Adaptive Wavelet Methods for SPDEs: Theoretical Analysis and Practical Realization

Stephan Dahlke

FB 12 Mathematics and Computer Sciences
Philipps–Universität Marburg

Workshop on Numerical Analysis of Multiscale Problems & Stochastic Modelling
December 12–16, 2011

Outline

Motivation

Theoretical Analysis
  Does Adaptivity Pay?
  The Model Equation
  SPDEs in Weighted Sobolev Spaces
  Besov Regularity

Practical Realization
  Discretization Scheme
  The Noise Model
  Stochastic Elliptic Equations
Outline

**Motivation**

Theoretical Analysis
- Does Adaptivity Pay?
- The Model Equation
- SPDEs in Weighted Sobolev Spaces
- Besov Regularity

Practical Realization
- Discretization Scheme
- The Noise Model
- Stochastic Elliptic Equations
Motivation: 

- Numerical treatment of SPDEs:

\[ du(t) = (A(u(t)) + F(t, u(t)))dt + \Sigma(t, u(t))dW_t, \]

in \( \mathcal{O} \subseteq \mathbb{R}^d \), bounded Lipschitz.
Motivation:

- Numerical treatment of SPDEs:

\[ du(t) = (A(u(t)) + F(t, u(t)))dt + \Sigma(t, u(t))dW_t, \]

in \( \mathcal{O} \subseteq \mathbb{R}^d \), bounded Lipschitz.

- Computational finance, epidemiology, population genetics...
Motivation:

- Numerical treatment of SPDEs:
  \[ du(t) = (A(u(t)) + F(t, u(t)))dt + \Sigma(t, u(t))dW_t, \]
  in \( \mathcal{O} \subseteq \mathbb{R}^d \), bounded Lipschitz.

- Computational finance, epidemiology, population genetics...

- as usual:
  - much is known concerning existence and uniqueness...
  - but how does the solution look like?
  - numerical approximation scheme needed!
Numerical Approaches:

- First natural idea: classical nonadaptive schemes.
Numerical Approaches:

- First natural idea: classical nonadaptive schemes.
  - based on uniform space/time refinements
  - approximation spaces/degrees of freedom a priori fixed
Numerical Approaches:

- First natural idea: classical **nonadaptive** schemes.
  - based on **uniform** space/time refinements
  - approximation spaces/degrees of freedom a priori fixed
  - ‘easy’ to implement/analyze
  - **but**: convergence might be slow!
Numerical Approaches:

- First natural idea: classical nonadaptive schemes.
  - based on uniform space/time refinements
  - approximation spaces/degrees of freedom a priori fixed
  - ‘easy’ to implement/analyze
  - but: convergence might be slow!

- Alternative: use adaptive schemes!
Numerical Approaches:

- First natural idea: classical nonadaptive schemes.
  - based on uniform space/time refinements
  - approximation spaces/degrees of freedom a priori fixed
  - ‘easy’ to implement/analyze
  - but: convergence might be slow!

- Alternative: use adaptive schemes!
  - nonuniform space/time refinements
  - updating strategy
  - degrees of freedom adjusted to the unknown solution
Numerical Approaches:

- First natural idea: classical nonadaptive schemes.
  - based on uniform space/time refinements
  - approximation spaces/degrees of freedom a priori fixed
  - ‘easy’ to implement/analyze
  - but: convergence might be slow!

- Alternative: use adaptive schemes!
  - nonuniform space/time refinements
  - updating strategy
  - degrees of freedom adjusted to the unknown solution
  - a posteriori error estimator, refinement strategy....
  - heavy to implement/analyze
  - but convergence might be faster!
Wavelets:

- Multiresolution Analysis \( \{V_j\}_{j \geq 0} \)

\[
V_0 \subset V_1 \subset V_2 \subset \ldots \quad \bigcup_{j=0}^{\infty} V_j = L_2(\mathcal{O})
\]
Wavelets:

- Multiresolution Analysis \( \{ V_j \}_{j \geq 0} \)

\[
V_0 \subset V_1 \subset V_2 \subset \ldots \quad \bigcup_{j=0}^{\infty} V_j = L_2(\mathcal{O})
\]

- \( V_{j+1} = V_j \oplus W_{j+1} \quad V_0 = W_0 \quad L_2(\mathcal{O}) = \bigoplus_{j=0}^{\infty} W_j \)

- \( W_j = \text{span}\{\psi_{j,k}, \ k \in J_j\} \)
Wavelets:

- Multiresolution Analysis \( \{V_j\}_{j \geq 0} \)

\[
V_0 \subset V_1 \subset V_2 \subset \ldots \quad \bigcup_{j=0}^{\infty} V_j = L_2(\mathcal{O})
\]

- \( V_{j+1} = V_j \oplus W_{j+1} \)

\[
V_0 = W_0 \quad L_2(\mathcal{O}) = \bigoplus_{j=0}^{\infty} W_j
\]

- \( W_j = \text{span}\{\psi_{j,k}, \ k \in J_j\} \)

- \( \lambda = (j, k), \ |\lambda| = j, \ \mathcal{J} = \bigcup_{j=0}^{\infty} (\{j\} \times J_j) \)
Basic Properties:

- $\text{diam} \left( \text{supp} \psi_\lambda \right) \sim 2^{-|\lambda|}, \quad \lambda \in \mathcal{J}$
Basic Properties:

- \( \text{diam} (\text{supp} \psi_\lambda) \sim 2^{-|\lambda|}, \quad \lambda \in J \)

- \[
\int_{\mathcal{O}} x^\gamma \psi_\lambda(x) dx = 0, \quad |\gamma| \leq N
\]
Basic Properties:

- \( \text{diam} \left( \text{supp} \psi_\lambda \right) \sim 2^{-|\lambda|}, \quad \lambda \in \mathcal{J} \)

- \[ \int_{\Omega} x^\gamma \psi_\lambda(x) \, dx = 0, \quad |\gamma| \leq N \]

- \[
\| f \|_{B_q^s(L_p(\Omega))} \sim \left( \sum_{|\lambda| = j_0}^{\infty} 2^{s|\lambda|} \left( \sum_{\lambda \in \mathcal{J}, |\lambda| = j} |\langle f, \tilde{\psi}_\lambda \rangle|^p \right)^{q/p} \right)^{1/q},
\]

where \( \tilde{\Psi} = \{ \tilde{\psi}_\lambda : \lambda \in \mathcal{J} \} \) satisfies \( \langle \psi_\lambda, \tilde{\psi}_\nu \rangle = \delta_{\lambda, \nu} \).
Outline

Motivation

Theoretical Analysis
  Does Adapativity Pay?
  The Model Equation
  SPDEs in Weighted Sobolev Spaces
  Besov Regularity

Practical Realization
  Discretization Scheme
  The Noise Model
  Stochastic Elliptic Equations
General Question: Does Adaptivity Really Pay?

- nonadaptive schemes

\[ E_j(u) = \inf_{g \in V_j} \| u - g \|_{L^2(\Omega)} \lesssim 2^{-sj} |u|_{W^s_2(\Omega)} \]

\[ E_j(u) = O(n_j^{-s/d}) \iff u \in W^s_2(\Omega) \]

(linear approximation)
General Question: Does Adaptivity Really Pay?

- nonadaptive schemes

\[ E_j(u) = \inf_{g \in V_j} \| u - g \|_{L^2(\mathcal{O})} \lesssim 2^{-sj} |u|_{W^s_2(\mathcal{O})} \]

\[ E_j(u) = O(n_j^{-s/d}) \iff u \in W^s_2(\mathcal{O}) \]

(linear approximation)

- adaptive schemes

“ideal” algorithm: best \( n \)-term approximation

( nonlinear approximation)

\[ \sigma_n(u)_{L^2(\mathcal{O})} := \| u - g_n \|_{L^2(\mathcal{O})} \]

\[ g_n = \sum_{(j,k) \in \Lambda_n} d_{j,k} \psi_{j,k}, \quad \Lambda_n \equiv n \text{ biggest wavelet coefficients} \]

\[ \sigma_n(u)_{L^2(\mathcal{O})} = O(n^{-s/d}) \iff u \in B^{s}_\tau(L^\tau(\mathcal{O})) \quad \frac{1}{\tau} = \left( \frac{s}{d} + \frac{1}{2} \right) \]
General Question: Does Adaptivity Really Pay?

- nonadaptive schemes

\[ E_j(u) = \inf_{g \in V_j} \| u - g \|_{L^2(O)} \lesssim 2^{-sj} |u|_{W^s_2(O)} \]

\[ E_j(u) = O(n_j^{-s/d}) \iff u \in W^s_2(O) \]

(linear approximation)

- adaptive schemes

“ideal” algorithm: best \( n \)–term approximation

(nonlinear approximation)

\[ \sigma_n(u)_{L^2(O)} := \| u - g_n \|_{L^2(O)} \]

\[ g_n = \sum_{(j,k) \in \Lambda_n} d_{j,k} \psi_{j,k}, \quad \Lambda_n \equiv n \text{ biggest wavelet coefficients} \]

\[ \sigma_n(u)_{L^2(O)} = O(n^{-s/d}) \iff u \in B^s_\tau(L^\tau(O)) \frac{1}{\tau} = \left( \frac{s}{d} + \frac{1}{2} \right) \]

- Question: \( u \in B^s_\tau(L^\tau(O)), \; 0 < s < s^* \)?
The Model Equation:

- Fix $T \in (0, \infty)$ and $\mathcal{O} \subseteq \mathbb{R}^d$ bounded Lipschitz domain, $(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow$ probability space.
The Model Equation:

- Fix $T \in (0, \infty)$ and $\mathcal{O} \subseteq \mathbb{R}^d$ bounded Lipschitz domain, $(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow$ probability space.
- Stochastic evolution equation

\[
du(t) = \sum_{\mu,\nu=1}^{d} a^{\mu\nu} u_{x_{x\mu}x_{x\nu}} dt + \sum_{k=1}^{\infty} g^k(t) d\omega^k_t,
\]

- $(a^{\mu\nu})_{1 \leq \nu, \mu \leq d} \in \mathbb{R}^{d \times d}$ symmetric positive definite, the Brownian motions $\omega^k_t$ are independent.
The Model Equation:

- Fix $T \in (0, \infty)$ and $\mathcal{O} \subseteq \mathbb{R}^d$ bounded Lipschitz domain, $(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow $ probability space.
- Stochastic evolution equation

$$d u(t) = \sum_{\mu, \nu=1}^{d} a^{\mu \nu} u_{x_\mu x_\nu} \, dt + \sum_{k=1}^{\infty} g^k(t) \, dw^k_t,$$

- $(a^{\mu \nu})_{1 \leq \nu, \mu \leq d} \in \mathbb{R}^{d \times d}$ symmetric positive definite, the Brownian motions $w^k_t$ are independent.
- $\varphi \in C_0^\infty(\mathcal{O}) \Rightarrow$

$$\langle u(t, \omega), \varphi \rangle = \langle u_0(\omega), \varphi \rangle + \sum_{\nu, \mu=1}^{d} \int_{0}^{t} \langle a^{\mu \nu} u_{x_\mu x_\nu}(s, \omega), \varphi \rangle \, ds$$

$$+ \sum_{k=1}^{\infty} \int_{0}^{t} \langle g^k(s), \varphi \rangle \, dw^k_s(t, \omega) \quad \mathbb{P}\text{-a.s.}$$

$$= \text{Int}^{w^k} \left( \langle g^k, \varphi \rangle \right)(t, \cdot)$$

convergence w.r.t. $\| \cdot \|_{\mathcal{M}_T^{2, c}}$
Weighted Sobolev Spaces:

- \( \rho(x) := \text{dist}(x, \partial \mathcal{O}) \) for \( x \in \mathcal{O} \)

- \( \gamma \in \mathbb{N}_0; \theta \in \mathbb{R} \)

\[ u \in H_{2, \theta}^\gamma(\mathcal{O}) :\Leftrightarrow \]

\[ \|u\|^2_{H_{2, \theta}^\gamma(\mathcal{O})} := \sum_{\alpha \in \mathbb{N}_0^d, |\alpha| \leq \gamma} \int_{\mathcal{O}} |\rho(x)|^{\alpha} |D^\alpha u(x)|^2 \rho(x)^{\theta-d} \, dx \]

is finite.
Weighted Sobolev Spaces for Sequences:

\[
g = (g_k)_{k \in \mathbb{N}} \in H_{2,\theta}^\gamma (\mathcal{O}; \ell_2)
\]

\[\iff \]

\[
\|g\|_{H_{2,\theta}^\gamma (\mathcal{O}; \ell_2)}^2 := \sum_{\alpha \in \mathbb{N}_0^d, |\alpha| \leq \gamma} \int_\mathcal{O} \left( \rho |\alpha| (D^\alpha g_k)_{k \in \mathbb{N}} \right)_{\ell_2}^2 \rho^{\theta-d} \, dx
\]

is finite.
Weighted Sobolev Spaces for Sequences:

1. \( \gamma \in \mathbb{N}_0, \theta \in \mathbb{R} \)

2. \( g = (g^k)_{k \in \mathbb{N}} \in H^{\gamma}_{2, \theta}(\mathcal{O}; \ell_2) \)

3. \[ \|g\|_{H^{\gamma}_{2, \theta}(\mathcal{O}; \ell_2)}^2 := \sum_{\alpha \in \mathbb{N}_0^d} \int_{\mathcal{O}} (\rho |\alpha| |D^\alpha g^k|_{\ell_2})^2 \rho^{\theta - d} \, dx \]

   is finite.

4. \( \gamma \in \mathbb{R} \): \( H^{\gamma}_{2, \theta}(\mathcal{O}) \) and \( H^{\gamma}_{2, \theta}(\mathcal{O}; \ell_2) \)

5. by complex interpolation

6. see [Lototsky(2000)]
Weighted Sobolev Space for Stochastic Processes:

\[ H^{\gamma}_{2,\theta}(O, T) := L_2([0, T] \times \Omega, dt \otimes P; H^{\gamma}_{2,\theta}(O)) \]

\[ \|u\|^2_{H^{\gamma}_{2,\theta}(O,T)} = \int_{\Omega} \int_0^T \|u(t, \omega, \cdot)\|^2_{H^{\gamma}_{2,\theta}(O)} \, dt \, P(d\omega) \]
Weighted Sobolev Space for Stochastic Processes:

- $\gamma \in \mathbb{R}, \theta \in \mathbb{R}$

$$H^\gamma_{2,\theta}(\mathcal{O}, T) := L_2([0, T] \times \Omega, dt \otimes P; H^\gamma_{2,\theta}(\mathcal{O}))$$

$$\|u\|_{H^\gamma_{2,\theta}(\mathcal{O}, T)}^2 = \int_\Omega \int_0^T \|u(t, \omega, \cdot)\|_{H^\gamma_{2,\theta}(\mathcal{O})}^2 dt \, P(d\omega)$$

- $\gamma \in \mathbb{R}, \theta \in \mathbb{R}$ for sequences

$$H^\gamma_{2,\theta}(\mathcal{O}, T; \ell_2) := L_2([0, T] \times \Omega, dt \otimes P; H^\gamma_{2,\theta}(\mathcal{O}; \ell_2))$$
Existence/Uniqueness of Solutions in $\mathbb{H}_{2,\theta-2}^\gamma$:

$$\begin{align*}
du(t) &= \sum_{\mu,\nu=1}^{d} a^{\mu\nu} u_{x_\mu x_\nu} \, dt + \sum_{k=1}^{\infty} g^k(t) \, dw^k_t 
\end{align*}\quad (\bullet)$$

Theorem (KIM 2008)

$$\exists \kappa = \kappa(d, \mathcal{O}) \in (0, 1),$$

such that

$$\forall \theta \in (d - \kappa, d + \kappa),$$

(\bullet) has a unique solution in the class $\mathbb{H}_{2,\theta-2}^\gamma(\mathcal{O}, T)$, provided

$$(g^k)_{k \in \mathbb{N}} \in \mathbb{H}_{2,\theta}^{-1}(\mathcal{O}, T; \ell_2) \quad \& \quad u_0 \in L_2(\Omega; H_{2,\theta}^{-1}(\mathcal{O})).$$
**Besov Regularity:**

**Theorem (2010)**

*If $\gamma \in \mathbb{N}$ and*

$$u \in L_2([0, T] \times \Omega; W_2^s(\mathcal{O}))$$

*for some*

$$s \in \left(0, \gamma \wedge 1 + \frac{d - \theta}{2}\right),$$

*then*

$$u \in L_\tau([0, T] \times \Omega; B_{\tau, \tau}^\alpha(\mathcal{O})), \quad \frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{2},$$

*for all*

$$\alpha \in \left(0, \gamma \wedge \frac{s d}{d - 1}\right).$$
Example: \( d = 2, \theta = d, \gamma = 2 \):

- Suppose that 
  \[ (g^k)_{k \in \mathbb{N}} \in H^1_{2,2}(\mathcal{O}, T; \ell_2) \] and 
  \[ u_0 \in L^2(\Omega, H^1_{2,2}(\mathcal{O})). \]

Kim's theorem \Rightarrow \exists ! \text{solution} 

\[ H^\gamma_{2,2}, \theta - 2, \bigl(\mathcal{O}, T\bigr) \]

\Rightarrow \[ u \in L^2\left(\ldots; W^{1,2}_{\tau}(O)\right) \]

Our theorem \Rightarrow \[ u \in L^{\tau\tau}(\ldots; B^\alpha_{\tau, \tau}(O)), \quad 1 \tau = \frac{\alpha^2}{2} + 1 \]

Adaptivity completely justified!
Example: \( d = 2, \theta = d, \gamma = 2 \):

- Suppose that 
\[(g^k)_{k \in \mathbb{N}} \in \mathbb{H}^{1}_{2,2}(\mathcal{O}, T; \ell_2) \text{ and } u_0 \in L_2(\Omega, \mathbb{H}^{1}_{2,2}(\mathcal{O})).\]

- Then: Kim’s theorem \( \Rightarrow \exists! \) solution \( \mathbb{H}^{\gamma}_{2,\theta-2}(\mathcal{O}, T) \)

\[\text{Adaptivity completely justified!}\]
Example: $d = 2$, $\theta = d$, $\gamma = 2$:

- Suppose that $(g^k)_{k \in \mathbb{N}} \in \mathcal{H}^{1}_{2,2}(\mathcal{O}, T; \ell_2)$ and $u_0 \in L^2(\Omega, H^1_{2,2}(\mathcal{O}))$.

- Then: Kim’s theorem $\implies \exists!$ solution $\mathcal{H}^\gamma_{2,\theta-2}(\mathcal{O}, T)$

- $\Rightarrow u \in L^2(\ldots; W^1_2(\mathcal{O}))$

Adaptivity completely justified!
Example: $d = 2$, $\theta = d$, $\gamma = 2$:

- Suppose that $(g^k)_{k \in \mathbb{N}} \in \mathbb{H}^1_{2,2}(\mathcal{O}, T; \ell_2)$ and $u_0 \in L_2(\Omega, H^1_{2,2}(\mathcal{O}))$.

- Then: Kim’s theorem $\implies$ $\exists$! solution $\mathbb{H}^\gamma_{2,\theta-2}(\mathcal{O}, T)$

- $\Rightarrow u \in L_2(\ldots; W^1_2(\mathcal{O}))$

- Our theorem $\Rightarrow$

$$u \in L^\tau(\ldots; B^\alpha_{\tau,\tau}(\mathcal{O})), \quad \frac{1}{\tau} = \frac{\alpha}{2} + \frac{1}{2} \quad \text{for all } \alpha < 2$$
Example: \( d = 2, \ \theta = d, \ \gamma = 2 \):

- Suppose that \((g^k)_{k \in \mathbb{N}} \in H_{2,2}^{1}(\mathcal{O}, T; \ell_2)\) and \(u_0 \in L_2(\Omega, H_{2,2}^{1}(\mathcal{O}))\).

- Then: Kim’s theorem \(\Longrightarrow \exists!\) solution \(H_{2,\theta-2}^{\gamma}(\mathcal{O}, T)\)

- \(\Rightarrow u \in L_2(\ldots; W_2^{1}(\mathcal{O}))\)

- Our theorem \(\Rightarrow\)

\[
u \in L_{\tau}(\ldots; B_{\tau,\tau}^{\alpha}(\mathcal{O})), \quad \frac{1}{\tau} = \frac{\alpha}{2} + \frac{1}{2} \quad \text{for all} \quad \alpha < 2
\]

- Adaptivity completely justified!
Idea of the Proof:

- Wavelet characterization of Besov spaces

\[
\|u\|_{B^s_{p,q}(\mathbb{R}^d)} \sim \left( \sum_{|\lambda| = j_0}^{\infty} 2^{|\lambda|(s+d(\frac{1}{2} - \frac{1}{p}))q} \left( \sum_{\lambda \in J} |\langle u, \tilde{\psi}_\lambda \rangle|^p \right)^{q/p} \right)^{1/q}
\]

- Weighted Sobolev space estimate (Kim 2008)

\[
\|u\|_{H^{\gamma-2}_{2,\theta}(\mathcal{O})} + \| \sum_{\mu, \nu = 1}^{d} a^{\mu\nu} u_{x_\mu x_\nu} \|_{H^{\gamma-2}_{2,\theta+2}(\mathcal{O})} \leq C \left( \|g\|_{H^{\gamma-1}_{2,\theta}(\mathcal{O})} + \|u_0\|_{L_2(H^{\gamma-1}_{2,\theta}(\mathcal{O}))} \right)
\]
Idea of the Proof:

- Wavelet characterization of Besov spaces

\[ \|u\|_{B^{s,q}_{p,q}(\mathbb{R}^d)} \sim \left( \sum_{|\lambda|=j_0}^{\infty} 2^{|\lambda|(s+d(\frac{1}{2}-\frac{1}{p}))q} \left( \sum_{\lambda \in J, |\lambda|=j} |\langle u, \tilde{\psi}_\lambda \rangle|^p \right)^{q/p} \right)^{1/q} \]

- Weighted Sobolev space estimate (KIM 2008)

\[ \|u\|_{H^{\gamma}_{2,\theta-2}(\mathcal{O})} + \left\| \sum_{\mu,\nu=1}^{d} a^{\mu\nu} u_{x_\mu x_\nu} \right\|_{H^{\gamma-2}_{2,\theta+2}(\mathcal{O})} \leq C \left( \|g\|_{H^{\gamma-1}_{2,\theta}(\mathcal{O})} + \|u_0\|_{L^2(H^{\gamma-1}_{2,\theta}(\mathcal{O}))} \right) \]

Generalizations:

- More general noise models, multiplicative noise
Generalizations:

- More general noise models, multiplicative noise

- Semilinear stochastic evolution equations [Cioica/D.]

Motivation

Theoretical Analysis
- Does Adaptivity Pay?
- The Model Equation
- SPDEs in Weighted Sobolev Spaces
- Besov Regularity

Practical Realization
- Discretization Scheme
- The Noise Model
- Stochastic Elliptic Equations
Possible Approaches:

- Vertical method of lines (first in space, then in time) [Gyöngy/Krylov/Millet/Morien], [Walsh], [Yan].... Hard to combine with adaptivity...
Possible Approaches:

- Vertical method of lines (first in space, then in time) [Gyöngy/Krylov/Millet/Morien], [Walsh], [Yan].... Hard to combine with adaptivity...

- Full space-time adaptive wavelet algorithms [Schwab/Stevenson]...
Possible Approaches:

- Vertical method of lines (first in space, then in time) [Gyöngy/Krylov/Millet/Morien],[Walsh], [Yan].... Hard to combine with adaptivity...

- Full space-time adaptive wavelet algorithms [Schwab/Stevenson]...

- Horizontal method of lines, Rothe method (first in time, then in space) [Debussche/Printems]
  - Abstract Cauchy problem
  - ODE in suitable functions spaces, ODE-solver with adaptive step-size control
Rothe Method:

- Stiff problem $\leadsto$ implicit discretization in time:

\[
(I - (t_{n+1} - t_n) A) U_{t_{n+1}} = U_{t_n} + (t_{n+1} - t_n) F(t_n, U_{t_n}) + \Sigma(t_n, U_{t_n})(W_{t_{n+1}} - W_{t_n})
\]
Rothe Method:

- Stiff problem $\Leftrightarrow$ implicit discretization in time:

$$
(I - (t_{n+1} - t_n) A) U_{t_{n+1}} = U_{t_n} + (t_{n+1} - t_n) F(t_n, U_{t_n}) + \Sigma(t_n, U_{t_n})(W_{t_{n+1}} - W_{t_n})
$$

- leads to elliptic subproblems, $\Leftrightarrow$ model problem:

$$
- \Delta V = X(\omega) \text{ in } \mathcal{O}, \quad V = 0 \text{ on } \partial\mathcal{O}
$$
Rothe Method:

- Stiff problem \( \rightsquigarrow \) implicit discretization in time:
  \[
  (I - (t_{n+1} - t_n) A) U_{t_{n+1}} = U_{t_n} + (t_{n+1} - t_n) F(t_n, U_{t_n}) + \Sigma(t_n, U_{t_n})(W_{t_{n+1}} - W_{t_n})
  \]

- leads to elliptic subproblems, \( \rightsquigarrow \) model problem:
  \[
  -\Delta V = X(\omega) \text{ in } \mathcal{O}, \quad V = 0 \text{ on } \partial \mathcal{O}
  \]

- Will be treated by (stochastic versions of) optimally convergent adaptive wavelet (frame) algorithms [Cohen/Dahmen/DeVore], [Stevenson/Schwab], [D./Fornasier/Raasch]....realize the convergence order of best \( n \)-term approximation!
The Noise Model:

- Usually: noise modeled by means of the Eigenfunctions of $A$.
- We need: noise model based on wavelets, control of Besov regularity.
The Noise Model:

- Usually: noise modeled by means of the Eigenfunctions of $A$.
- We need: noise model based on wavelets, control of Besov regularity.
- Choose parameters $\alpha > 0$ and $0 \leq \beta \leq 1$ s.t. $\alpha + \beta > 1$
- Let $Y_\lambda, (w_\lambda^t)_{t \in [0,T]}, \lambda \in \mathcal{J}$ be independent, $Y_\lambda \sim B(1, 2^{-\beta jd})$. Set

$$g^\lambda(\omega, t, \cdot) := \sigma_j Y_\lambda \psi_\lambda(\cdot), \quad \sigma_j := (j-(j_0-2)) \frac{cd}{2} 2^{-\frac{\alpha(j-j_0-1)d}{2}}$$
The Noise Model:

- Usually: noise modeled by means of the Eigenfunctions of $A$.
- We need: noise model based on wavelets, control of Besov regularity.
- Choose parameters $\alpha > 0$ and $0 \leq \beta \leq 1$ s.t. $\alpha + \beta > 1$
- Let $Y_\lambda$, $(w^\lambda_t)_{t\in[0,T]}$, $\lambda \in \mathcal{J}$ be independent, $Y_\lambda \sim B(1, 2^{-\beta jd})$. Set

$$g^\lambda(\omega, t, \cdot) := \sigma_j Y_\lambda \psi_\lambda(\cdot), \quad \sigma_j := (j - (j_0 - 2)) \frac{cd}{2} 2^{-\alpha(j-j_0-1)/2}$$

- in each time step

$$-\Delta V = X(\omega) \text{ in } \mathcal{O}, \quad V = 0 \text{ on } \partial\mathcal{O}$$

$$X = \sum_{\lambda \in \mathcal{J}} Y_\lambda Z_\lambda \psi_\lambda, \quad Z_\lambda \sim N(0, 2^{-\alpha|\lambda|d})$$
The Noise Model:

\[ X = \sum_{\lambda \in \mathcal{J}} Y_{\lambda} Z_{\lambda} \cdot \psi_{\lambda} \]  

\[ (\ast) \]
The Noise Model:

\[ X = \sum_{\lambda \in \mathcal{J}} Y_{\lambda} Z_{\lambda} \cdot \psi_{\lambda} \] (\(\ast\))

- Note that \(X\) is Gaussian iff \(\beta = 0\).
The Noise Model:

\[ X = \sum_{\lambda \in \mathcal{J}} Y_\lambda Z_\lambda \cdot \psi_\lambda \]  

\[ \text{Note that } X \text{ is Gaussian iff } \beta = 0. \]

\[ \text{Moreover, } (\ast) \text{ is the Karhunen-Loève expansion iff } (\psi_\lambda)_{\lambda \in \mathcal{J}} \text{ is an ONB} \]
Adaptive Wavelet Methods for SPDEs

Stephan Dahlke

Motivation

Theoretical Analysis

Does Adaptivity Pay?

The Model Equation

SPDEs in Weighted Sobolev Spaces

Besov Regularity

Practical Realization

Discretization Scheme

The Noise Model

Stochastic Elliptic Equations

Besov Regularity:

\[ \| f \|_{B^s_q(L_p(O))} \sim \left( \sum_{|\lambda| = j_0}^{\infty} 2^{|\lambda| (s + d(\frac{1}{2} - \frac{1}{p}))q} \left( \sum_{\lambda \in J, |\lambda| = j} |\langle f, \tilde{\psi}_\lambda \rangle|^p \right)^{q/p} \right)^{1/q}, \]

where \( \tilde{\Psi} = \{ \tilde{\psi}_\lambda : \lambda \in J \} \) satisfies \( \langle \psi_\lambda, \tilde{\psi}_\nu \rangle = \delta_{\lambda,\nu} \),

Theorem

\[ X \in B^s_q(L_p(O)) \text{ P-a.s. iff } \]

\[ \frac{\alpha - 1}{2} + \frac{\beta}{p} > \frac{s}{d}, \]

in which case \( \mathbb{E} \left( \| X \|_{B^s_q(L_p(D))}^q \right) < \infty. \)

[Abramovich/Sapatinas/Silverman], [Bochkina] for \( d = 1 \) and \( p, q \geq 1. \)
### Besov Regularity:

<table>
<thead>
<tr>
<th>Corollary</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X \in W^s_2(\mathcal{O}) \ P-a.s. \text{ iff} )</td>
</tr>
<tr>
<td>[ s &lt; d \left( \frac{\alpha + \beta - 1}{2} \right) ]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Corollary</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let ( 1/\tau = s/d + 1/2 ). ( X \in B^s_\tau(L_\tau(\mathcal{O})) \ P-a.s. \text{ iff} )</td>
</tr>
<tr>
<td>[ s &lt; d \left( \frac{\alpha + \beta - 1}{2(1 - \beta)} \right) ]</td>
</tr>
</tbody>
</table>

▶ \( \beta \) is a sparsity parameter
### Besov Regularity:

#### Corollary

\[ X \in W_2^s(O) \text{ P-a.s. iff } \]

\[ s < d \left( \frac{\alpha + \beta - 1}{2} \right) \]

#### Corollary

Let \( 1/\tau = s/d + 1/2 \). \( X \in B_\tau^s(L_\tau(O)) \text{ P-a.s. iff } \]

\[ s < d \left( \frac{\alpha + \beta - 1}{2(1 - \beta)} \right) \]

- \( \beta \) is a sparsity parameter
- \( \alpha + \beta \) fixed, \( \beta \to 1 \) Sobolev smoothness fixed, **arbitrary high Besov regularity!**
Realizations:

(a) $\alpha = 2.0$, $\beta = 0.0$

(b) $\alpha = 2.0$, $\beta = 0.0$

(c) $\alpha = 1.8$, $\beta = 0.2$

(d) $\alpha = 1.8$, $\beta = 0.2$
Realizations:

(e) $\alpha = 1.5, \beta = 0.5$

(f) $\alpha = 1.5, \beta = 0.5$

(g) $\alpha = 1.2, \beta = 0.8$

(h) $\alpha = 1.2, \beta = 0.8$
Stochastic Elliptic Equations:

- Rothe method leads to elliptic subproblem:

\[-\Delta V = X \text{ in } \mathcal{O}, \quad V = 0 \text{ on } \partial \mathcal{O}\]
Stochastic Elliptic Equations:

- Rothe method leads to elliptic subproblem:
  \[-\Delta V = X \text{ in } \mathcal{O}, \quad V = 0 \text{ on } \partial\mathcal{O}\]

- Will be treated by (stochastic versions of) optimally convergent adaptive wavelet (frame) algorithms [Cohen/Dahmen/DeVore], [Stevenson/Schwab], [D./Fornasier/Raasch]....
Stochastic Elliptic Equations:

- Rothe method leads to elliptic subproblem:

\[-\Delta V = X \text{ in } \mathcal{O}, \quad V = 0 \text{ on } \partial \mathcal{O}\]

- Will be treated by (stochastic versions of) optimally convergent adaptive wavelet (frame) algorithms [Cohen/Dahmen/DeVore], [Stevenson/Schwab], [D./Fornasier/Raasch]....

- Optimal in the energy norm,

\[\eta(g) := \#\{\lambda \in \mathcal{J} : c_\lambda \neq 0, g = \sum_{\lambda \in \mathcal{J}} c_\lambda \psi_\lambda\}\]

\[e(\hat{V}) = \left( \mathbb{E} \| V - \hat{V} \|_{H^1(\mathcal{O})}^2 \right)^{1/2}\]

\[e_{n,H^1}(V) = \inf \left\{ e(\hat{V}_n), \quad \eta(\hat{V}_n) \leq n \right\} \text{ a.s.}\]
Approximation Rates:

**Theorem**

Let $d \in \{2, 3\}$. Put

$$\rho = \min \left( \frac{1}{2(d - 1)}, \frac{\alpha + \beta - 1}{6} + \frac{2}{3d} \right).$$

For $n$-term approximation of $V$, with any $\epsilon > 0$,

$$e_{n,H^1}(V) \preceq n^{-\rho + \epsilon}.$$

- Uniform discretizations yield $n^{-1/(2d)}$ on general Lipschitz domains. We have $\rho > 1/(2d)$. 
Approximation Rates:

Theorem

Let $d \in \{2, 3\}$. Put

$$\rho = \min \left( \frac{1}{2(d-1)}, \frac{\alpha + \beta - 1}{6} + \frac{2}{3d} \right).$$

For $n$-term approximation of $V$, with any $\epsilon > 0$,

$$e_{n,H^1}(V) \leq n^{-\rho + \epsilon}.$$  

- Uniform discretizations yield $n^{-1/(2d)}$ on general Lipschitz domains. We have $\rho > 1/(2d)$.
- Better results for more specific domains, e.g. polygonal $\Omega$. 
Approximation Rates:

**Theorem**

Let \( d \in \{2, 3\} \). Put

\[
\rho = \min \left( \frac{1}{2(d - 1)}, \frac{\alpha + \beta - 1}{6} + \frac{2}{3d} \right).
\]

For \( n \)-term approximation of \( V \), with any \( \epsilon > 0 \),

\[
e_{n, H^1}(V) \leq n^{-\rho + \epsilon}.
\]

- Uniform discretizations yield \( n^{-1/(2d)} \) on general Lipschitz domains. We have \( \rho > 1/(2d) \).
- Better results for more specific domains, e.g. polygonal \( \mathcal{O} \).
- Convergence order realized by adaptive wavelet algorithms.
Numerical Results (1D):

Specifically

- $D = [0, 1]$,

- Problem completely regular, Sobolev/Besov smoothness only depends on smoothness of the right-hand side!

- ‘Exact’ solution via master computation.

- Adaptive wavelet scheme $\leftrightarrow$ uniform scheme

- The $W_2^s$-regularity

\[ s < \frac{\alpha + \beta - 1}{2} \]

of $X$ kept constant, Besov smoothness varies
Convergence Rates: $\alpha = 0.9$, $\beta = 0.2$

Orders of convergence:
upper bounds $1.05$ and $19/16 = 1.1875$. 
Comparison:

Convergence Rates: $\alpha = 0.4$, $\beta = 0.7$
Comparison:

Convergence Rates: $\alpha = -0.87$, $\beta = 0.97$
Example in 2D:

(a) exact solution
(b) exact right-hand side
(c) $\alpha = 1.0, \beta = 0.1$
(d) $\alpha = 1.0, \beta = 0.9$
Summary:

- Adaptive numerical treatment of SPDEs
Summary:

- Adaptive numerical treatment of SPDEs
- Theoretical analysis: Besov regularity
Summary:

- Adaptive numerical treatment of SPDEs
- Theoretical analysis: Besov regularity
  - $u \in B^s_\tau(L_\tau(\Omega))$, $1/\tau = s/d + 1/2$, $0 < s < s^*$?
Summary:

- Adaptive numerical treatment of SPDEs
- Theoretical analysis: Besov regularity
  - \( u \in B^s_\tau(L_\tau(\Omega)) \), \( 1/\tau = s/d + 1/2, \ 0 < s < s^* \)?
  - weighted Sobolev estimates + wavelet expansions \( \leadsto \)
  - new regularity results, \( s^* \) sufficiently large
Summary:

- Adaptive numerical treatment of SPDEs
- Theoretical analysis: Besov regularity
  - $u \in B^s_{\tau}(L_{\tau}(\Omega)), \quad 1/\tau = s/d + 1/2, \quad 0 < s < s^*$
  - weighted Sobolev estimates + wavelet expansions $\leadsto$ new regularity results, $s^*$ sufficiently large
- Practical realization
Summary:

- Adaptive numerical treatment of SPDEs
- Theoretical analysis: Besov regularity
  - \( u \in B^s_\tau(L_\tau(\Omega)), \quad 1/\tau = s/d + 1/2, \quad 0 < s < s^*? \)
  - weighted Sobolev estimates + wavelet expansions \( \sim \)
    new regularity results, \( s^* \) sufficiently large
- Practical realization
  - Rothe method, implicit discretization scheme
Summary:

- Adaptive numerical treatment of SPDEs
- Theoretical analysis: Besov regularity
  - \( u \in B^s_\tau(L_\tau(\Omega)), \quad 1/\tau = s/d + 1/2, \quad 0 < s < s^* \)?
  - weighted Sobolev estimates + wavelet expansions \( \sim \)
    new regularity results, \( s^* \) sufficiently large
- Practical realization
  - Rothe method, implicit discretization scheme
  - new noise model, prescribed Besov regularity
Summary:

- Adaptive numerical treatment of SPDEs

- Theoretical analysis: Besov regularity
  - $u \in B^s_\tau(L_\tau(\Omega))$, $1/\tau = s/d + 1/2$, $0 < s < s^*$?
  - weighted Sobolev estimates + wavelet expansions $\sim$ new regularity results, $s^*$ sufficiently large

- Practical realization
  - Rothe method, implicit discretization scheme
  - new noise model, prescribed Besov regularity
  - elliptic subproblems, solved by adaptive wavelet algorithms
Summary:

- Adaptive numerical treatment of SPDEs
- Theoretical analysis: Besov regularity
  - \( u \in B^s_\tau(L_\tau(\Omega)), \quad 1/\tau = s/d + 1/2, \quad 0 < s < s^* ? \)
  - weighted Sobolev estimates + wavelet expansions \( \sim \) new regularity results, \( s^* \) sufficiently large
- Practical realization
  - Rothe method, implicit discretization scheme
  - new noise model, prescribed Besov regularity
  - elliptic subproblems, solved by adaptive wavelet algorithms
  - numerical experiments
Adaptive Wavelet Methods for SPDEs
Stephan Dahlke

Motivation

Theoretical Analysis
Does Adaptivity Pay?
The Model Equation
SPDEs in Weighted Sobolev Spaces
Besov Regularity

Practical Realization
Discretization Scheme
The Noise Model
Stochastic Elliptic Equations


Adaptive Wavelet Methods for SPDEs

Stephan Dahlke

Motivation

Theoretical Analysis

Does Adaptivity Pay?
The Model Equation
SPDEs in Weighted Sobolev Spaces
Besov Regularity

Practical Realization

Discretization Scheme
The Noise Model
Stochastic Elliptic Equations

Thank you for the attention!
The DeVore-Triebel Diagram:

\[ W_{2}^{3/2}(\mathcal{O}) \]

\[ \frac{1}{\tau} = \frac{s}{2} + \frac{1}{2} \]
Extensions

More general linear equations of the type:

\[ du = \sum_{i,j=1}^{d} \left( a^{ij} u_{x^i x^j} + b^i u_{x^i} + cu + f \right) \, dt \]

\[ + \sum_{k=1}^{\infty} \left( \sigma^{ik} u_{x^i} + \eta^k u + g^k \right) \, dw^k_t, \]

\[ u(0, \cdot) = u_0, \]

with random functions \( a^{ij}, b^i, c, \sigma^{ik}, \eta^k, f \) and \( g^k \) depending on \( t \) and \( x \)
Adaptive Wavelet Methods for SPDEs

Stephan Dahlke

Motivation

Theoretical Analysis

Does Adaptivity Pay?

The Model Equation

SPDEs in Weighted Sobolev Spaces

Besov Regularity

Practical Realization

Discretization Scheme

The Noise Model

Stochastic Elliptic Equations

Weighted Sobolev Spaces

Set $\rho(x) := \text{dist}(x, \partial \mathcal{O})$ for $x \in \mathcal{O}$.

1. $(\zeta_n)_{n \in \mathbb{Z}} \subseteq C_0^\infty(\mathcal{O})$, such that:
   1. $\sum_{n \in \mathbb{Z}} \zeta_n(x) = 1$, $x \in \mathcal{O}$,
   2. $\text{supp} \, \zeta_n \subseteq \mathcal{O}_n := \{x \in \mathcal{O} : 2^{-n-1} < \rho(x) < 2^{-n+1}\}$,
   3. $|D^m \zeta_n| \leq N(m) 2^{mn}$, $m \in \mathbb{N}_0$, $n \in \mathbb{Z}$.

2. For $\gamma \in \mathbb{R}$ and $\theta \in \mathbb{R}$ define:
   $$H_{2, \theta}^{\gamma}(\mathcal{O}) := \left\{ u \in \mathcal{D}'(\mathcal{O}) : \|u\|_{H_{2, \theta}^{\gamma}(\mathcal{O})}^2 < \infty \right\}$$

3. with:
   $$\|u\|_{H_{2, \theta}^{\gamma}(\mathcal{O})}^2 := \sum_{n \in \mathbb{Z}} 2^n \theta \|\zeta_n(2^n \cdot) u(2^n \cdot)\|_{H_2^\gamma(\mathbb{R}^d)}^2$$

4. and $\|f\|_{H_2^\gamma(\mathbb{R}^d)} = \|(1 - \Delta)^{\gamma/2} f\|_{L_2(\mathbb{R}^d)}$. 
Weighted Sobolev Spaces for Sequences

- For $\gamma \in \mathbb{R}$ and $\theta \in \mathbb{R}$ define:

$$H_{2,\theta}^\gamma(O; \ell_2) := \left\{ g = (g^k)_{k \in \mathbb{N}} \in [\mathcal{D}'(O)]^\mathbb{N} : \right.$$  

$$\left. \| g \|_{H_{2,\theta}^\gamma(O; \ell_2)} < \infty \right\}$$

- Where

$$\| g \|_{H_{2,\theta}^\gamma(O; \ell_2)}^2 := \sum_{n \in \mathbb{Z}} 2^{n\theta} \| \zeta_n(2^n \cdot) g(2^n \cdot) \|_{H_2^\gamma(\mathbb{R}^d; \ell_2)}^2$$

- And $f = (f^k)_{k \in \mathbb{N}} \in H_{2,\theta}^\gamma(\mathbb{R}^d; \ell_2) \Leftrightarrow$

  - $f^k \in H_2^\gamma$ for all $k \in \mathbb{N}$, and
  - $\| f \|_{H_{2,\theta}^\gamma(\mathbb{R}^d; \ell_2)} := \| \left( (1 - \Delta)^{\gamma/2} f^k \right)_{k \in \mathbb{N}} \|_{\ell_2} \|_{L_2(\mathbb{R}^d)} < \infty$.  

Motivation

Theoretical Analysis

Does Adaptivity Pay?

The Model Equation

SPDEs in Weighted Sobolev Spaces

Besov Regularity

Practical Realization

Discretization Scheme

The Noise Model

Stochastic Elliptic Equations