Atanassov's methods for low-discrepancy sequences

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Discrepancy

For a point set $\mathcal{P}_N = \{X_1, X_2, ..., X_N\}$ in $I^s = [0,1)^s$ and an interval J of I^s, we define the discrepancy function as

$$E(J;\mathcal{P}_N) = A(J;\mathcal{P}_N) - NV(J) \quad \text{where}$$

$A(J;\mathcal{P}_N) = \#\{n; 1 \leq n \leq N, X_n \in J\}$ and $V(J)$ is the volume of J.

Then, the star (extreme) discrepancy D^* and the (extreme) discrepancy D of $\mathcal{P}_N$ are defined as

$$D^*(\mathcal{P}_N) = \sup_{J^*}|E(J^*;\mathcal{P}_N)| \quad \text{and} \quad D(\mathcal{P}_N) = \sup_J|E(J;\mathcal{P}_N)|$$

where J^* (resp. J) is of the form $\prod_{i=1}^s [0, y_i)$ (resp. $\prod_{i=1}^s [y_i, z_i)$).

- For an infinite sequence X, we write $D(N, X)$, $D^*(N, X)$ and $E(J; N; X) = A(J; N; X) - NV(J)$, for the first N points of X .

- A sequence satisfying $D^*(N, X) \in O((\log N)^s)$ is usually considered to be a low-discrepancy sequence. More explicitly this is achieved by :

$$D^*(N, X) \leq c_s(\log N)^s + d_s(\log N)^{s-1} \text{ for all } N \geq N_0. \qquad (1)$$

- A common approach to compare low-discrepancy sequences is to focus on the asymptotic behavior of $D^*(N, X)/(\log N)^s$, and compare constants c_s.

- Improving the leading term c_s is still of great interest from a number theory point of view, even though d_s can be so huge that c_s can lose any significance in the non-asymptotic regime for QMC methods.

- Hence comparisons with the full LHS of (1) make sense too!

Sequences

- The van der Corput sequence S_b in base b is defined as ($n \geq 1$)

$$S_b(n) = \sum_{r=0}^{\infty} \frac{a_r(n)}{b^{r+1}} \text{ with } n-1 = \sum_{r=0}^{\infty} a_r(n)\ b^r \text{(b-adic expansion).}$$

- An s-dimensional Halton sequence H in $I^s = [0,1)^s$ is defined as

$$H(n) = (S_{b_1}(n), \dots, S_{b_s}(n)), \text{ with pairwise coprime bases } b_j.$$

- A generalized van der Corput sequence associated with a sequence $\Sigma = (\sigma_r)_{r\geq 0}$ of permutations of $Z_b = \{0, 1, \dots, b-1\}$ is defined as

$$S_b^{\Sigma}(n) = \sum_{r=0}^{\infty} \frac{\sigma_r\Big(a_r(n)\Big)}{b^{r+1}}.$$

• A generalized Halton sequence GH, associated with s sequences of permutations, $\Sigma_j = (\sigma_{j,r})_{r \geq 0}$ of Z_{b_j}, $j = 1, \ldots, s$, is defined as

$$GH(n) = (S_{b_1}^{\Sigma_1}(n), \ldots, S_{b_s}^{\Sigma_s}(n)), \ n \geq 1.$$

Pairwise coprime bases b_j are usually chosen as the first s primes.

• The concept of (t,s)-sequences in base b introduced by Niederreiter to extend former constructions (Sobol', HF) requires two definitions:

– An elementary interval E in I^s is defined as (a_i, d_i are integers)

$$E = \prod_{i=1}^{s} [a_i b^{-d_i}, (a_i + 1) b^{-d_i}) \text{ with } 0 \leq a_i \leq b^{d_i} \text{ for } 1 \leq i \leq s.$$

– Given integers t, m with $0 \leq t \leq m$, a (t,m,s)-net in base b is an s-dimensional set with b^m points such that any elementary interval in base b with volume b^{t-m} contains exactly b^t points of the set.

- An s-dimensional sequence X in I^s is a (t,s)-sequence (in the narrow sense) if the subset $\{X(n) : kb^m < n \leq (k+1)b^m\}$ is a (t,m,s)-net in base b for all integers $k \geq 0$ and $m \geq t$.

For short, from now on, we will use the notation X_n instead of $X(n)$.

- Truncation : Let $x = \sum_{i=1}^{\infty} x_i b^{-i}$ be a b-adic expansion of $x \in [0,1]$, with the possibility that $x_i = b-1$ for all but finitely many i. For every integer $m \geq 1$, we define the m-truncation of x by $[x]_{b,m} = \sum_{i=1}^{m} x_i b^{-i}$.

If $X \in I^s$, $[X]_{b,m}$ means an m-truncation is applied to all coordinates.

- An s-dimensional sequence $(X_n)_{n\geq 1}$, with prescribed $b-$adic expansions for each coordinate, is a *(t,s)-sequence* (in the broad sense) if the subset $\{[X_n]_{b,m}; kb^m < n \leq (k+1)b^m\}$ is a (t,m,s)-net in base b for all integers $k \geq 0$ and $m \geq t \geq 0$.

- Notice that the truncation is necessary to give sense to further constructions of Halton sequences and (t,s)-sequences.

Basic ingredients of the first Atanassov's method

• Atanassov introduced what we call his "first method" for the study of original Halton sequences. Then he developped a "second method", using the first one as a main ingredient, to obtain a drastic improvement for the discrepancy of a specific class of generalized Halton sequences to be seen later.

• Let $\mathcal{P}_N(X)$ be the set of the first N points (truncated if necessary) of a given Halton sequence or (t,s)-sequence: $X = (X^{(1)}, \ldots, X^{(s)})$.

• The idea is to split the discrepancy function $E(J; \mathcal{P}_N(X))$, where $J = \prod_{i=1}^{s} [0, z^{(i)})$, by using b_i(or b)-adic expansions of $z^{(i)}$ giving a decomposition of J in elementary intervals and then bound each term.

• To this end, one uses numeration systems in base b with signed digits and signed splittings of J in order to keep the $1/2^s$ factor previously obtained by symmetrisation in the classical methods:

(i) Numeration system in base b with signed digits: every $z \in [0,1)$ has a unique expansion of the form

$$z = \sum_{j=0}^{\infty} a_j b^{-j} \begin{cases} \text{with } |a_j| \leq \frac{b-1}{2} \text{ if } b \text{ is odd} \\ \text{with } |a_j| \leq \frac{b}{2} \text{ and } |a_j| + |a_{j+1}| \leq b-1 \text{ if } b \text{ is even.} \end{cases}$$

(ii) Signed splitting of an interval $J \in I^s$: any collection of intervals $J_1, \ldots, J_n$ and respective signs $\epsilon_1, \ldots, \epsilon_n$ (± 1), such that for any additive function ν on intervals in I^s, we have $\nu(J) = \sum_{i=1}^{n} \epsilon_i \nu(J_i)$.

- Signed splittings $I(\boldsymbol{j}) = I(j_1, \ldots, j_s)$ deduced from (i) are used to get a decomposition of the discrepancy function ($n_i = \lfloor \log N / \log b_i \rfloor$)

$$E(J; \mathcal{P}_N(X)) = \sum_{j_1=0}^{n_1} \cdots \sum_{j_s=0}^{n_s} \epsilon(\boldsymbol{j}) E\left(I(\boldsymbol{j}); \mathcal{P}_N(X)\right) =: \Sigma_1 + \Sigma_2$$

where $\boldsymbol{j} \in \Sigma_1 \Leftrightarrow b_1^{j_1} \ldots b_s^{j_s} \leq N$ and Σ_2 is the complementary sum.

• Σ_1 will give the leading term $\in O((\log N)^s)$ and Σ_2 the complementary term $\in O((\log N)^{s-1})$ in the bound (1).

• Thanks to the fundamental property of Halton (with coprime b_i) and (t,s)-sequences (with $b_i = b$) , one can easily bound the discrepancy function:

$$| E(I(\boldsymbol{j}); \mathcal{P}_N(X)) | \leq (b^t) \prod_{i=1}^{s} (v_i - u_i) \quad \text{where} \quad I(\boldsymbol{j}) := \prod_{i=1}^{s} [u_i b_i^{-d_i}, v_i b_i^{-d_i}).$$

• It remains to deal with the sums involved in Σ_1 and Σ_2, i.e. with the sum on $\{\boldsymbol{j} = (j_1, \ldots, j_s); j_i \geq 0 \text{ and } b_1^{j_1} \ldots b_s^{j_s} \leq N\}$ and the complementary sum. For Halton sequences this is achieved using an essential argument from diophantine geometry introduced by Atanassov:

$$\# \left\{ (j_1, \ldots, j_k); j_i > 0 \text{ and } b_1^{j_1} \ldots b_k^{j_k} \leq N \right\} \leq \frac{1}{k!} \prod_{i=1}^{k} \frac{\log N}{\log b_i}. \qquad (2)$$

• Thanks to (2), in the case of Halton sequences, Atanassov gets the following fundamental Lemma 1 (Lemma 3.3 of his paper) to bound Σ_1 (with $k = s$ and $c = \lfloor b/2 \rfloor$) and similarly Σ_2 (with $1 \le k \le s-1$):

Lemma 1 *Let $N \ge 1$, $k \ge 1$ and $b_1, \ldots, b_k \ge 2$ be integers. For integers $j \ge 0$ and $1 \le i \le k$, let $c_j^{(i)} \ge 0$ be given numbers satisfying $c_0^{(i)} \le 1$ and $c_j^{(i)} \le c_i$ for $j \ge 1$, where c_i are some fixed numbers. Then*

$$\sum_{\{(j_1,\ldots,j_k)\,;\, j_i \ge 0 \text{ and } b_1^{j_1} \cdots b_k^{j_k} \le N\}} \prod_{i=1}^{k} c_{j_i}^{(i)} \le \frac{1}{k!} \prod_{i=1}^{k} \left(c_i \frac{\log N}{\log b_i} + k \right).$$

This is the reason for the $1/s!$ factor in the constant c_s obtained by Atanassov in the forthcoming Theorem 2.1 of his paper.

• The necessary adaptations of Atanassov's method for (t, s)-sequences will follow in Part II, Part I being devoted to Halton sequences.

Part I : Extending Atanassov's methods to scrambled Halton sequences

1. Atanassov's result for Generalized Halton seq.

$$D^*(N,GH) \leq \frac{1}{s!}\prod_{j=1}^{s}\left(\frac{(b_j-1)\log N}{2\log b_j}+s\right)+\sum_{k=0}^{s-1}\frac{b_{k+1}}{k!}\prod_{j=1}^{k}\left(\left\lfloor\frac{b_j}{2}\right\rfloor\frac{\log N}{\log b_j}+k\right)$$

if all b_j's are odd and with an additional term u if b_j is the even base:

$$u = \frac{b_j}{2(s-1)!}\prod_{1\leq i\leq s, i\neq j}\left(\frac{(b_i-1)\log N}{2\log b_i}+s-1\right) \quad \text{(Theorem 2.1, 2004).}$$

Hence the leading constant is $c_s = \frac{1}{s!}\prod_{j=1}^{s}\frac{b_j-1}{2\log b_j}$ (the old one/$s!$).

2. Atanassov's result for Modified Halton seq.

- In his paper, Atanassov obtains a further improvement for the leading constant c_s of a sub-class of Generalized Halton sequences which he calls Modified Halton sequences. Here begins what we call the "second Atanassov's method". Requires an auxiliary definition:

- Admissible integers. Let $p_1, \ldots, p_s$ be distinct primes. The integers $k_1, \ldots, k_s$ are called admissible for $p_1, \ldots, p_s$ if $p_i \nmid k_i$ and for each set of integers $b_1, \ldots, b_s$ such that $p_i \nmid b_i$, we have

$$k_i^{\alpha_i} \prod_{1 \le j \le s, j \ne i} p_j^{\alpha_j} = b_i \pmod{p_i}, \quad i = 1, \ldots, s.$$

- Sets of admissible integers are easy to obtain (amounts to solving systems of linear equations in integers).

- A modified Halton sequence (MH) in bases (p_i) with admissible integers (k_i) is a generalized Halton sequence $GH = (S_{p_1}^{\Sigma_1}, \ldots, S_{p_s}^{\Sigma_s})$ where the sequence of permutations $\Sigma_i = (\sigma_{i,r})_{r \geq 0}$ is defined by

$$\sigma_{i,r}(a) = ak_i^r \pmod{p_i} \quad \text{for all } 0 \leq a < p_i,\ r \geq 0,\ i = 1, \ldots, s.$$

- Such sequences are low discrepancy sequences with upper bound

$$D^*(N, MH) \leq \frac{1}{s!} \sum_{i=1}^{s} \log p_i \prod \frac{p_i(1+\log p_i)}{(p_i - 1)\log p_i} (\log N)^s + O((\log N)^{s-1}).$$

- The leading constant c_s is very low (MH sequences compete with Niederreiter–Xing (t,s)-sequences with $\log(c_s) \approx -s \log s$), but the constant in the complementary term is very huge. In applications taking $r \geq 1$ (instead of $r \geq 0$) is highly recommended.

- For the sake of completeness, we give the full upper bound:

Theorem 2.3 *(Atanassov, 2004)*

$$
\begin{aligned}
D^*(N, MH) \;\leq\; & \left(\frac{1}{s!} \prod_{i=1}^{s} \left(\frac{\log N}{K \log p_i} + s \right) \right) \cdot K^s \cdot \\
& \left(1 + \sum_{i=1}^{s} \log p_i \prod_{i=1}^{s} \frac{p_i}{p_i - 1} \left(\prod_{i=1}^{s} (1 + \log p_i) - 1 \right) \right) \\
& + \sum_{k=0}^{s-1} \frac{p_{k+1}}{k!} \prod_{i=1}^{k} \left(\left\lfloor \frac{p_i}{2} \right\rfloor \frac{\log N}{\log p_i} + k \right) \\
& + \sum_{i=1}^{s} \frac{1}{(s-1)!} \prod_{\substack{k=1 \\ k \neq i}}^{s} \left(\frac{p_k}{2} \frac{\log N}{\log p_k} + s - 1 \right),
\end{aligned}
$$

where $K = \prod_{i=1}^{s} (p_i - 1)$.

Some comments on Theorem 2.3

- We give one hint to link the proof with trigonometrical sums involved in fundamental Proposition 4.1 of Atanassov paper:

Lemma 4.2. *Let* $\mathbf{p} = (p_1, \ldots, p_s)$ *and let* $\omega = (\omega_n^{(1)}, \ldots, \omega_n^{(s)})_{n=0}^{\infty}$ *be a sequence in* $\mathbb{Z}^s$. *Let* $\mathbf{b}, \mathbf{c}$ *be fixed elements in* $\mathbb{Z}^s$, *such that* $0 \leq b_i < c_i \leq p_i$, *for* $1 \leq i \leq s$. *For* $C \geq 1$, *denote by* $a_C(\mathbf{b}, \mathbf{c})$ *the number of terms of* ω *among the first* C *terms such that for all* $1 \leq i \leq s$, *we have* $b_i \leq \omega_n^{(i)} \pmod{p_i} < c_i$. *Then*

$$\sup_{\mathbf{b},\mathbf{c}} \left| a_C(\mathbf{b}, \mathbf{c}) - C \prod_{i=1}^{s} \frac{c_i - b_i}{p_i} \right| \leq \sum_{\mathbf{j} \in M(\mathbf{p})} \frac{|S_C(\mathbf{j}, \omega)|}{R(\mathbf{j})},$$

where $S_C(\mathbf{j}, \omega) = \sum_{n=0}^{C-1} e\left(\sum_{k=1}^{s} \frac{j_k \omega_n^{(k)}}{p_k}\right)$ *in which* $e(x) = \exp(2i\pi x)$, *and where* $M(\mathbf{p}) = \{\mathbf{j} \,|\, 0 \leq j_i \leq p_i - 1, j_1 + \cdots + j_s > 0\}$.

- This result is applied to the counting function $A(J;\mathcal{P}_N)$ in place of $a_C(\mathbf{b},\mathbf{c})$. Hence, the discrepancy function can be estimated by means of a trigonometrical sum, which in turn gives an essential part of the upper bound for $D^*(N, MH)$ in Proposition 4.1. This relation with trigonometrical sums is possible thanks to the fine periodicity properties of Halton sequences, properties that are not shared by (t, s)-sequences.

- An attempt by Tezuka to obtain an analog of Theorem 2.3 for polynomial arithmetic Halton sequences by using Walsh functions instead of trigonometrical functions has been so far unsuccessful.

3. Scrambling Halton sequences with matrices

- This is the analog of generalized $(0, s)$-sequences from Tezuka (1994), first mentioned by Lemieux (Book, App. B, 2009).

- A linearly scrambled Halton (LSH) sequence X based on nonsingular lower triangular (NLT) matrices $A_1, \ldots, A_s$, where A_i has entries in $\mathbb{Z}_{p_i}$, is defined as

$$X = (S_{p_1}^{A_1}, \ldots, S_{p_s}^{A_s}),$$

in which, for an $\infty \times \infty$ matrix $C = (C_{r,l})_{r \geq 0, l \geq 0}$ with elements in $\mathbb{Z}_b$ (b prime), S_b^C is the sequence

$$S_b^C(n) = \sum_{r=0}^{\infty} y_{n,r} b^{-(r+1)} \quad \text{with} \quad y_{n,r} = \sum_{l=0}^{\infty} C_{r,l} a_l(n),$$

(the $a_l(n)$ are the digits of the b-adic expansion of $n - 1$ as stated previously). Requires truncation as for generalized $(0, s)$-sequences.

Theorem 1 *(FLW, 2012) A LSH sequence satisfies the discrepancy bound of a GH sequence (thus with the same leading constant c_s):*

$$D^*(N, LSH) \leq \frac{1}{s!} \prod_{j=1}^{s} \left(\frac{(b_j - 1) \log N}{2 \log b_j} + s \right) + \sum_{k=0}^{s-1} \frac{b_{k+1}}{k!} \prod_{j=1}^{k} \left(\left\lfloor \frac{b_j}{2} \right\rfloor \frac{\log N}{\log b_j} + k \right)$$

if all b_j's are odd and with additional term u if b_j is the even base

$$u = \frac{b_j}{2(s-1)!} \prod_{1 \leq i \leq s, i \neq j} \left(\frac{(b_i - 1) \log N}{2 \log b_i} + s - 1 \right).$$

The proof follows closely that of Atanassov, but requires a new fundamental lemma (instead of Lemma 3.1) taking into account the action of NLT matrices and the necessity of the truncation operator.

4. Scrambling Halton sequences with admissible matrices

- This is the generalisation of modified Halton sequences, using powers of admissible integers on the diagonal entries.
- Let $A_1, \ldots, A_s$ be NLT matrices in distinct prime bases $p_1, \ldots, p_s$ and let $k_1, \ldots, k_s$ be admissible integers for $p_1, \ldots, p_s$. Then the matrices $A_1, \ldots, A_s$ are admissible if they have the form

$$\begin{pmatrix} k_i^{\beta_i} & 0 & 0 & \cdots \\ * & k_i^{\beta_i+1} & 0 & \cdots \\ * & * & k_i^{\beta_i+2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $\beta_1, \ldots, \beta_s$ are non-negative integers. A LSH sequence based on admissible matrices $A_1, \ldots, A_s$ is called a modified linearly scrambled Halton (MLSH) sequence.

- Original Atanassov's modified Halton sequences correspond to diagonal admissible matrices A_j and to $\beta_j = 0$ for all j.
- Sequences used in experiments by Atanassov–Durchova (2003) and Faure–Lemieux (2009) correspond to diagonal admissible matrices A_j and to integers $\beta_j = 1$.

Theorem 2 *(FLW, 2012) A modified linearly scrambled Halton sequence associated to distinct primes $p_1, \ldots, p_s$ with admissible matrices $A_1, \ldots, A_s$ and to non-negative integers $\beta_1, \ldots, \beta_s$ satisfies the same discrepancy bound as a modified Halton sequence, in short:*

$$D^*(N, MLSH) \leq \frac{1}{s!} \sum_{i=1}^{s} \log p_i \prod \frac{p_i(1 + \log p_i)}{(p_i - 1) \log p_i} (\log N)^s + O((\log N)^{s-1}).$$

- The proof is adapted from that of Atanassov to take into account the addition of non-negative integers $\beta_1, \ldots, \beta_s$ and the truncation involved by NLT matrices.

Part II: Extending first Atanassov's method to (t, s)-sequences

1. Reminders on constants c_s for (t, s)-sequences

Various constants c_s below refer to inequality (1) for low-disc.seq.

$$c_s^{Ni} = \frac{b^t}{s!} \frac{b-1}{2\lfloor \frac{b}{2} \rfloor} \left(\frac{\lfloor \frac{b}{2} \rfloor}{\log b} \right)^s \quad \text{(Niederreiter, 1987).}$$

$$c_s^{Kr} = \frac{b^t}{s!} \frac{1}{2} \left(\frac{b-1}{2 \log b} \right)^s \text{ if } b \text{ is an odd base and}$$

$$c_s^{Kr} = \frac{b^t}{s!} \frac{b-1}{2(b+1)} \left(\frac{b}{2 \log b} \right)^s \text{ if } b \text{ is an even base (Kritzer, 2006).}$$

- More recently $c_s^{FL} = \frac{b^t}{s!}\left(\frac{b-1}{2\log b}\right)^s$ if b is even (FL 2012) and

$$c_s^{FK} = \frac{b^t}{s!}\frac{b^2}{2(b^2-1)}\left(\frac{b-1}{2\log b}\right)^s \text{ if } b \text{ is even (FK 2013).}$$

- Hence c_s^{Kr} for an odd base and c_s^{FK} for an even base are currently the best constants c_s for general (t,s)-sequences.

- Constants c_s^{Ni}, c_s^{Kr} and c_s^{FK} are all obtained with the same approach based on the study of (t,m,s)-nets prior to (t,s)-sequences. It involves a double recursion on both m and s (hence the $1/s!$ factor) along with a method of symmetrisation (hence the $1/2^s$ factor).

- On the contrary, c_s^{FL} is directly obtained for (t,s)-sequences and results from an extension of the first Atanassov's method.

2. First Atanassov's method for (t, s)-sequences

- The basic ingredients of the method (signed numeration systems and signed splittings of intervals) are used to get the decomposition of the discrepancy function with elementary intervals $I(\boldsymbol{j}) = I(j_1, \ldots, j_s)$:

$$E(J; [\mathcal{P}_N(X)]) = \sum_{j_1=0}^{n} \cdots \sum_{j_s=0}^{n} \epsilon(\boldsymbol{j}) E\left(I(\boldsymbol{j}); [\mathcal{P}_N(X)]\right) =: \Sigma_1 + \Sigma_2$$

where $\boldsymbol{j} \in \Sigma_1 \Leftrightarrow b^{j_1} \ldots b^{j_s} \leq N$ and Σ_2 is the complementary sum.

- Here $n = \lfloor \log N / \log b \rfloor$ and $[\mathcal{P}_N(X)]$ is the set of truncated points $([X_k^{(1)}]_{b,n+1}, \ldots, [X_k^{(s)}]_{b,n+1})$ $(1 \leq k \leq N)$ of the given (t, s)-sequence $X = (X^{(1)}, \ldots, X^{(s)})$.

• Thanks to the fundamental property of (t,s)-sequences on elementary intervals, we bound the discrepancy function:

$$| E\,(I(\boldsymbol{j}); [\mathcal{P}_N(X)]) | \leq b^t \prod_{i=1}^{s} (c_i - b_i) \text{ where } I(\boldsymbol{j}) := \prod_{i=1}^{s} [b_i b^{-d_i}, c_i b^{-d_i}).$$

• It remains to deal with the sums involved in Σ_1 and Σ_2, i.e. with the sum on $\{\boldsymbol{j} = (j_1, \ldots, j_s); b^{j_1} \ldots b^{j_s} \leq N\}$ and the complementary sum. This is achieved using the argument from diophantine geometry for base b instead of bases $b_1, \ldots, b_k$ for Halton sequences:

$$\#\left\{(j_1, \ldots, j_k); b^{j_1} \ldots b^{j_k} \leq N\right\} \leq \frac{1}{k!}\left(\frac{\log N}{\log b}\right)^k, \text{ i.e.}$$

$$\#\,\{(j_1, \ldots, j_k); j_1 + \cdots + j_k \leq n\} \leq n^k/k! \qquad (3)$$

• Thanks to (3), in the same way as Atanassov did for Halton sequences, we get the following basic Lemma 2 to bound Σ_1 (with $k = s$ and $c = \lfloor b/2 \rfloor$) and similarly Σ_2 (with $1 \leq k \leq s-1$):

Lemma 2 *Let $b \geq 2$, $N \geq 1$ and $k \geq 1$ be integers. For integers $j \geq 0$ and $1 \leq i \leq k$, let $c_j^{(i)} \geq 0$ be given numbers satisfying $c_0^{(i)} \leq 1$ and $c_j^{(i)} \leq c$ for $j \geq 1$, where c is some fixed number. Then*

$$\sum_{\{(j_1,\ldots,j_k)\,;\,b^{j_1}\cdots b^{j_k} \leq N\}} \prod_{i=1}^{k} c_{j_i}^{(i)} \leq \frac{1}{k!}\left(c\frac{\log N}{\log b} + k\right)^k.$$

• This is the reason for the $1/s!$ factor in the constant c_s when using an extension of Atanassov's method for (t, s)-sequences.

• We postpone the statement of the resulting theorem to introduce now a variant of our method.

3. Variant of Atanassov's method for (t,s)-seq.

Based on a combinatorial argument in place of (3) and on a careful analysis of the property of signed digits in an even base:

$$\#\{(j_1,\ldots,j_s); 0 \le j_1+\cdots+j_k \le n\} = \binom{n+k}{k} \tag{4}$$

Lemma 2-Variant *Let $b \ge 2$, $n \ge 0$, $k \ge 1$ be integers. For integers $j \ge 0$ and $1 \le i \le k$, let $c_j^{(i)} \ge 0$ be given numbers such that $c_{2h+1}^{(i)} \le c$ and $c_{2h}^{(i)} \le c'$ for any $h \ge 0$, where $c, c' \ge 0$ are fixed numbers. Then*

$$\sum_{(j_1,\ldots,j_k)\in S} \prod_{i=1}^{k} c_{j_i}^{(i)} \le \frac{1}{k!}\left(\frac{c+c'}{2}\right)^k \prod_{l=1}^{k}(n+2l),$$

where $S = \{(j_1,\ldots,j_k)\,;\, 0 \le j_1+\cdots+j_k \le n\}$.

- Thanks to (4), applying step by step the method above, we obtain a new slightly improved bound in the case where b is odd .

- In the case of an even base, using the variant of Lemma 2, we get an improved bound with a simpler complementary term.

- Another interest of this variant of Atanassov's method is to give effective explicit bounds that appear very efficient in the non-asymptotic regime, where sequences are compared via bounds for $D^*(N, X)$ instead of asymptotic constants c_s in the leading terms.

- In the next slides we show old bounds (with Lemma 2) and new bounds (with Lemma 2-Variant) for (t, s)-sequences.

Theorem 3 *For any* (t, s)*-sequence* X *in odd base* b *and for any* $N \geq 1$

$$D^*(N, X) \leq \frac{b^t}{s!}\left(\frac{b-1}{2}\frac{\log N}{\log b} + s\right)^s + b^t \sum_{k=1}^{s-1} \frac{b}{k!}\left(\frac{b-1}{2}\frac{\log N}{\log b} + k\right)^k \quad \textit{(oldFL)}$$

Theorem 4 *For any* (t, s)*-sequence* X *in odd base* b *and for any* $N \geq 1$

$$D^*(N, X) \leq \frac{b^t}{s!}\left(\frac{b-1}{2}\right)^s \prod_{k=1}^{s}\left(\frac{\log N}{\log b} + k\right) +$$

$$b^t \sum_{k=1}^{s-1} \frac{b}{k!}\left(\frac{b-1}{2}\right)^k \prod_{l=1}^{k}\left(\frac{\log N}{\log b} + l\right) \quad \textit{(newFL)}$$

Constant c_s remains the same but newFL gives lower effective bounds especially for small bases.

Theorem 5 *For any* (t,s)*-sequence* X *in even base* b *and any* $N \geq b^s$

$$D^*(N,X) \leq \frac{b^t}{s!}\left(\frac{b-1}{2}\frac{\log N}{\log b} + s\right)^s +$$
$$sb^t\left(\frac{b}{2}\right)^s\left(\frac{\log N}{\log b}\right)^{s-1} + b^t\sum_{k=1}^{s-1}\frac{b}{k!}\left(\frac{b}{2}\frac{\log N}{\log b} + k\right)^k \quad \textit{(oldFL)}$$

Theorem 6 *For any* (t,s)*-sequence* X *in even base* b *and any* $N \geq 1$

$$D^*(N,X) \leq \frac{b^t}{s!}\left(\frac{b-1}{2}\right)^s\prod_{k=1}^{s}\left(\frac{\log N}{\log b} + 2k\right) +$$
$$b^t\sum_{k=1}^{s-1}\frac{b}{k!}\left(\frac{b-1}{2}\right)^k\prod_{l=1}^{k}\left(\frac{\log N}{\log b} + 2l\right) \quad \textit{(newFL)}$$

Constant c_s remains the same but no condition on N and one term less. Effective bounds much better especially for small bases.

Part III: Atanassov's Variant for $(t, \mathbf{e}, s)$-seq.

- The notion of $(t, \mathbf{e}, s)$-sequences (where $\mathbf{e} = (e_1, \ldots, e_s)$ is an s-tuple of positive integers) was recently introduced by Tezuka (J. Comp. 2013) and it already has important consequences for new constructions of low-discrepancy sequences (by Niederreiter & all).

- Given integers t, m with $0 \leq t \leq m$, a $(t, m, \mathbf{e}, s)$-*net in base b* is an s-dimensional point set with b^m points such that any elementary interval $E = \prod_{i=1}^{s} [a_i b^{-d_i}, (a_i + 1) b^{-d_i})$ with $0 \leq a_i \leq b^{d_i}$ and $e_i | d_i$ for $1 \leq i \leq s$, and $V(E) = b^{t-m}$, contains exactly b^t points of the set. (Notice that these conditions imply that $m - t$ is of the form $j_1 e_1 + \cdots + j_s e_s$).

- A stronger version of this definition has been further introduced by Hofer and Niederreiter with the condition $V(E) \geq b^{t-m}$ (and $b^m V(E)$ points in E instead of b^t) in order to avoid problems with propagatoin rules and other reasons.

• The definition of $(t, \mathbf{e}, s)$-sequences is the same as for (t, s)-sequences with $(t, m, \mathbf{e}, s)$-nets in place of usual (t, m, s)-nets.
Of course, (t, s)-sequences are $(t, \mathbf{e}, s)$-sequences with $\mathbf{e} = (1, \ldots, 1)$.

• In this framework, Tezuka proves a first theorem: *generalized Niederreiter sequences (as defined by Tezuka in 1993) are $(0, \mathbf{e}, s)$-sequences where e_i is the degree of the i-th polynomial in the definition of these sequences*. Hence, so are usual (t, s)-sequences.

• Then, using Atanassov's method with signed numeration systems in bases b^{e_i}, signed splittings of intervals, and the argument from diophantine geometry introduced by Atanassov, adapted to bases b^{e_i}, Tezuka proves the following remarkable new type of bounds for low-discrepancy sequences:

• **Theorem 7** *For any* $(t, \mathbf{e}, s)$*-sequence* X *in base* b *and any* $N > b^t$

$$D^*(N, X) \le \frac{b^t}{s!} \prod_{i=1}^{s} \left(\frac{\lfloor b^{e_i}/2 \rfloor}{e_i} \left(\frac{\log N}{\log b} - t \right) + s \right) + \sum_{k=0}^{s-1} \frac{b^{t+e_{k+1}}}{k!} \prod_{i=1}^{k} \left(\frac{\lfloor b^{e_i}/2 \rfloor}{e_i} \left(\frac{\log N}{\log b} - t \right) + k \right) \quad \text{(Tez bound)}$$

• **Corollary** *The constant* c_s *resulting from (Tez bound) for* $(0, \mathbf{e}, s)$*-sequences is*

$$c_s^{Tez} = \frac{1}{s!} \prod_{i=1}^{s} \frac{\lfloor b^{e_i}/2 \rfloor}{e_i \log b}.$$

• In base two, with a result of Pollack on the prime number theorem for polynomials over a finite field, one has $\lim_{s \to \infty} c_s^{Tez} = 0$ (with irreducible polynomials of degree e_i in non-decreasing order of degrees).

• Tezuka proves Theorem 7 with an analog of Lemma 1 for Halton sequences which reads as follows (Lemma 2 in his paper):

Let k, n and $e_1, \dots, e_k$ be positive integers. For $j \geq 0$ and $1 \leq i \leq k$, let $c_j^{(i)} \geq 0$ be given numbers such that $c_0^{(i)} \leq 1$, $c_j^{(i)} \leq f_i$ for all $j \geq 1$, where the f_i's are some numbers. Then

$$\sum_{(j_1,\dots,j_k)\in S'} \prod_{i=1}^{k} c_{j_i}^{(i)} \leq \frac{1}{k!} \prod_{i=1}^{k} \left(f_i \frac{n}{e_i} + k \right).$$

where $S' = \{(j_1, \dots, j_k)\,;\, j_i \geq 0 \, and \, e_1 j_1 + \cdots + e_k j_k \leq n\}$.

• Using an adaptation of our variant of Atanassov's method, we are able to improve this lemma in an even base. The difference is that we take into account the conditions on signed numeration systems in even bases b which state that $|a_j| \leq \frac{b}{2}$ and $|a_j| + |a_{j+1}| \leq b - 1$. Hence two consecutive digits cannot be $\pm\frac{b}{2}$. We obtain successively:

Lemma 3 *Let* k*,* n *and* $e_1, \ldots, e_k$ *be positive integers. For integers* $j \geq 0$ *and* $1 \leq i \leq k$*, let* $c_j^{(i)} \geq 0$ *be given numbers such that* $c_0^{(i)} \leq 1$*,* $c_{2h+1}^{(i)} \leq f_i$ *for any* $h \geq 0$ *and* $c_{2h}^{(i)} \leq f_i'$ *for any* $h \geq 1$*, where* f_i *and* f_i' *are some fixed nonnegative numbers. Then*

$$\sum_{(j_1,\ldots,j_k)\in S'} \prod_{i=1}^{k} c_{j_i}^{(i)} \leq \frac{1}{k!} \prod_{i=1}^{k} \left(\frac{f_i + f_i'}{e_i} \left\lfloor \frac{n}{2} \right\rfloor + k \right).$$

Theorem 8 *Let* $b \geq 2$ *be an arbitrary integer. The star discrepancy for the first* $N \geq 1$ *points of a* $(t, \mathbf{e}, s)$*-sequence in base* b *satisfies*

$$D^*(N, X) \leq \frac{b^t}{s!} \prod_{i=1}^{s} \left(\frac{b^{e_i} - 1}{2e_i} \frac{\log N}{\log b} + s \right) + \sum_{k=0}^{s-1} \frac{b^{t+e_{k+1}}}{k!} \prod_{i=1}^{k} \left(\frac{b^{e_i} - 1}{2e_i} \frac{\log N}{\log b} + k \right) \quad \textit{(FL bound).}$$

- Therefore the constant c_s resulting from Theorem 8 for $(0, \mathbf{e}, s)$-sequences (hence for generalized Niederreiter sequences) is

$$c_s^{FL} = \frac{1}{s!} \prod_{i=1}^{s} \frac{b^{e_i} - 1}{2e_i \log b} \quad \text{whereas} \quad c_s^{Tez} = \frac{1}{s!} \prod_{i=1}^{s} \frac{\lfloor b^{e_i}/2 \rfloor}{e_i \log b}$$

- When b is odd, the two constants are equal, but in an even base we see that $c_s^{FL} < c_s^{Tez}$. In the table below, we provide a numerical comparison of the two constants when $b = 2$ for some values of s.

Comparison of constants c_s^{FL} and c_s^{Tez} for $(0, \mathbf{e}, s)$-sequences in base 2

	Values of s						
	1	2	3	4	5	6	7
c_s^{Tez}	1.4	1.0	5.0e-1	2.4e-1	9.2e-2	4.4e-2	1.8e-2
c_s^{FL}	7.2e-1	2.6e-1	9.3e-2	3.9e-2	1.3e-2	5.9e-3	2.3e-3
	8	9	10	20	30	40	50
c_s^{Tez}	6.6e-3	3.4e-3	1.5e-3	2.2e-7	6.4e-11	3.3e-15	2.1e-18
c_s^{FL}	7.8e-4	3.8e-4	1.7e-4	1.9e-8	5.1e-12	2.4e-16	1.5e-19

Two questions in the continuation of the preceding results:

- We use the following bound for the proof of Lemma 3:

Let k, n and $e_1, \ldots, e_k$ be positive integers. Then the number of positive integer solutions of the inequality $e_1 x_1 + \cdots + e_k x_k \leq n$ is bounded by $\frac{1}{k!} \prod_{i=1}^{k} \frac{n}{e_i}$. Find a better bound or even equality (seems hopeless!) like we get in our variant.

- Prove Atanassov Theorem 2.3 for polynomial Halton sequences in prime base b, where polynomials p_i with degree e_i take place of prime numbers p_i. That is overcome the difficulty found by Tezuka in trying to apply Walsh functions instead of trigonometrical functions.

Conclusion

- We have extended Atanassov's methods for Halton sequences to linearly scrambled Halton sequences by means of NLT matrices.

- We have adapted first Atanassov's method to (t, s)-sequences, obtaining new improved bounds in the case of an even base.

- Thanks to a variant of Atanassov's method applied to (t, s)-sequences and $(t, \mathbf{e}, s)$-sequences, we have improved preceding bounds obtained with this approach, especially in the case of even bases.

- New approach by Tezuka with $(t, \mathbf{e}, s)$-sequences (2013) add up a new scope to the fruitful contribution of Atanassov in the area of low discrepancy sequences.

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