.

Arrays of functions
With a u.d.d. function system $\mathcal{F} = \{\xi_{\mathbf{k}} : \mathbf{k} \in \Lambda\}$ on $[0,1)^s$ and a weight function ρ on Λ associate the array of weighted functions

$$(\rho(\mathbf{k})\xi_{\mathbf{k}})_{\mathbf{k}\in\Lambda}\,.$$

The operator $S_N(\cdot,\omega)$ is linear, hence

$$S_N((\rho(\mathbf{k})\xi_{\mathbf{k}})_{\mathbf{k}\in\Lambda}\,,\omega) = (\rho(\mathbf{k})S_N(\xi_{\mathbf{k}},\omega))_{\mathbf{k}\in\Lambda}\,.$$

Spectral test

Definition
$\mathcal{F} = \{\xi_{\mathbf{k}} : \mathbf{k} \in \Lambda\}$ a u.d.d. function system on $[0,1)^s$, ρ be a weight function on Λ, and $\omega = (\mathbf{x}_n)_{n\geq 0}$ in $[0,1)^s$.

The spectral test $\sigma_N(\omega)$ of the first N elements of ω, with respect to $\mathcal{F}$ and ρ:

$$\begin{aligned}\sigma_N(\omega) &= \|\,(\rho(\mathbf{k}))_{\mathbf{k}\in\Lambda}\,\|_\infty^{-1}\;\|\,(\rho(\mathbf{k})S_N(\xi_{\mathbf{k}},\omega))_{\mathbf{k}\in\Lambda}\,\|_\infty \\ &= \sup_{\mathbf{k}\in\Lambda}\{\rho(\mathbf{k})\}^{-1}\sup_{\mathbf{k}\in\Lambda}\{\rho(\mathbf{k})\,|S_N(\xi_{\mathbf{k}},\omega)|\}\,.\end{aligned}$$

Let $\alpha > 1$ be a given real number. If the weight function ρ fulfills the additional condition

$$\|\,(\rho(\mathbf{k}))_{\mathbf{k}\in\Lambda}\,\|_\alpha = \left(\sum_{\mathbf{k}\in\Lambda}\rho(\mathbf{k})^\alpha\right)^{1/\alpha} < \infty,$$

then the L^α-diaphony $F_N^{(\alpha)}(\omega)$ of the first N elements of ω, with respect to $\mathcal{F}$ and ρ:

$$\begin{aligned}F_N^{(\alpha)}(\omega) &= \|\,(\rho(\mathbf{k}))_{\mathbf{k}\in\Lambda}\,\|_\alpha^{-1}\;\|\,(\rho(\mathbf{k})S_N(\xi_{\mathbf{k}},\omega))_{\mathbf{k}\in\Lambda}\,\|_\alpha \\ &= \left(\sum_{\mathbf{k}\in\Lambda}\rho(\mathbf{k})^\alpha\right)^{-1/\alpha}\left(\sum_{\mathbf{k}\in\Lambda}\rho(\mathbf{k})^\alpha\,|S_N(\xi_{\mathbf{k}},\omega)|^\alpha\right)^{1/\alpha}.\end{aligned}$$

Properties

Theorem
Let $\mathcal{F}$, ρ and $\sigma_N(\omega)$ be as above. Then

1. The quantity $\sigma_N(\omega)$ is a maximum.
2. The sequence ω is uniformly distributed in $[0,1)^s$ if and only if
$$\lim_{N\to\infty} \sigma_N(\omega) = 0.$$

Theorem
Let $\mathcal{F}$, ρ and $\sigma_N(\omega)$ be as above and let K denote an arbitrary positive integer. Then

$$\sigma_N(\omega) \leq \|\,(\rho(\mathbf{k}))_{\mathbf{k}\in\Lambda}\,\|_\infty^{-1} \max\left\{ \max_{0\leq\|\mathbf{k}\|\leq K} \{\rho(\mathbf{k})\,|S_N(\xi_\mathbf{k},\omega)|\}\,,\ \sup_{\|\mathbf{k}\|>K} \{\rho(\mathbf{k})\} \right\}.$$

Theorem
Let $\mathcal{F}$, ρ and $F_N^{(\alpha)}(\omega)$ be as above. Then the sequence ω is uniformly distributed in $[0,1)^s$ if and only if
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Examples: lattices

Examples: figures of merit for lattices
Choose $\mathcal{F} = \mathcal{T}^{(s)}$ (hence $\Lambda = \mathbb{Z}^s \setminus \{\mathbf{0}\}$),
and consider $\omega_{\mathbf{a}} = (\{(n/N)\mathbf{a}\})_{n=0}^{N-1}$, where $\mathbf{a} \in \mathbb{Z}^s$, $\gcd(a_i, N) = 1$, $1 \leq i \leq s$.
Then

1. if $\rho(\mathbf{k}) = ||\mathbf{k}||_2^{-1}$:
   $\sigma_N(\omega_{\mathbf{a}})$ is equal to the classical spectral test applied to the integration lattice

$$L(\omega_{\mathbf{a}}) = \bigcup_{n=0}^{N-1} \left(\{(n/N)\mathbf{a}\} + \mathbb{Z}^s\right),$$

2. if $\rho(\mathbf{k}) = r(\mathbf{k})^{-1}$:
   $\sigma_N(\omega_{\mathbf{a}}) = \varkappa(\mathbf{a}, N)$, the Babenko-Zaremba index,
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b-adic intervals

b-adic intervals
Let $\mathbf{b} = (b_1, \ldots, b_s)$, with not necessarily distinct integers $b_i \geq 2$.
A **b**-adic interval with resolution $\mathbf{g} = (g_1, \ldots, g_s) \in \mathbb{N}_0^s$:

$$\prod_{i=1}^{s} \left[u_i b_i^{-g_i}, v_i b_i^{-g_i}\right), \; 0 \leq u_i < v_i \leq b_i^{g_i}, \; u_i, v_i \in \mathbb{N}_0, \; 1 \leq i \leq s \,.$$

$\mathcal{J}_{\mathbf{b},\mathbf{g}}$: the class of all **b**-adic intervals with resolution **g**,
$\mathcal{J}^*_{\mathbf{b},\mathbf{g}}$: the subclass of those **b**-adic intervals anchored at the origin,
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Remark
We have the following discretization:

$$D_N(\omega) = \sup_{I \in \mathcal{J}_{\mathbf{b}}} \left| S_N(\mathbf{1}_I - \lambda_s(I), \omega) \right| ,$$

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Indexing $\mathcal{J}_{\mathbf{b},\mathbf{g}}$

The procedure

For a given resolution $\mathbf{g} \in \mathbb{N}_0^s$, define

$$\Delta_{\mathbf{b}}(\mathbf{g}) = \left\{\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s : \ 0 \le k_i < b_i^{g_i}, 1 \le i \le s\right\}.$$

Then

$$\mathcal{J}_{\mathbf{b},\mathbf{g}} = \{I_{\mathbf{a},\mathbf{d};\mathbf{g}} : (\mathbf{a}, \mathbf{d}) \in \Delta_{\mathbf{b}}(\mathbf{g}) \times \Delta_{\mathbf{b}}(\mathbf{g})\},$$

$$I_{\mathbf{a},\mathbf{d};\mathbf{g}} = \prod_{i=1}^{s} [\varphi_{b_i}(a_i), \varphi_{b_i}(d_i) + b_i^{-g_i}),$$

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The intervals in $\mathcal{J}^*_{\mathbf{b},\mathbf{g}}$ are of the form $I_{\mathbf{0},\mathbf{d};\mathbf{g}}$, with $\mathbf{d} \in \Delta_{\mathbf{b}}(\mathbf{g})$.

Further, let

$$\Lambda = \mathbb{N}_0^s \times \mathbb{N}_0^s = \bigcup_{\mathbf{g} \in \mathbb{N}_0^s} (\Delta_{\mathbf{b}}(\mathbf{g}) \times \Delta_{\mathbf{b}}(\mathbf{g})).$$

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$$\Lambda = \mathbb{N}_0^s \times \mathbb{N}_0^s = \bigcup_{\mathbf{g} \in \mathbb{N}_0^s} (\Delta_{\mathbf{b}}(\mathbf{g}) \times \Delta_{\mathbf{b}}(\mathbf{g})).$$

Indexing $\mathcal{J}_{\mathbf{b},\mathbf{g}}$

The procedure

For a given resolution $\mathbf{g} \in \mathbb{N}_0^s$, define

$$\Delta_{\mathbf{b}}(\mathbf{g}) = \left\{\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s : \; 0 \le k_i < b_i^{g_i}, 1 \le i \le s\right\}.$$

Then

$$\mathcal{J}_{\mathbf{b},\mathbf{g}} = \{I_{\mathbf{a},\mathbf{d};\mathbf{g}} : (\mathbf{a}, \mathbf{d}) \in \Delta_{\mathbf{b}}(\mathbf{g}) \times \Delta_{\mathbf{b}}(\mathbf{g})\},$$

$$I_{\mathbf{a},\mathbf{d};\mathbf{g}} = \prod_{i=1}^{s} [\varphi_{b_i}(a_i), \varphi_{b_i}(d_i) + b_i^{-g_i}),$$

$\mathbf{a} = (a_1, \ldots, a_s), \mathbf{d} = (d_1, \ldots, d_s)$.

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Admissible indices

Definition
Let $\Lambda = \mathbb{N}_0^s \times \mathbb{N}_0^s$.
Admissible index point $(\mathbf{a}, \mathbf{d}) \in \Lambda$: if there exists $\mathbf{g} \in \mathbb{N}_0^s$ s.t. $(\mathbf{a}, \mathbf{d}) \in \Delta_\mathbf{b}(\mathbf{g}) \times \Delta_\mathbf{b}(\mathbf{g})$ and $I_{\mathbf{a},\mathbf{d};\mathbf{g}} \in \mathcal{J}_{\mathbf{b},\mathbf{g}}$.
Non-admissible index point: otherwise.

A u.d.d. function system
Put $\mathcal{F} = \{\xi_{(\mathbf{a},\mathbf{d})} : (\mathbf{a}, \mathbf{d}) \in \Lambda\}$:
for an admissible index $(\mathbf{a}, \mathbf{d}) \in \Lambda$,

$$\xi_{(\mathbf{a},\mathbf{d})} = \mathbf{1}_{I_{\mathbf{a},\mathbf{d};\mathbf{g}}} - \lambda_s(I_{\mathbf{a},\mathbf{d};\mathbf{g}}),$$

for non-admissible indices $(\mathbf{a}, \mathbf{d})$: $\xi_{(\mathbf{a},\mathbf{d})} \equiv 0$.

$\mathcal{F}$ contains all functions $1_I - \lambda_s(I)$, $I \in \mathcal{J}_\mathbf{b}$.

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Relation between $\sigma_N(\omega)$ and $D_N(\omega)$

Weight function

Fix $\mathbf{g} \in \mathbb{N}^s$. If $k \in \mathbb{N}_0$, $k = k_0 + k_1 b + \cdots$,

$$v_b(k) = \begin{cases} 0, & \text{if } k = 0, \\ 1 + \max\{j : k_j \neq 0\}, & \text{if } k \geq 1. \end{cases}$$

For $(\mathbf{a}, \mathbf{d}) \in \Lambda$, let

$$\rho_{\mathbf{g}}((\mathbf{a}, \mathbf{d})) = \begin{cases} 1, & \text{if } (\mathbf{a}, \mathbf{d}) \in \Delta_{\mathbf{b}}(\mathbf{g}) \times \Delta_{\mathbf{b}}(\mathbf{g}), \\ \prod_{i=1}^{s} b_i^{-(v_{b_i}(a_i) + v_{b_i}(d_i))}, & \text{otherwise.} \end{cases} \tag{1}$$

Further, choose the maximum norm on $\mathbb{R}^s$. Then $\rho_{\mathbf{g}}$ is a weight function on Λ.

Theorem

Λ and $\mathcal{F}$ be as above.

For every sequence ω in $[0,1)^s$, and for every $\epsilon > 0$, there exists an integer vector $\mathbf{g} \in \mathbb{N}^s$ such that for the spectral test w.r.t. $\mathcal{F}$ and $\rho_{\mathbf{g}}$,

$$|D_N(\omega) - \sigma_N(\omega)| < \epsilon.$$

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Final slide

Thank you for your attention!