# Metric number theory, lacunary series and systems of dilated functions

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Uniform distribution and quasi-Monte Carlo methods
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# The equidistribution theorem

The sequence of fractional parts $(\{kx\})_{k \geq 1}$ is uniformly distributed modulo 1 if and only if $x \notin \mathbb{Q}$ (Bohl–Sierpinski–Weyl, 1909–1910).

A sequence $(x_k)_{k \geq 1}$ is called uniformly distributed modulo 1 if for all $a \in [0,1]$ the relation

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_{[0,a)}(x_k) = a$$

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# The equidistribution theorem

For a sequence $(n_k)_{k \geq 1}$ of distinct integers, the sequence of fractional parts $(\{n_k x\})_{k \geq 1}$ is uniformly distributed modulo 1 for almost all $x$. (Weyl, 1916)

In general the exceptional set cannot be determined explicitly (for example for $n_k = 2^k, \ k \geq 1$.)

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# Discrepancy

For a finite sequence $x_1, \dots, x_N$ of points in the unit interval the star-discrepancy $D_N^*$ is defined as

$$D_N^*(x_1, \dots, x_N) = \sup_{0 \leq a \leq 1} \left| \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_{[0,a)}(x_k) - a \right|.$$

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# Closely related problems

- Find upper bounds for

$$\left| \sum_{k=1}^{N} \mathbb{1}_{[0,a)}(\{n_k x\}) - Na \right|$$

for *fixed a* and a.e. $x$.

- Find upper bounds for

$$\left| \sum_{k=1}^{N} f(n_k x) \right|$$

for $f$ being centered, 1-periodic and of bounded variation, for a.e. $x$.

- Find criteria for the a.e. convergence of

$$\sum_{k=1}^{\infty} c_k f(n_k x)$$

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# Remarks I: Known results

In a few special cases the answers are known.

- If $n_k = k, \ k \geq 1$ (connection with continued fractions expansion).
- If $(n_k)_{k \geq 1}$ is increasing very quickly (*lacunary sequences* - almost independent behavior).

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# Remarks II: Probabilistic interpretation

- It suffices to consider $x \in [0, 1]$.
- The space $([0, 1], \mathcal{B}, \lambda)$ is a probability space.
- The functions $\{n_k x\}$ and $f(n_k x)$ are *random variables*.
- For each $k$, the random variable $\{n_k x\}$ has distribution $\mathcal{U}([0, 1])$.
- The discrepancy is the Kolmogorov–Smirnov statistic.

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# Tools I

**Erdős–Turán inequality:**

For any positive integer $H$

$$D_N^*(x_1, \dots, x_N) \leq \frac{3}{H} + 3 \sum_{h=1}^{H} \frac{1}{h} \left| \frac{1}{N} \sum_{k=1}^{N} e^{2\pi i h x_k} \right|.$$

# Tools II

**Koksma's inequality:**

For any function $f$ which has bounded variation $\mathrm{Var}(f)$ in the unit interval we have

$$\left| \frac{1}{N} \sum_{k=1}^{N} f(x_k) - \int_0^1 f(x)\ dx \right| \leq \mathrm{Var}(f) \cdot D_N^*(x_1, \dots, x_N).$$

# Tools III

**Carleson's theorem:**

The Fourier series of a function in $L^2([0,1])$ is a.e. convergent.

Equivalently: there exists an absolute constant $c$ such that for any $f \in L^2([0,1])$

$$\|S^* f\|_2 \leq c \|f\|_2.$$

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# Tools IV

**Gál's theorem:**

There is an absolute constant $c$ such that for any distinct integers $n_1, \dots, n_N$ we have

$$\sum_{k,l=1}^{N} \frac{(\gcd(n_k, n_l))^2}{n_k n_l} \le cN(\log \log N)^2.$$

# An upper bound for trigonometric sums

For any increasing sequence of integers $(n_k)_{k\geq 1}$ we have

$$\left|\sum_{k=1}^{N} \cos(2\pi n_k x)\right| = \mathcal{O}\left(\sqrt{N}(\log N)^{1/2+\varepsilon}\right) \qquad \text{a.e.}$$

This upper bound is optimal (Berkes–Philipp, 1994).

For the discrepancy we get

$$D_N^*(\{n_1 x\}, \ldots, \{n_N x\}) = \mathcal{O}\left(\frac{(\log N)^{3/2+\varepsilon}}{\sqrt{N}}\right) \qquad \text{a.e.}$$

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# An upper bound for sums of dilated functions

How about sums $\sum f(n_k x)$?

We have to estimate

$$\left\| \max_{1 \leq M \leq N} \left| \sum_{k=1}^{M} f(n_k x) \right| \right\|_2 .$$

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For $f$ being centered, 1-period and of bounded variation, and for $(n_k)_{k\geq 1}$ being an increasing sequence of integers, we have

$$\left|\sum_{k=1}^{N} f(n_k x)\right| = \mathcal{O}\left(\sqrt{N}(\log N)^{3/2+\varepsilon}\right) \qquad \text{a.e.}$$

# An upper bound for sums of dilated functions II

**Theorem (A–Berkes–Seip, 2013, to appear in Journal of the EMS)**

*Let $f$ be a centered, 1-periodic function of bounded variation. Let $(n_k)_{k\geq 1}$ be an increasing sequence of positive integers. Then*

$$\left| \sum_{k=1}^{N} f(n_k x) \right| = \mathcal{O}\left( \sqrt{N} (\log N)^{1/2} (\log \log N)^{5/2+\varepsilon} \right) \qquad \text{a.e.}$$

## Theorem (A–Berkes–Seip, 2013)

*Let $f$ be a centered, 1-periodic function of bounded variation, and let $(n_k)_{k\geq 1}$ be an increasing sequence of positive integers. Let $(c_k)_{k\geq 1}$ be a real sequence satisfying*

$$\sum_{k=3}^{\infty} c_k^2 (\log \log k)^{4+\varepsilon} < \infty.$$

*Then the series*

$$\sum_{k=1}^{\infty} c_k f(n_k x)$$

*is a.e. convergent.*

# Problems I

We can estimate

$$\left\| \max_{1 \leq M \leq N} \left| \sum_{k=1}^{M} f(n_k x) \right| \right\|_2 .$$

For the discrepancy we need

$$\left\| \max_{1 \leq M \leq N} \sup_{a \in [0,1]} \left| \sum_{k=1}^{M} \mathbb{1}_{[0,a)}(\{n_k x\}) - Ma \right| \right\|_2 .$$

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# Problems II

- Investigate the a.e. convergence problem for

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