Proper Generalized Decomposition for Linear and Non-Linear Stochastic Models

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1 Motivation
- Linear elliptic problem
- Probabilistic framework
- Stochastic Galerkin problem
- Stochastic Discretization

2 Proper Generalized Decomposition
- Optimal Decomposition
- Algorithms
- An example

3 Nonlinear problems : N-S
- Stochastic Navier-Stokes equations
- Discretization
- Results
Parametric model uncertainty:

- A model $\mathcal{M}$ involving uncertain input parameters $D$
- Treat uncertainty in a probabilistic framework: $D(\theta) \in (\Theta, \Sigma, d\mu)$
- Assume $D = D(\xi(\theta))$, where $\xi \in \mathbb{R}^N$ with known probability law

The model solution is stochastic and solves:

$$\mathcal{M}(U(\xi); D(\xi)) = 0 \quad \text{a.s.}$$

Uncertainty in the model solution:

- $U(\xi)$ can be high-dimensional
- $U(\xi)$ can be analyzed by sampling techniques, solving multiple deterministic problems (e.g. MC)
- We would like to construct a functional approximation of $U(\xi)$

$$U(\xi) \approx \sum_k u_k \psi_k(\xi)$$
Consider the deterministic linear scalar elliptic problem (in $\Omega$)

Find $u \in V$ s.t. : $a(u, v) = b(v), \quad \forall v \in V$

where

$$a(u, v) \equiv \int_{\Omega} k(x) \nabla u(x) \cdot \nabla v(x) \, dx \quad \text{(bilinear form)}$$

$$b(v) \equiv \int_{\Omega} f(x) v(x) \, dx \quad \text{(linear form)}$$

$\epsilon < k(x)$ and $f(x)$ given \hspace{1cm} \text{(problem data)}

$V (= H^1_0(\Omega))$ deterministic space \hspace{1cm} \text{(vector space)}.
Stochastic elliptic problem:

- Conductivity $k$, source field $f$ (and BCs) uncertain
- Considered as random:
- Probability space $(\Theta, \Sigma, d\mu)$:

$$
\mathbb{E}[h] \equiv \int_{\Theta} h(\theta) d\mu(\theta), \quad h \in L^2(\Theta, d\mu) \implies \mathbb{E}[h^2] < \infty.
$$

- Assume $0 < \epsilon_0 \leq k$ a.e. in $\Theta \times \Omega$, $k(\mathbf{x}, \cdot) \in L^2(\Theta, d\mu)$ a.e. in $\Omega$ and $f \in L^2(\Omega, \Theta, d\mu)$

Variational formulation:

Find $U \in V \otimes L^2(\Theta, d\mu)$ s.t.

$$
A(U, V) = B(V) \quad \forall V \in V \otimes L^2(\Theta, d\mu),
$$

where $A(U, V) \doteq \mathbb{E}[a(U, V)]$ and $B(V) \doteq \mathbb{E}[b(V)]$. 
Stochastic Galerkin problem

Stochastic expansion:

- Let \( \{\psi_0, \psi_1, \psi_2, \ldots \} \) be an orthonormal basis of \( L^2(\Theta, d\mu) \)
- \( W \in V \otimes L^2(\Theta, d\mu) \) has for expansion

\[
W(x, \theta) = \sum_{\alpha=0}^{+\infty} w_\alpha(x) \psi_\alpha(\theta), \quad w_\alpha(x) \in V
\]

- Truncated expansion:

\[
\text{span } \{\psi_0, \ldots, \psi_P\} = S^P \subset L^2(\Theta, d\mu)
\]

- Galerkin problem:

Find \( U \in V \otimes S^P \) s.t.

\[
A(U, V) = B(V) \quad \forall V \in V \otimes S^P,
\]

with \( U = \sum_{\alpha=0}^{P} u_\alpha \psi_\alpha \) and \( V = \sum_{\alpha=0}^{P} v_\alpha \psi_\alpha \)
Stochastic Galerkin problem

Stochastic expansion:

- Let \( \{\psi_0, \psi_1, \psi_2, \ldots\} \) be an orthonormal basis of \( L^2(\Theta, d\mu) \)
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\]

- Truncated expansion:

\[
\text{span} \{\psi_0, \ldots, \psi_P\} = S^P \subset L^2(\Theta, d\mu)
\]

- Galerkin problem:

Find \( \{u_0, \ldots, u_P\} \) s.t. for \( \beta = 0, \ldots, P \)

\[
\sum_{\alpha} a_{\alpha,\beta}(u_\alpha, v_\beta) = b_\beta(v_\beta), \quad \forall v_\beta \in V
\]

with \( a_{\alpha,\beta}(u, v) := \int_{\Omega} \mathbb{E}[k \psi_\alpha \psi_\beta] \nabla u \cdot \nabla v dx \),

\( b_\beta(v) := \int_{\Omega} \mathbb{E}[f \psi_\beta] v(x) dx \).
Stochastic Galerkin problem

**Stochastic expansion :**

- Let \( \{ \psi_0, \psi_1, \psi_2, \ldots \} \) be an orthonormal basis of \( L^2(\Theta, d\mu) \)
- \( W \in V \otimes L^2(\Theta, d\mu) \) has for expansion

\[
W(x, \theta) = \sum_{\alpha=0}^{+\infty} w_\alpha(x) \psi_\alpha(\theta), \quad w_\alpha(x) \in V
\]

- **Truncated expansion :**

\[
\text{span} \{ \psi_0, \ldots, \psi_P \} = \mathcal{S}^P \subset L^2(\Theta, d\mu)
\]

- **Galerkin problem :**

Find \( \{ u_0, \ldots, u_P \} \) s.t. for \( \beta = 0, \ldots, P \)

\[
\sum_\alpha a_{\alpha,\beta}(u_\alpha, v_\beta) = b_\beta(v_\beta), \quad \forall v_\beta \in V
\]

Large system of coupled linear problem, **globally SDP.**
Stochastic parametrization

- Parameterization using $N$ independent $\mathbb{R}$-valued r.v.
  \[ \xi(\theta) = (\xi_1 \cdots \xi_N) \]
- Let $\Xi \subseteq \mathbb{R}^N$ be the range of $\xi(\theta)$ and $p_\xi$ its pdf
- The problem is solved in the image space $(\Xi, \mathcal{B}(\Xi), p_\xi)$

\[
U(\theta) \equiv U(\xi(\theta)) \quad \text{Stochastic basis : } \psi_\alpha(\xi)
\]

- Spectral polynomials (Hermite, Legendre, Askey scheme, …)
  
  [Ghanem and Spanos, 1991], [Xiu and Karniadakis 2001]

- Piecewise continuous polynomials (Stochastic elements, multiwavelets, …)
  
  [Deb et al, 2001], [olm et al, 2004]

- Truncature w.r.t. polynomial order : advanced selection strategy
  
  [Nobile et al, 2010]

Size of $\dim S^p$ - Curse of dimensionality
Stochastic Galerkin solution

\[ U(x, \xi) \approx \sum_{\alpha=0}^{P} u_{\alpha}(x)\psi_{\alpha}(\xi) \]

Find \( \{ u_0, \ldots u_P \} \) s.t. \( \sum_{\alpha} a_{\alpha,\beta}(u_{\alpha}, v_{\beta}) = b_{\beta}(v_{\beta}), \forall v_{\beta}=0,\ldots, P \in V \)

- A priori selection of the subspace \( S^P \)
- Is the truncature / selection of the basis well suited ?
- Size of the Galerkin problem scales with \( P + 1 \) : iterative solver
- Memory requirements may be an issue for large bases

Paradigm :

- Decouple the modes computation (smaller size problems, complexity reduction)
- Use reduced basis representation : find important components in \( U \) (reduce complexity and memory requirements)

Proper Generalized Decomposition

* Also GSD : Generalized Spectral Decomposition
Motivation
- Linear elliptic problem
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- An example

Nonlinear problems: N-S
- Stochastic Navier-Stokes equations
- Discretization
- Results
The *m*-terms PGD approximation of $U$ is 

$$U(x, \theta) \approx U^m(x, \theta) = \sum_{\alpha=1}^{m<P} u_\alpha(x) \lambda_\alpha(\theta), \quad \lambda_\alpha \in S^p, \; u_\alpha \in V.$$ 

separated representation

**Interpretation:** $U$ is approximated on

- the stochastic reduced basis $\{\lambda_1, \ldots, \lambda_m\}$ of $S^p$
- the deterministic reduced basis $\{u_1, \ldots, u_m\}$ of $V$

none of which is selected *a priori*

The questions are then:

- how to **define** the (deterministic or stochastic) reduced basis?
- how to **compute** the reduced basis and the *m*-terms PGD of $U$?
Optimal Decomposition

Optimal $L_2$-spectral decomposition :

$$U^m(x, \theta) = \sum_{\alpha=1}^{m} u_\alpha(x) \lambda_\alpha(\theta) \text{ minimizes } \mathbb{E} \left[ \| U^m - U \|_{L^2(\Omega)}^2 \right]$$

The modes $u_\alpha$ are the $m$ dominant eigenvectors of the kernel $\mathbb{E} [U(x, \cdot) U(y, \cdot)]$ :

$$\int_{\Omega} \mathbb{E} [U(x, \cdot) U(y, \cdot)] u_\alpha(y) dy = \beta u_\alpha(x), \quad \| u_\alpha \|_{L^2(\Omega)} = 1.$$ 

The modes are orthonormal :

$$\lambda_\alpha(\theta) = \int_{\Omega} U(x, \theta) u_\alpha(x) dx$$

However $U(x, \theta)$, so $\mathbb{E} [u(x, \cdot) u(y, \cdot)]$ is not known !
Optimal \( L_2 \)-spectral decomposition:

\[
U^m(x, \theta) = \sum_{\alpha=1}^{m} u_\alpha(x) \lambda_\alpha(\theta) \text{ minimizes } \mathbb{E} \left[ \| U^m - U \|_{L^2(\Omega)}^2 \right]
\]

- Solve the Galerkin problem in \( V^h \otimes S^P' \lt P \) to construct \( \{ u_\alpha \} \), and then solve for the \( \{ \lambda_\alpha \in S^P \} \).

- Solve the Galerkin problem in \( V^H \otimes S^P \) to construct \( \{ \lambda_\alpha \} \), and then solve for the \( \{ u_\alpha \in V^h \} \) with \( \text{dim } V^H \ll \text{dim } V^h \).

See works by groups of Ghanem and Matthies.
Alternative definition of optimality

$A(\cdot, \cdot)$ is symmetric positive definite, so $U$ minimizes the energy functional

$$J(V) \equiv \frac{1}{2} A(V, V) - B(V)$$

We define $U_m$ through

$$J(U_m) = \min_{\{u_\alpha\}, \{\lambda_\alpha\}} J \left( \sum_{\alpha=1}^{m} u_\alpha \lambda_\alpha \right).$$

- Equivalent to minimizing a Rayleigh quotient
- **Optimality w.r.t the $A$-norm** (change of metric) :

$$\|V\|_A^2 = \mathbb{E} [a(V, V)] = A(V, V)$$
Sequential construction:
For $i = 1, 2, 3 \ldots$

$$J(\lambda_i u_i) = \min_{v \in V, \beta \in S^p} J \left( \beta v + \sum_{j=1}^{i-1} \lambda_j u_j \right) = \min_{v \in V, \beta \in S^p} J (\beta v + U^{i-1})$$

The optimal couple $(\lambda_i, u_i)$ solves simultaneously

- **a) deterministic problem**
  
  $$u_i = D(\lambda_i, U^{i-1})$$
  
  $$A(\lambda_i u_i, \lambda_i v) = B(\lambda_i v) - A(U^{i-1}, \lambda_i v), \quad \forall v \in V$$

- **b) stochastic problem**
  
  $$\lambda_i = S(u_i, U^{i-1})$$
  
  $$A(\lambda_i u_i, \beta u_i) = B(\beta u_i) - A(U^{i-1}, \beta u_i), \quad \forall \beta \in S^p$$
Optimal Decomposition

- **Deterministic problem:** \( u_i = D(\lambda_i, U^{i-1}) \)

\[
\int_{\Omega} E \left[ \lambda_i^2 k \nabla u_i \cdot \nabla v d\mathbf{x} \right] = E \left[ - \int_{\Omega} \lambda_i k \nabla U^{i-1} \cdot \nabla v d\mathbf{x} + \int_{\Omega} \lambda_i f v d\mathbf{x} \right], \quad \forall v.
\]

- **Stochastic problem:** \( \lambda_i = S(u_i, U^{i-1}) \)

\[
E \left[ \lambda_i \beta \int_{\Omega} k \nabla u_i \cdot \nabla u_i d\mathbf{x} \right] = E \left[ - \beta \int_{\Omega} k \nabla U^{i-1} \cdot \nabla u_i d\mathbf{x} + \int_{\Omega} f u_i d\mathbf{x} \right], \quad \forall \beta.
\]
Properties:

- The couple \((\lambda_i, u_i)\) is a fixed-point of:

  \[
  \lambda_i = S \circ D(\lambda_i, \cdot), \quad u_i = D \circ S(u_i, \cdot)
  \]

- Homogeneity property:

  \[
  \frac{\lambda_i}{c} = S(cu_i, \cdot), \quad \frac{u_i}{c} = D(c\lambda_i, \cdot), \quad \forall c \in \mathbb{R} \setminus \{0\}.
  \]

  \Rightarrow \text{arbitrary normalization of one of the two elements.}

  Algorithms inspired from dominant subspace methods
  Power-type, Krylov/Arnoldi, . . .
Power Iterations

1. Set $l = 1$
2. initialize $\lambda$ (e.g. randomly)
3. While not converged, repeat (power iterations)
   a) Solve: $u = D(\lambda, U^{l-1})$
   b) Normalize $u$
   c) Solve: $\lambda = S(u, U^{l-1})$
4. Set $u_l = u$, $\lambda_l = \lambda$
5. $l \leftarrow l + 1$, if $l < m$ repeat from step 2

Comments:

- Convergence criteria for the power iterations (subspace with dim $> 1$ or clustered eigenvalues) [Nouy, 2007,2008]
- Usually few (4 to 5) inner iterations are sufficient
Power Iterations with Update

1. Same as Power Iterations, but after \((u_l, \lambda_l)\) is obtained (step 4) update of the stochastic coefficients:
   - Orthonormalize \(\{u_1, \ldots, u_l\}\) (optional)
   - Find \(\{\lambda_1, \ldots, \lambda_l\}\) s.t.
     \[
     A \left( \sum_{i=1}^{l} u_i \lambda_i, \sum_{i=1}^{l} u_i \beta_i \right) = B \left( \sum_{i=1}^{l} u_i \beta_i \right), \quad \forall \beta_{i=1,\ldots,l} \in \times S^P
     \]

2. Continue for next couple

Comments:
- Improves the convergence
- **Low dimensional stochastic linear system** \((l \times l)\)
- Cost of update increases linearly with the order \(l\) of the reduced representation
Arnoldi (Full Update version)

1. Set \( l = 0 \)
2. Initialize \( \lambda \in S^p \)
3. For \( l' = 1, 2, \ldots \) (Arnoldi iterations)
   - Solve deterministic problem \( u' = \mathcal{D}(\lambda, U^l) \)
   - Orthogonalize: \( u_{l+l'} = u' - \sum_{j=1}^{l+l'-1} (u', u_j)_{\Omega} \)
   - If \( \|u_{l+l'}\|_{L^2(\Omega)} \leq \epsilon \) or \( l + l' = m \) then break
   - Normalize \( u_{l+l'} \)
   - Solve \( \lambda = \mathcal{S}(u_{l'}, U^l) \)
4. \( l \leftarrow l + l' \)
5. Find \( \{\lambda_1, \ldots, \lambda_l\} \) s.t. (Update)
   \[
   A \left( \sum_{i=1}^{l} u_i \lambda_i, \sum_{i=1}^{l} u_i \beta_i \right) = B \left( \sum_{i=1}^{l} u_i \beta_i \right), \quad \forall \beta_{i=1,\ldots,l} \in S^p
   \]
6. If \( l < m \) return to step 2.
Summary

- Resolution of a sequence of deterministic elliptic problems, with elliptic coefficients $\mathbb{E} [\lambda^2 k]$ and modified (deflated) rhs
  
  dimension is $\dim V^h$

- Resolution of a sequence of linear stochastic equations
  
  dimension is $\dim S^P$

- Update problems: system of linear equations for stochastic random variables

  dimension is $m \times \dim S^P$

- To be compared with the Galerkin problem dimension

  $\dim V^h \times \dim S^P$

Weak modification of existing (FE/FV) codes
(weakly intrusive)
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Example definition

- Rectangular domain $25,000 \times 695$ (m)
- 4 Geological layers: D (Dogger), C (Clay), L (Limestone) and M (Marl)
Test case definition (cont.) : uncertain Dirichlet boundary conditions

<table>
<thead>
<tr>
<th>Δ Head (m)</th>
<th>Expectation</th>
<th>Range</th>
<th>distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Δh_{1,2}</td>
<td>+51</td>
<td>±10</td>
<td>Uniform</td>
</tr>
<tr>
<td>Δh_{1,3}</td>
<td>+21</td>
<td>±5</td>
<td>Uniform</td>
</tr>
<tr>
<td>Δh_{1,6}</td>
<td>-3</td>
<td>±2</td>
<td>Uniform</td>
</tr>
<tr>
<td>Δh_{2,5}</td>
<td>-110</td>
<td>±10</td>
<td>Uniform</td>
</tr>
<tr>
<td>Δh_{3,4}</td>
<td>-160</td>
<td>±20</td>
<td>Uniform</td>
</tr>
</tbody>
</table>

Heads at boundaries are taken independent
Example definition (cont.) : Uncertain conductivities

<table>
<thead>
<tr>
<th>Layer</th>
<th>$k_i$ median</th>
<th>$k_i$ min</th>
<th>$k_i$ max</th>
<th>distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dogger</td>
<td>25</td>
<td>5</td>
<td>125</td>
<td>LogUniform</td>
</tr>
<tr>
<td>Clay</td>
<td>$3 \times 10^{-6}$</td>
<td>$3 \times 10^{-7}$</td>
<td>$3 \times 10^{-5}$</td>
<td>LogUniform</td>
</tr>
<tr>
<td>Limestone</td>
<td>6</td>
<td>1.2</td>
<td>30</td>
<td>LogUniform</td>
</tr>
<tr>
<td>Marl</td>
<td>$3 \times 10^{-5}$</td>
<td>$1 \times 10^{-5}$</td>
<td>$1 \times 10^{-4}$</td>
<td>LogUniform</td>
</tr>
</tbody>
</table>

Conductivities are taken independent

Parameterization

- 9 independent r.v. $\{\xi_1, \ldots, \xi_9\} \sim U[0, 1]^9$
- Stochastic space $S^P :$ Legendre polynomial up to order $N_0$
- $\text{dim } S^P = P + 1 = (9 + N_0)!/(9!N_0)!$
Deterministic discretization:

- $\mathcal{P} - 1$ finite-element
- Mesh conforming with the geological layers

- $N_e \approx 30,000$ finite elements
- $\dim(V^h) \approx 15,000$
- Dimension of Galerkin problem: $8.2 \times 10^5$ ($\text{No} = 2$), $3.3 \times 10^6$ ($\text{No} = 3$)
Norm of Galerkin residual as a function of $m$ ($N_0 = 3$)
Norm of error (vs Galerkin) as a function of $m$ ($N_0 = 3$)
CPU times ($No = 3$)
CPU times of PGD algorithms \((\text{No} = 3)\)
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Extension to general problems

- **Non-definite / non symmetric linear and nonlinear problems**: no optimality results
  
  [Nouy, 2008]

- **Algorithms**: from numerical experiments methods & algorithms can be safely applied
  
  [Nouy & olm, 2009]

- Illustration on the **Incompressible Navier-Stokes equations**
Test problem: steady, two dimensional, Newtonian fluid in a square domain $\Omega$:

\[
U \nabla U + \nabla P - \nu \nabla^2 U = F, \quad \text{in } \Omega
\]
\[
\nabla \cdot U = 0, \quad \text{in } \Omega
\]
\[
U = 0, \quad \text{on } \partial \Omega.
\]

- $U$ the velocity field,
- $P$ the pressure field,
- uncertainty in the viscosity $\nu = \nu(\theta)$ and forcing $F = F(x, \theta)$. 
Functional spaces:
$L^2(\Omega)$ the space of functions that are square integrable over $\Omega$, equipped with the inner product and norm

$$(p, q)_\Omega := \int_\Omega pq \, d\Omega, \quad \|q\|_{L^2(\Omega)} = (q, q)_\Omega^{1/2}.$$

Extended to vector valued functions by summation over vectors components, and to the constrained space

$L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q \, d\Omega = 0 \right\}.$

$H^1(\Omega)$, Sobolev space of vector valued functions with all components and their first derivatives being square integrable over $\Omega$, and $H_0^1(\Omega)$ the constrained space of such vector functions vanishing on $\partial \Omega$,

$H_0^1(\Omega) = \left\{ v \in H^1(\Omega), \ v = 0 \text{ on } \partial \Omega \right\}.$
Weak formulation:
Find $U \in H^1_0(\Omega) \otimes S$ and $P \in L^2_0(\Omega) \otimes S$, such that

$$C(U, U, V) + D(V, P) + A(U, V) = \mathbb{E}[(F, V)_\Omega] \quad \forall V \in H^1_0(\Omega) \otimes S,$$

$$D(U, Q) = 0 \quad \forall Q \in L^2_0(\Omega) \otimes S.$$

where:

- $C(U, V, W) \doteq \mathbb{E}[c(U, V, W)] = \mathbb{E}[\int_\Omega (U \nabla V) \cdot W \, dx]$
- $D(V, Q) \doteq \mathbb{E}[d(V, Q)] = \mathbb{E}[\int_\Omega Q \nabla V \, dx]$
- $A(U, V) \doteq \mathbb{E}[\nu a(U, V)] = \mathbb{E}[\nu \int_\Omega \nabla V : \nabla V \, dx]$
Weak formulation in (weakly) Divergence Free space: Let $H_{0; \text{div}}^1(\Omega) \otimes \mathcal{S}$

$$H_{0; \text{div}}^1(\Omega) = \{ \mathbf{v} \in H_0^1(\Omega), \; d(\mathbf{v}, q) = 0 \; \forall q \in L_0^2(\Omega) \}.$$  

The problem becomes:

Find $\mathbf{U} \in H_{0; \text{div}}^1(\Omega) \otimes \mathcal{S}$ such that

$$C(\mathbf{U}, \mathbf{U}, \mathbf{V}) + A(\mathbf{U}, \mathbf{V}) = \mathbb{E}[(\mathbf{F}, \mathbf{V})_\Omega] \quad \forall \mathbf{V} \in H_{0; \text{div}}^1(\Omega) \otimes \mathcal{S}. $$

- Remove the pressure
- Difficulties in the discretization of $H_{0; \text{div}}^1(\Omega)$
Stochastic Navier-Stokes equations

PGD approximation in $H_{0;\text{div}}^1(\Omega)$:

$$U(x, \theta) \approx U^m(x, \theta) = \sum_{k=1}^{m} u_k(x) \lambda_k(\theta),$$

- The deterministic functions $u_k \in H_{0;\text{div}}^1(\Omega)$ form a reduced basis of $H_{0;\text{div}}^1(\Omega)$
- The stochastic coefficients $\lambda_k \in S$ form a reduced basis of $S$
- None of the two reduced bases are selected *a priori*
- Sequential construction of the approximation $U^0 \to U^1 \to U^2 \to \cdots$ with Arnoldi algorithm
- $U^{m+1} = U^m + \lambda u$ : successive couples $(\lambda, u)$ are defined as previously
The couple \((\lambda_{m+1}, u_{m+1})\) is sought as the fixed point of

\[
\lambda = S \circ D(\lambda; U^m), \quad u = D \circ S(u; U^m),
\]

where

**Deterministic problem:** \(u = D(\lambda; U^m) \in H^1_{0; \text{div}}(\Omega)\) is the solution of

\[
C(\lambda u, \lambda u, \lambda v) + C(\lambda u, U^m, \lambda v) + C(U^m, \lambda u, \lambda v) + A(\lambda u, \lambda v) = \mathbb{E}[(F, \lambda v)_{\Omega}] - C(U^m, U^m, \lambda v) - A(U^m, \lambda v), \quad \forall v \in H^1_{0; \text{div}}(\Omega).
\]

**Stochastic problem:** \(\lambda = S(u; U^m) \in S\) is the solution of

\[
C(\lambda u, \lambda u, \beta u) + C(\lambda u, U^m, \beta u) + C(U^m, \lambda u, \beta u) + A(\lambda u, \beta u) = \mathbb{E}[(F, \beta u)_{\Omega}] - C(U^m, U^m, \beta u) - A(U^m, \beta u), \quad \forall \beta \in S.
\]
The couple \((\lambda_{m+1}, u_{m+1})\) is sought as the fixed point of

\[
\lambda = S \circ D(\lambda; U^m), \quad u = D \circ S(u; U^m),
\]

where

**Deterministic problem** : \(u = D(\lambda; U^m) \in H^1_{0;\text{div}}(\Omega)\) is the solution of

\[
\mathbb{E} \left[ \lambda^3 \right] c(u, u, v) + \mathbb{E} \left[ \lambda^2 \nu \right] a(u, v) + c(u, \tilde{u}, v) + c(\tilde{u}, u, v) = (\tilde{f}, v)_\Omega, \quad \forall v \in H^1_{0;\text{div}}(\Omega),
\]

with \(\tilde{u} = \mathbb{E} \left[ \lambda^2 U^m \right]\)

**Stochastic problem** : \(\lambda = S(u; U^m) \in S\) is the solution of

\[
\lambda = S(u; U^m) \iff \mathbb{E} \left[ (\tilde{c}\lambda^2 + G\lambda + R)\beta \right] = 0, \quad \forall \beta \in S,
\]

with \(\tilde{c} \in \mathbb{R}\) and \(G, R \in S\).
**Stochastic Navier-Stokes equations**

**Update problem:** fix one of the two reduced bases and solve the Galerkin problem for the other component.

**Updating of the stochastic coefficients:**

Find \( \{ \lambda_k \in S \}_{k=1}^m \) such that

\[
C \left( \sum_i \lambda_i u_i, \sum_j \lambda_j u_j, \beta u_k \right) + A \left( \sum_i \lambda_i u_i, \beta u_k \right) = \mathbb{E} \left[ (F, \beta u_k) \right],
\]

\( \forall \beta \in S \) and \( k = 1, \ldots, m \).

**System of \( m \) quadratic stochastic equations:**

\[
\mathbb{E} \left[ \left( \sum_{i,j=1}^m \tilde{q}_{k,i,j} \lambda_i \lambda_j + \sum_{i=1}^m G_{k,i} \lambda_i - R_k \right) \beta_k \right] = 0, \quad \forall \beta_k \in S.
\]
Stochastic discretization:

- **Parametrization** of $\nu(\theta)$ and $F(\theta)$ using $N$ i.i.d. random variables:
  \[
  \xi = \{\xi_1, \ldots, \xi_N\}
  \]

- **Stochastic basis**: Polynomial Chaos
  \[
  \lambda(\theta) = \sum_{\alpha} \lambda_{\alpha} \Psi_{\alpha}(\xi(\theta)),
  \]

  where the $\Psi$s are orthonormal polynomials

  \[
  \mathbb{E} [\Psi_{\alpha}(\xi) \Psi_{\beta}(\xi)] = \delta_{\alpha,\beta}.
  \]

- **Truncature** to (total) polynomial degree $No$:
  \[
  \dim S^p = \frac{(No + N)!}{No!N!}.
  \]
Spatial discretization

- $P_n$-$P_{n-2}$ Spectral Element Method

$$H^1(\Omega) \ni u(x) \approx u^h(x) = \sum_{i,j=1}^{n} u_{i,j} \phi_{i,j}(x) \in V^h,$$

$$L^2(\Omega) \ni p(x) \approx p^h(x) = \sum_{i,j=1}^{n-2} p_{i,j} \phi_{i,j}(x) \in \Pi^h.$$

(Nodal basis on tensorized Gauss-Lobatto grid)

- Resolution of the deterministic problem

$$\mathbb{E} [\lambda^3] c(u^h, u^h, v^h) + \mathbb{E} [\lambda^2 \nu] a(u^h, v^h) + c(u^h, \tilde{u}^h, v^h) + c(\tilde{u}^h, u^h, v^h) + d(v^h, p^h) = (\tilde{f}, v^h)_\Omega$$

$$d(u^h, q^h) = 0 \quad \forall v^h \in V^h_0, \quad \forall q \in \Pi^h.$$
Case of a deterministic forcing and a random (Log-normal) viscosity :

\[ \nu(\theta) = \frac{1}{200} \exp \left( \frac{\sigma_N}{\sqrt{N}} \sum_{i=1}^{N} \xi_i(\theta) \right) \left( +10^{-4} \right), \quad \xi_i \sim N(0, 1) \text{ i.i.d.} \]

Same problem but for parametrization involving N Gaussian R.V.
Galerkin solution for $N = 1$ and $N_0 = 10$ (Wiener-Hermite expansion)

Mean and standard deviation of $U^G$ rotational.
Results

POD modes of the Galerkin solution for \( N = 1 \) and \( N_\theta = 10 \)

Rotational of the spatial POD modes of \( U^G \).
First PGD-Arnoldi modes for $N = 1$ and $N_0 = 10$
POD modes of the PGD solution for $m = 15$, $N = 1$ and $N_0 = 10$

Rotational of the spatial POD modes of $U^{m=15}$.
Convergence of PGD solution $N = 1$ and $N_0 = 10$

Comparison of the norms of the POD coefficients at $m = 15$ (left), residual norm (center), error norm (right).
Convergence of PGD solutions $N = 1, 2$ and $3$ at $No = 10$

Comparison of the norms of the POD coefficients at $m = 15$ (left), residual norm (center), error norm (right).

PGD captures the essential features of the stochastic solution
Stochastic forcing $F$: Hodge's decomposition

\[ F(x, \theta) = \nabla \Phi(x, \theta) + \nabla \wedge \Psi(x, \theta) = \nabla \wedge \Psi(x, \theta)k \]

Define $F_\omega \doteq (\nabla \wedge F) \cdot k = -\triangle \psi$ and assume:

\[ F_\omega(x, \theta) = f_\omega^0 + F'_\omega(x, \theta), \quad \mathbb{E}[F'_\omega(x, \cdot)] = 0, \quad F'_\omega(x, \cdot) \sim N(0, \sigma^2), \]
\[ \mathbb{E}[F'_\omega(x, \cdot)F'_\omega(x', \cdot)] = \sigma^2 \exp(-|x - x'|/L). \]

KL expansion of $F_\omega$:

\[ F_\omega(x, \theta) \approx F^N_\omega(x, \xi(\theta)) = f_\omega^0 + \sum_{k=0}^{N} \sqrt{\gamma_k} f^k_\omega(x) \xi_k(\theta) \]

It comes

\[ F(x, \theta) \approx F^N(x, \xi(\theta)) = f^0 + \sum_{k=0}^{N} \sqrt{\gamma_k} f^k(x) \xi_k(\theta), \]

with $f^i = \nabla \wedge (\triangle^{-1} f^i_\omega)k$. 
### KL modes of the forcing:

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Forcing modes for $L = 1$, $\sigma/f_\omega^0 = 0.2$
**Results**

**PGD solution** for $\bar{\nu} = 1/50$, $N = 11$, $No = 3$ and $m = 45$

Mean and standard deviation of $\nabla \otimes U^{m=45}$
First PGD-Arnoldi modes
Results at $\nu = 1/50$ : $N_0 = 3$, $N = 11$, $P = 364$

Residual (left), $\| U^m - U^G \|$ (center) and norm of POD modes for $m = 45$ (right).
Results for $\bar{\nu} = 1/50$: No = 3, N = 11, P = 364

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$m = 26$  
$m = 43$  
$m = 45$

Spatial distribution of the stochastic residual.
Residual computation:

- Residual computation in $H_{0,\text{div}}^1(\Omega)$ is difficult.
- Projection of the $H_0^1(\Omega)$-residual into $H_{0,\text{div}}^1(\Omega)$ amounts to solving a stochastic linear equation for $P^m$.
- PGD can be used for the approximation of $P^m$.
- Re-use of the reduced basis of Arnoldi Lagrange multipliers $\{p_i\}$ associated to $\{u_i\}$ was found an effective approach.
- However, monitoring convergence using the decay of $||\lambda_i||$ appears effective.
Residual computation:

Comparison of different error measures of the PGD solution at $\nu = 1/10$, 1/50 and 1/100 (from left to right).
On-going work:

- Evolution problems: PGD on each time step or global PGD with possibly alternative separated forms

\[
u(x, t, \theta) \approx \sum_{\alpha} u_{\alpha}(x, t) \lambda_{\alpha}(\theta) \approx \sum_{\alpha} \tilde{u}_{\alpha}(x) \beta_{\alpha}(t, \theta) \\
\approx \sum_{\alpha} \hat{u}_{\alpha}(x) \tau_{\alpha}(t) \gamma_{\alpha}(\theta)
\]

- Nested separation?

- Other definition of optimality for non definite linear operators (least square residual minimization).
Thanks for your attention

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