Some combinatorial aspects of perfect codes.

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based on joint work with S. Costa

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A q-ary code of length $n$: $C \subseteq \mathbb{Z}_q^n$. 

In 1958 C.Y. Lee proposed the use of a metric in $\mathbb{Z}_q^n$ (Lee metric), appropriate to correct errors in certain types of channels. For $n = 1$:

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The resurgence of Lee Codes

Engineering applications

- Constrained and partial-response channels.
  

- Interleaving schemes.
  

- Multidimensional burst-error-correction.
  

- Error-correction for flash memories.
  
The resurgence of Lee Codes

Theoretical research

- **Enumerating and decoding perfect linear Lee codes.**
  
  B. AlBdaiwi, P. Horak, L. Milazzo. Enumerating and decoding perfect linear Lee codes.

- **Dense Lee Codes.**
  
  T. Etzion, A. Vardy, E. Yaakobi. Dense error-correcting codes in the Lee metric.
  Information Theory Workshop (ITW), 2010 IEEE.

- **Special constructions for perfect Lee codes.**
  
  T. Etzion. Product constructions for perfect Lee codes.

- **Diameter perfect Lee codes.**
  
  P. Horak, B.F. AlBdaiwi. Diameter perfect Lee codes.
Let $C \subseteq \mathbb{Z}_q^n$ be a $q$-ary code.

**Definitions**

- As in the case of the Hamming metric, $C$ is a perfect Lee code when

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  where $e$ is the packing radius and the balls are Lee-balls.
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  - $PL(n, e, q) = \{ C \subseteq \mathbb{Z}_q^n : C \text{ is } e\text{-perfect} \}$
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Main problem

Characterize the triplets \((n, e, q)\) for which \(PL(n, e, q) \neq \emptyset\).
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- For \(e = 1\) we have \(PL(n, 1) \neq \emptyset\) for all \(n\).
- For \(n = 2\) we have \(PL(2, e) \neq \emptyset\) for all \(e\).
- For each \(n\) there exists \(e_n\) s.t. \(PL(n, e) = \emptyset\) for all \(e \geq e_n\).
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Proof of Golomb and Welch for the bidimensional case:

Golomb and Welch present the codes $D_e = \langle (e, e+1) \rangle \subset \mathbb{Z}_q^2$ for $q = 2e^2 + 2e + 1$ and prove that these codes are perfect, then $PL(2, e, q) \neq \emptyset$ for $q = 2e^2 + 2e + 1$ ($\Rightarrow PL(2, e) \neq \emptyset$).
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Example: $D_2 = \langle (2, 3) \rangle \subset \mathbb{Z}_{13}^2$
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Related questions.

- For which \((e, q)\) we have \(PL(2, e, q) \neq \emptyset\)? In that case, is it possible to describe all these codes? (Remark: we are considering linear and non-linear codes.)

- What are the possible structures as abelian groups of these codes?
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We can use the geometry of polyominoes and combinatorial arguments.

Theorem

For $e, q \in \mathbb{Z}^+$ we define: $q_e = e^2 + (e + 1)^2$, $\nu_1 = (e, e + 1)$, $\nu_2 = -(e + 1), e$, $\eta_1 = (1, -(2e + 1))$, $\eta_2 = (0, q_e) \in \mathbb{Z}_q^2$, $D_e = \nu_1 \mathbb{Z} + \nu_2 \mathbb{Z}$ and $\overline{v} = (-x, y)$ is the conjugate of $v = (x, y)$. 
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2. (Characterization) \( C \in PL(2, e, q) \iff C = c + D_e \) or \( C = c + \overline{D_e} \) for any \( c \in C \) (in particular \( C - c \) is a group).
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3. (Structure) Let $C \in PL(2, e, q)$ and $G_C = C - c$ the group assoc. with $C$.
   
   i) $G_C$ is cyclic iff $q = q_e$. In this case $G_C \cong \mathbb{Z}_q$ with generator $\nu_1 = (e, e + 1)$ if $G_C = D_e$ or $\overline{\nu}_1$ if $G_C = \overline{D_e}$.

   ii) If $q = h q_e$ com $h > 1$ then $G_C \cong \mathbb{Z}_q \times \mathbb{Z}_h$.

   Moreover, $G_C = \eta_1 \mathbb{Z} \oplus \eta_2 \mathbb{Z}$ if $G_C = D_e$ or $G_C = \overline{\eta}_1 \mathbb{Z} \oplus \eta_2 \mathbb{Z}$ if $G_C = \overline{D_e}$.
Sketch of the proof

Impossible configuration

Let \( I = \{(-1, -1), (-1, 0), (-1, 1), (-1, 2), (0, -1), (0, 2)\} \subseteq \mathbb{Z}_q^2 \).

If \( C \in PL(2, e, q) \) and \( c, c' \in C \Rightarrow \nexists \, x \in \mathbb{Z}_q^2 \) such that

- \( x \in B(c, e) \cup B(c', e) \).
- \( x + C \subseteq B(c, e) \cup B(c', e) \)
Decoding of special points

If $C \in PL(2, e, q)$ and $c \in C$, the point $c + (0, e + 1)$ can only be decoded in two ways. These possibilities are $c + (-e, e + 1)$ and $c + (e, e + 1)$. 
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For $C \in PL(2, e, q)$ and $c \in C$ we define the set $\omega(c) = \{v_1, \ldots, v_\tau\}$ where the adjacent balls of $B(c, e)$ are exactly $B(c + v_i, e)$ for $1 \leq i \leq \tau$. 
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Kissing Lemma

If \( C \in \text{PL}(2, e, q) \) the set \( \omega(c) \) does not depend on \( c \). Moreover we have only two possibilities: \( \omega(c) = \{ \pm \nu_1, \pm \nu_2 \} \) (type 1) or \( \omega(c) = \{ \pm \nu_1, \pm \nu_2 \} \) (type 2), where \( \nu_1 = (e, e + 1) \), \( \nu_2 = (-(e + 1), e) \).
Sketch of the proof

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If $C \in PL(2, e, q)$ the set $\omega(c)$ does not depend on $c$. Moreover we have only two possibilities: $\omega(c) = \{\pm \nu_1, \pm \nu_2\}$ (type 1) or $\omega(c) = \{\pm \overline{\nu}_1, \pm \overline{\nu}_2\}$ (type 2), where $\nu_1 = (e, e + 1)$, $\nu_2 = (- (e + 1), e)$. 
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(Qureshi - Campinas University, Brazil)
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\omega(c) = \{\nu_1, \nu_2, 
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Sketch of the proof

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\[
\omega(c) = \{\pm \nu_1, \pm \nu_2, \ldots\}
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For \( e, q \in \mathbb{Z}^+ \) we define: \( q_e = e^2 + (e + 1)^2, \nu_1 = (e, e + 1), \nu_2 = (-e + 1, e), \eta_1 = (1, -(2e + 1)), \eta_2 = (0, q_e) \in \mathbb{Z}_q^2, D_e = \nu_1 \mathbb{Z} + \nu_2 \mathbb{Z} \) and \( \overline{v} = (-x, y) \) is the conj. of \( v = (x, y) \).

1. **(Existence)** \( PL(2, e, q) \neq \emptyset \iff q \equiv 0 \pmod{q_e} \).

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   ii) If \( q = h q_e \) com \( h > 1 \) then \( G_C \simeq \mathbb{Z}_q \times \mathbb{Z}_h \).

Moreover, \( G_C = \eta_1 \mathbb{Z} \oplus \eta_2 \mathbb{Z} \) if \( G_C = D_e \) or \( G_C = \overline{\eta}_1 \mathbb{Z} \oplus \eta_2 \mathbb{Z} \) if \( G_C = \overline{D_e} \).
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Let $C \in PL(2, q, e)$ and fix any $c \in C$. 
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For the other inclusion, if $c' \in C$ it is sufficient to consider a chain of adjacent balls from $B(c', e)$ to $B(c, e)$ and use kissing lemma again.
Sketch of the proof

For \( e, q \in \mathbb{Z}^+ \) we define: \( q_e = e^2 + (e + 1)^2, \nu_1 = (e, e + 1), \nu_2 = (-(e + 1), e), \eta_1 = (1, -2(e+1)), \eta_2 = (0, q_e) \in \mathbb{Z}^2, D_e = \nu_1 \mathbb{Z} + \nu_2 \mathbb{Z} \) and \( \overline{\nu} = (-x, y) \) is the conj. of \( \nu = (x, y) \).

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$(\Rightarrow)$ Let $C \in PL(2, q, e)$ and fix any $c \in C$. We can suppose

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$q = |\nu_1 \mathbb{Z}| \cdot |\nu_1 \mathbb{Z} + \nu_2 \mathbb{Z}| = \#C \Rightarrow \#C = qh$ for some $h \in \mathbb{Z}^+$. 

(⇐) Golomb and Welch. □
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By the sphere packing condition:

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We can suppose that $C$ is linear and type 1, that is $C = \nu_1 \mathbb{Z} + \nu_2 \mathbb{Z}$ where

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As

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\begin{pmatrix}
-1 & -1 \\
e + 1 & e
\end{pmatrix}
\begin{pmatrix}
\nu_1 \\
\nu_2
\end{pmatrix}
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\eta_1 \\
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and

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then $C = \eta_1 \mathbb{Z} + \eta_2 \mathbb{Z}$ where $\eta_1 = (1, -(2e + 1))$ and $\eta_2 = (0, q_e)$ (in $\mathbb{Z}_q^2$).

Clearly $\eta_1 \mathbb{Z} \cap \eta_2 \mathbb{Z} = (0)$ (therefore $C = \eta_1 \mathbb{Z} \oplus \eta_2 \mathbb{Z}$)) and $|\eta_1 \mathbb{Z}| = q \cdot e$

$|\eta_2 \mathbb{Z}| = \frac{q}{q_e} = h$ from where we can conclude that

$C = \eta_1 \mathbb{Z} \oplus \eta_2 \mathbb{Z} \simeq \mathbb{Z}_q \times \mathbb{Z}_h$. 
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$$\begin{pmatrix} -1 & -1 \\ e + 1 & e \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

and

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As $h|q$, it is clear that $C$ is cyclic iff $h = 1$. 
For $q = 1105$. There are exactly 5 codes in $\mathbb{Z}_{1105} \times \mathbb{Z}_{1105}$ up to translations and conjugation ($1105 = 5 \cdot 13 \cdot 17$). One of these codes is cyclic and the others are non-cyclic. These codes are given by:

- $C_1 = (1, -3)\mathbb{Z}_{1105} \oplus (0, 5)\mathbb{Z}_{1105}$ ($e = 1$)
- $C_2 = (1, -5)\mathbb{Z}_{1105} \oplus (0, 13)\mathbb{Z}_{1105}$ ($e = 2$)
- $C_3 = (1, -13)\mathbb{Z}_{1105} \oplus (0, 85)\mathbb{Z}_{1105}$ ($e = 6$)
- $C_4 = (1, -21)\mathbb{Z}_{1105} \oplus (0, 221)\mathbb{Z}_{1105}$ ($e = 10$)
- $C_5 = (23, 24)\mathbb{Z}_{1105}$ ($e = 23$)
More relevant results related to the Golomb-Welch conjecture

- $PL(2, e, e) \neq \emptyset$, $PL(n, 1, 2n + 1) \neq \emptyset$, $PL(3, 2) = \emptyset$, $PL(n, e) = \emptyset$ for $e \geq e_n$.
  

- $PL(n, e, q) = \emptyset$ for $3 \leq n \leq 5$, $e \geq n - 1$, $q \geq 2e + 1$ and for $n \geq 6$, $e \geq \frac{2n-3}{2\sqrt{2}} - \frac{1}{2}$
  

- $PL(n, 2, q) = \emptyset$ for $q = 13$, $q$ not divisible by a prime $4 + 1$, and $q = p^k$ with $p$
  prime, $p \neq 13$ and $p < \sqrt{n^2 + (n + 1)^2}$.
  

- $PL(3, e) = \emptyset$ for $e \geq 2$.
  

- $PL(4, e) = \emptyset$ for $e \geq 2$.
  
More relevant results related to the Golomb-Welch conjecture

- $PL(n, 2) = \emptyset$ for $5 \leq n \leq 12$ for linear codes


P. Horak, O. Grosek. A new approach towards the Golomb-Welch conjecture. preprint

Future work

- Approach the classification of $n$-dimensional perfect single-error-correcting Lee codes using the geometry of polyominoes and combinatorial arguments.

- Prove the non-existence of $e$-perfect Lee codes in some dimension $n > 3$ using these techniques.

- Construct quasi-perfect Lee codes and dense codes using this approach.
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  - S. Gravier, M. Mollard, C. Payan succeed for $n = 3$.


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Some references in polyominoes

Thanks for your attention!