Discrepancy theory and harmonic analysis

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“Uniform distribution and quasi-Monte Carlo methods”
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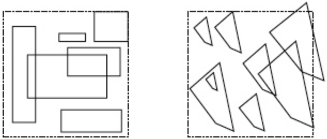
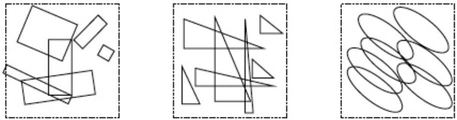
Geometric discrepancy

- No rotations: discrepancy $\approx (\log N)^{\alpha}$
  (K. Roth, W. Schmidt, van der Corput)



- All rotations: discrepancy $\approx N^{\beta}$
  (J. Beck, H. Montgomerry)



Pictures taken from Matoušek's "Geometric Discrepancy: An Illustrated Guide"

Discrepancy bounds in dimension $d = 2$

	LOWER BOUND	UPPER BOUND
	Axis-parallel rectangles	
L^∞	$\log N$	$\log N$
L^2	$\log^{\frac{1}{2}} N$	$\log^{\frac{1}{2}} N$
	Rotated rectangles	
	$N^{1/4}$	$N^{1/4}\sqrt{\log N}$
	Circles	
	$N^{1/4}$	$N^{1/4}\sqrt{\log N}$
	Convex Sets	
	$N^{1/3}$	$N^{1/3}\log^4 N$

Higher dimensions: $d \geq 3$

	LOWER BOUND	UPPER BOUND
	Axis-parallel boxes	
L^∞	$(\log N)^{\frac{d-1}{2}+\eta}$	$(\log N)^{d-1}$
L^2	$(\log N)^{\frac{d-1}{2}}$	$(\log N)^{\frac{d-1}{2}}$
	Rotated boxes	
	$N^{\frac{1}{2}-\frac{1}{2d}}$	$N^{\frac{1}{2}-\frac{1}{2d}}\sqrt{\log N}$
	Balls	
	$N^{\frac{1}{2}-\frac{1}{2d}}$	$N^{\frac{1}{2}-\frac{1}{2d}}\sqrt{\log N}$
	Convex Sets	
	$N^{1-\frac{2}{d+1}}$	$N^{1-\frac{2}{d+1}}\log^c N$

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Discrepancy function $D_N(x) = \#\{\mathcal{P}_N \cap [0,x)\} - Nx_1x_2$	Lacunary Fourier series $f(x) \sim \sum_{k=1}^{\infty} c_k \sin n_k x$, $\frac{n_{k+1}}{n_k} > \lambda > 1$
$\|D_N\|_2 \gtrsim \sqrt{\log N}$ (Roth, '54)	$\|f\|_2 \equiv \sqrt{\sum \|c_k\|^2}$
$\|D_N\|_\infty \gtrsim \log N$ (Schmidt, '72; Halász, '81) Riesz product: $\prod \left(1 + cf_k\right)$	$\|f\|_\infty \gtrsim \sum \|c_k\|$ (Sidon, '27) Riesz product: $\prod \left(1 + \cos(n_k x + \phi_k)\right)$
$\|D_N\|_1 \gtrsim \sqrt{\log N}$ (Halász, '81) Riesz product: $\prod \left(1 + i \cdot \frac{c}{\sqrt{\log N}} f_k\right)$	$\|f\|_1 \gtrsim \|f\|_2$ (Sidon, '30) Riesz product: $\prod \left(1 + i \cdot \frac{\|c_k\|}{\|f\|_2} \cos(n_k x + \theta_k)\right)$

Table : Discrepancy function and lacunary Fourier series

Sidon's theorem and non-dense orbits

- Sidon's theorem:
  Let $\{n_j\} \subset \mathbb{N}$ be lacunary with $\frac{n_{j+1}}{n_j} > 1 + \varepsilon$.
  Then there exists $\theta \in [0,1]$ such that

$$\left| \sum_j c_j \sin(2\pi n_j x) \right| \gtrsim \varepsilon |\log \varepsilon|^{-1} \sum_j |c_j|.$$

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  Let $\{n_j\} \subset \mathbb{N}$ be lacunary with $\frac{n_{j+1}}{n_j} > 1 + \varepsilon$.
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  $$\|n_j \theta\| \gtrsim \varepsilon |\log \varepsilon|^{-1}.$$
- History:
  - Erdős posed the question in 1976.
  - Khintchine (1926): $\varepsilon^2 |\log \varepsilon|^{-2}$.
  - Pollington (1979), de Mathan (1980): $\varepsilon^4 |\log \varepsilon|^{-1}$.
  - Katznelson (2001): $\varepsilon^2 |\log \varepsilon|^{-1}$.
  - Akhunzhanov, Moschevitin (2004): ε^2.

Connections

- Another question of Erdős (1987):
  Let $\mathcal{S} = \{n_j\} \subset \mathbb{N}$ be lacunary with $\frac{n_{j+1}}{n_j} > 1 + \varepsilon$.
  Define the graph G on $\mathbb{Z}$: (n, m) is an edge iff $|n - m| \in \mathcal{S}$.
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- There exists a set $A \subset \mathbb{N}$ with upper density $d^*(A) \gtrsim \varepsilon |\log \varepsilon|^{-1}$ such that

$$(A - A) \cap \mathcal{S} = \emptyset.$$

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  ($\mathcal{S} = \{n_j\}$ is not an *intersective* set).
- For $H \subset \mathbb{N}$, define

$$\delta_H = \sup\{d^*(A) : \ (A - A) \cap H = \emptyset\}.$$

$$\gamma_H = \inf\{a_0 : \ T(x) = a_0 + \sum_{n \in H} a_h \cos(2\pi n x) \geq 0, \ T(0) = 1\}$$

  Rusza (1984): $\qquad \gamma_H \geq \delta_H.$

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- Given $\Omega \subset [0,1]$, does there exists α such that

$$\left| (\alpha - \theta) - \frac{p}{q} \right| > \frac{c}{q^2} \text{ for all } \quad \theta \in \Omega \text{ ???}$$

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- Given any θ_1, θ_2, there exists α such that

$$\left|(\alpha-\theta_1)-\frac{p}{q}\right| > \frac{c}{q^2} \text{ and } \left|(\alpha-\theta_2)-\frac{p}{q}\right| > \frac{c}{q^2}.$$

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Lemma (Cassels (1956); Davenport (1964))

For all $n > 0$ there exists $c = c(n) > 0$ so that for any $\theta_1, \theta_2, ..., \theta_n$ there exists α such that for all $j = 1, ..., n$

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for all $p \in \mathbb{Z}$, $q \in \mathbb{N}$. $c(n) \approx 1/n^2$

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- Given $\Omega \subset [0,1]$, does there exists α such that

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- Lacunary sets of finite order:
  - Lacunary of order $M+1$ = union of a set $\mathcal{S}$ lacunary of order M and lacunary sequences converging to to each point of $\mathcal{S}$
  - Directional maximal function M_Ω is bounded in $L^p(\mathbb{R}^2)$ iff Ω is a finite union of lacunary sets of finite order (Bateman, 2007)
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- Sets of Minkowski dimension $0\le d\le 1$
- "Superlacunary" sets, e.g. $\{2^{-2^n}\}$.

1. Cassels-Davenport iteration

- Construct a sequence $\ldots I_{n-1} \supset I_n \supset \ldots$ with $|I_n| \to 0$ so that

$$\left|(\alpha - \theta) - \frac{p}{q}\right| \geq \frac{c(n)}{q^2} \qquad (*)$$

for all $\alpha \in I_n$, $\theta \in \Omega$, and $R(n) \leq q \leq R(n+1)$

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Theorem (DB, X. Ma, J. Pipher, C. Spencer)

There exists θ such that for all $\phi \in \Omega$

$$\left|(\alpha - \theta) - \frac{p}{q}\right| \gtrsim F^{-1}\left(F^{-1}\left(\frac{1}{q^2}\right)\right)$$

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- Ω finite: $N = \#\Omega$, $F(x) \approx Nx$,

$$F^{-1}(F^{-1}(1/q^2)) \approx \frac{1}{N^2 q^2}$$

- Ω lacunary: $N(x) \approx \log \frac{1}{x}$, $F(x) \approx x \cdot \log \frac{1}{x}$, $F^{-1}(x) \approx \frac{x}{\log(1/x)}$,

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- Ω "superlacunary", $F^{-1}(F^{-1}(1/q^2)) \approx \frac{1}{q^2 (\log \log q)^2}$

2. Trivial argument

- Let $h(q)$ be an increasing function of q such that $\sum \frac{1}{q \cdot h(q)} < \infty$. ($h(q) = q^{\varepsilon}$, $h(q) = \log^{1+\varepsilon} q$ etc.)

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- Ω has upper Minkowski dimension d: $N(x) \lesssim \left(\frac{1}{x}\right)^{d+\epsilon}$, $F(x) \lesssim x^{1-d-\epsilon}$, $F^{-1}(x) \gtrsim x^{\frac{1}{1-d}-\epsilon}$,

$$F^{-1}(1/q^{2+\varepsilon}) \gtrsim q^{-\frac{2}{1-d}-\epsilon}$$

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- Peres–Schlag 2010:
  Let $\{n_j\} \subset \mathbb{N}$ be lacunary with $\frac{n_{j+1}}{n_j} > 1 + \varepsilon$.
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- Moschevitin,... 2010-2012:
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- Einsiedler, Katok, Lindenstrauss (2006):
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Theorem (DB, Ma, Pipher, Spencer)

Assume that for every $Q \in \mathbb{N}$ *we have* $\sum_{q=Q}^{1/\delta(Q)} q \cdot F(\delta(q)) \lesssim 1$. *Then there exists* α *such that for all* $\theta \in \Omega$

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- Ω lacunary of order M: $N(x) \approx \log^M \frac{1}{x}$, $F(x) \approx x \cdot \log^M \frac{1}{x}$,

$$\delta(q) \approx \frac{1}{q^2 \log^{M+1} q}$$

A. Erdős-Turan inequality

- For a sequence $\omega = (\omega_n) \subset [0,1]$:

$$D_N(\omega) = \sup_{x \subset [0,1]} \left| \#\{n \leq N : \omega_n \leq x\} - Nx \right|$$

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Theorem (Erdős-Turan)

For any sequence $\omega \subset [0,1]$ we have

$$D_N(\omega) \lesssim \frac{N}{m} + \sum_{h=1}^{m} \frac{1}{h} \left| \sum_{n=1}^{N} e^{2\pi i h \omega_n} \right|$$

for all natural numbers m.

A. Erdős-Turan inequality

- For a sequence $\omega = (\omega_n) \subset [0,1]$:

$$D_N(\omega) = \sup_{x \subset [0,1]} \left| \#\{n \leq N : \omega_n \leq x\} - Nx \right|$$

Theorem (Erdős-Turan)

For any sequence $\omega \subset [0,1]$ we have

$$D_N(\omega) \lesssim \frac{N}{m} + \sum_{h=1}^{m} \frac{1}{h} \left| \sum_{n=1}^{N} e^{2\pi i h \omega_n} \right|$$

for all natural numbers m.

- For the Kronecker sequence $\omega = \{n\alpha\}$, we have
$$\left| \sum_{n=1}^{N} e^{2\pi i h n \alpha} \right| \leq \frac{1}{|\sin \pi h\alpha|} \lesssim \frac{1}{\|h\alpha\|}.$$

A. Erdős-Turan inequality

Lemma

For the sequence $\omega = \{n\alpha\}$ *we have*

$$D_N(\omega) \lesssim \frac{N}{m} + \sum_{h=1}^{m} \frac{1}{h \cdot \|h\alpha\|}$$

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We have

$$\sum_{h=1}^{m} \frac{1}{h\|h\alpha\|} \lesssim \psi(2m) \log m + \sum_{h=1}^{m} \frac{\psi(2h) \log h}{h}$$

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$$\sum_{h=1}^{m} \frac{1}{h\|h\alpha\|} \lesssim \log^2 m + \psi(m) + \sum_{h=1}^{m} \frac{\psi(h)}{h}$$

B. Partial quotients

- For the Kronecker sequence $\omega = \{n\alpha\}$

$$D_N(\omega) \lesssim \sum_{j=1}^{m+1} a_j,$$

where $\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots}}} = [a_0;\ a_1,\ a_2,\ a_3, \ldots]$,
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- Ω "superlacunary", e.g. $\Omega = \left\{2^{-2^m}\right\}$:
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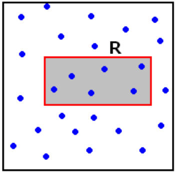
- One can deduce that

$$D_N(\omega) \lesssim \log N \cdot (\log\log N)^2,$$

while Erdős-Turan would only yield $\log^2 N$.

Directional Discrepancy: Setting

$\mathcal{P}_N$ – a set of N points in $[0,1]^d$
$R \subset [0,1]^d$ – a measurable set.



$$D(\mathcal{P}_N, R) = \sharp\{\mathcal{P}_N \cap R\} - N \cdot vol\,(R)$$

- Let $\Omega \subset [0, \pi/2)$
- $\mathcal{A}_\Omega = \{$rectangles pointing in directions of $\Omega\}$

$$D_\Omega(N) = \inf_{\mathcal{P}_N} \sup_{R \in \mathcal{A}_\Omega} |D(\mathcal{P}_N, R)|$$

All rotations

- $\Omega = [0, \pi/2)$

Theorem (J. Beck, 1987-88)

$$N^{1/4} \lesssim D_\Omega(N) \lesssim N^{1/4}\sqrt{\log N}$$

No rotations: axis-parallel rectangles

- $\Omega = \{0\}$

Theorem (Lerch, 1904; Schmidt, 1972)

$$D_\Omega(N) \approx \log N$$

Theorem (L^2: Roth, 1954; Davenport, 1956)

$$D_\Omega^{(2)}(N) \approx \sqrt{\log N}$$

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- Ω finite $\longrightarrow$ same (Beck-Chen, Chen-Travaglini)

Irrational Lattice

Example

Let $\alpha \in [0, \pi]$. Denote by $\mathcal{P}_N^\alpha$ the lattice $\frac{1}{\sqrt{N}}\mathbb{Z}^2$ rotated by α and intersected with $[0,1]^2$.

Theorem

If $\tan \alpha$ *is badly approximable, i.e.*

$$\left|\tan(\alpha) - \frac{p}{q}\right| \gtrsim \frac{1}{q^2}$$

for all $p \in \mathbb{Z}$, $q \in \mathbb{N}$, *then the discrepancy of* $\mathcal{P}_N^\alpha$ *with respect to axis-parallel rectangles satisfies*

$$D(\mathcal{P}_N^\alpha) \approx \log N.$$

Directional Discrepancy

$$D_\Omega(\mathcal{P}_N^\alpha) \lesssim \sup_{\theta \in \Omega} D_{c\sqrt{N}}(\{n \cdot \tan(\alpha - \theta)\})$$

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Theorem (DB, Ma, Pipher, Spencer)

- *Lacunary directions:* $D_\Omega(N) \lesssim \log^3 N$
- *Lacunary of order M:* $D_\Omega(N) \lesssim \log^{M+2} N$
- *"Superlacunary" sequence*

$$D_\Omega(N) \lesssim \log N \cdot (\log \log N)^2$$

- Ω *has upper Minkowski dimension* $0 \le d < 1$

$$D_\Omega(N) \lesssim N^{\frac{d}{d+1} + \varepsilon}.$$

Critical dimension

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- So $(*)$ is interesting only for $\frac{d}{d+1} < \frac{1}{4}$, i.e.

$$d < \frac{1}{3}.$$