Dynamical Inverse Scattering

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Outline

Introduction

Orthogonality Sampling

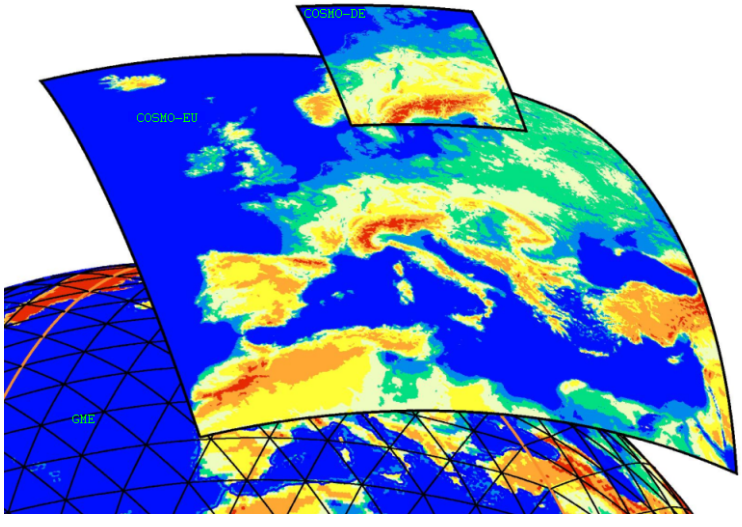
Time-Domain Probe Method

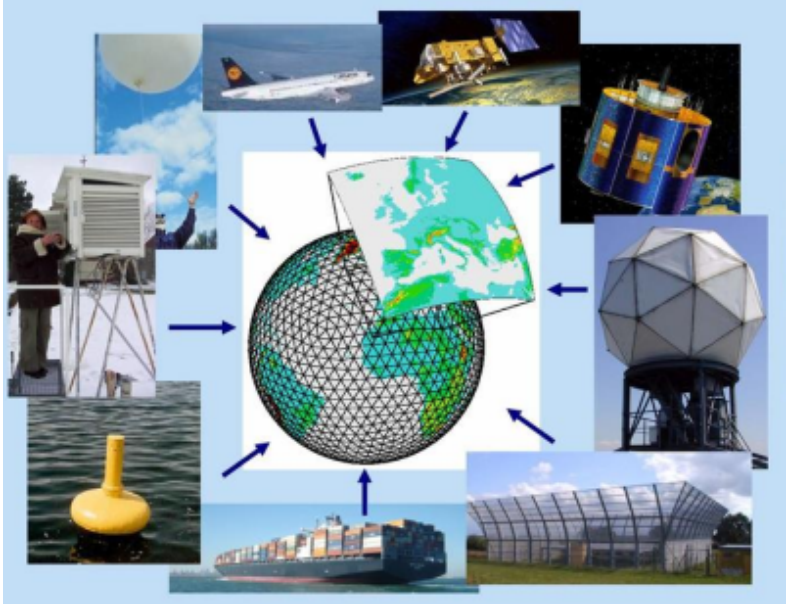
- Field & Shape Reconstruction
- Time-Domain Probe

Data Assimilation

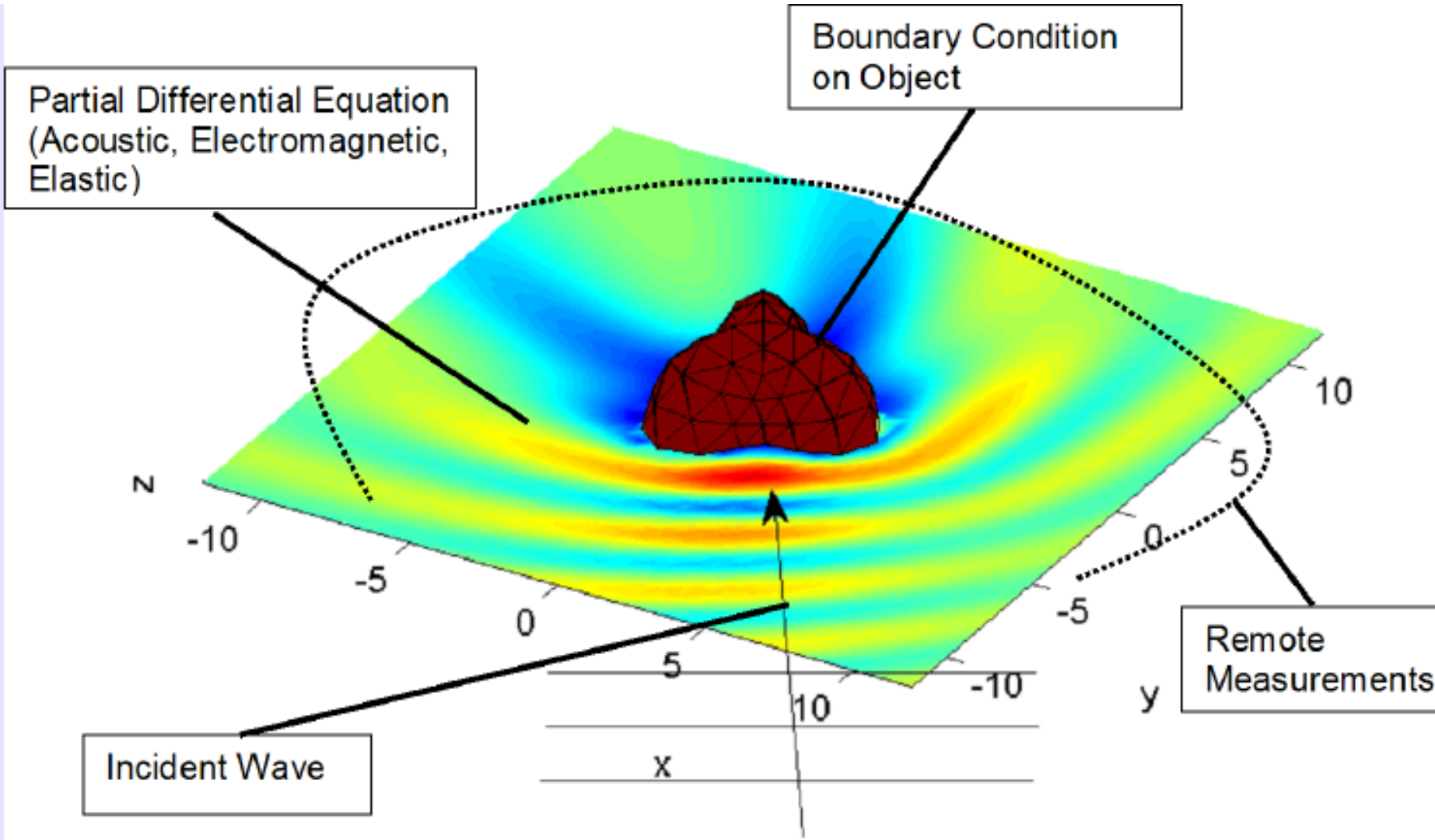
- 3dVar
- 4dVar
- Dynamic Inverse Scattering

Introduction





Setup


Partial Differential Equation (Acoustic, Electromagnetic, Elastic)
Boundary Condition on Object
Incident Wave
Remote Measurements
N
-10
-5
0
5
10
x
y
10
5
0
-5
-10

Dynamical Inverse Scattering, Survey

1. Static scatterer and wave, i.e. one frequency time-harmonic wave
2. Multi-Frequency scattering, static scatterer
3. Dynamical wave field, i.e. time-dependent pulse
4. Moving Scatterer, i.e. constant speed, accelerating, rotating
5. Scatterer is evolving, i.e. changing its location or shape, we get repeated measurements for various time-slices
6. Fully coupled time-space-wave dynamic problem, where the time-dependent wave and the time-dependent scatterer are interacting

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   Orthogonality Sampling (P. 2010)

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   Time-Domain Probe Method (Burkard, P. 2009)

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   Doppler Effect (Standard)

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Time-Domain to Frequency Domain

- We assume that we take a windowed Fast Fourier Transform of our available time-domain data.
- This leads to a large set of wave numbers for which data in the frequency domain is available.
- Usually we will have a
  - low number of directions of incidence (sources)
  - larger number of measurement points for the scattered or total wave.
  - many wavenumbers

$$u^\infty(\hat{x}_j, d_\ell, \kappa_\xi), \quad j = 1, \ldots, N, \quad \ell = 1, \ldots, M, \quad \xi = 1, \ldots, Q. \tag{1}$$

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Orthogonality Sampling Method

Algorithm (One-wave OS, Multi-wave OS)

For fixed wave number κ one-wave orthogonality sampling calculates

$$\mu(y,\kappa) = \left| \int_{\mathbb{S}} e^{i\kappa\hat{x}\cdot y} u^{\infty}(\hat{x})\, ds(\hat{x}) \right| \tag{2}$$

on a grid $\mathcal{G}$ of points $\tilde{y} \in \mathbb{R}^m$ from the knowledge of the far field pattern u^{∞} on the unit sphere $\mathbb{S}$.

For fixed wave number κ multi-direction orthogonality sampling calculates

$$\mu(y,\kappa) = \int_{\mathbb{S}} \left| \int_{\mathbb{S}} e^{i\kappa\hat{x}\cdot y} u^{\infty}(\hat{x},\theta)\, ds(\hat{x}) \right| ds(\theta) \tag{3}$$

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Multi-frequency Orthogonality Sampling

ALGORITHM (MULTI-FREQUENCY)

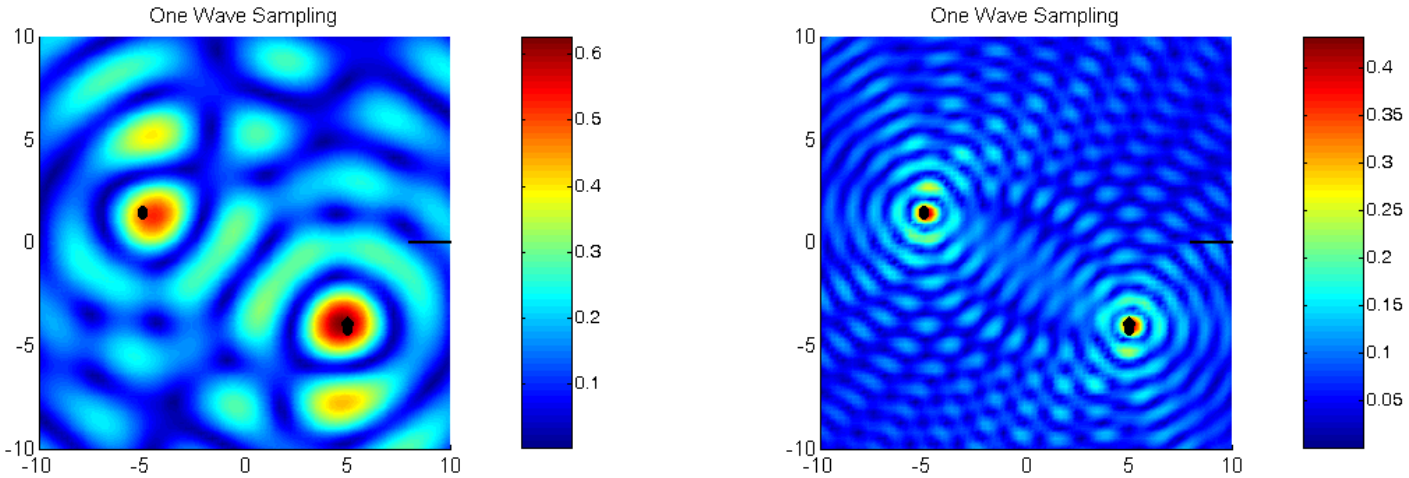
The multi-frequency orthogonality sampling calculates

$$\mu(y,\theta) = \int_{\kappa_0}^{\kappa_1} \left| \int_{\mathbb{S}} e^{i\kappa\hat{x}\cdot y} u^{\infty}(\hat{x},\theta)\, ds(\hat{x}) \right| d\kappa \tag{4}$$

on a grid $\mathcal{G}$ of points $\tilde{y} \in \mathbb{R}^m$ from the knowledge of the far field pattern $u_\kappa^\infty(\hat{x})$ for $\hat{x} \in \mathbb{S}$ and $\kappa \in [\kappa_0, \kappa_1]$.

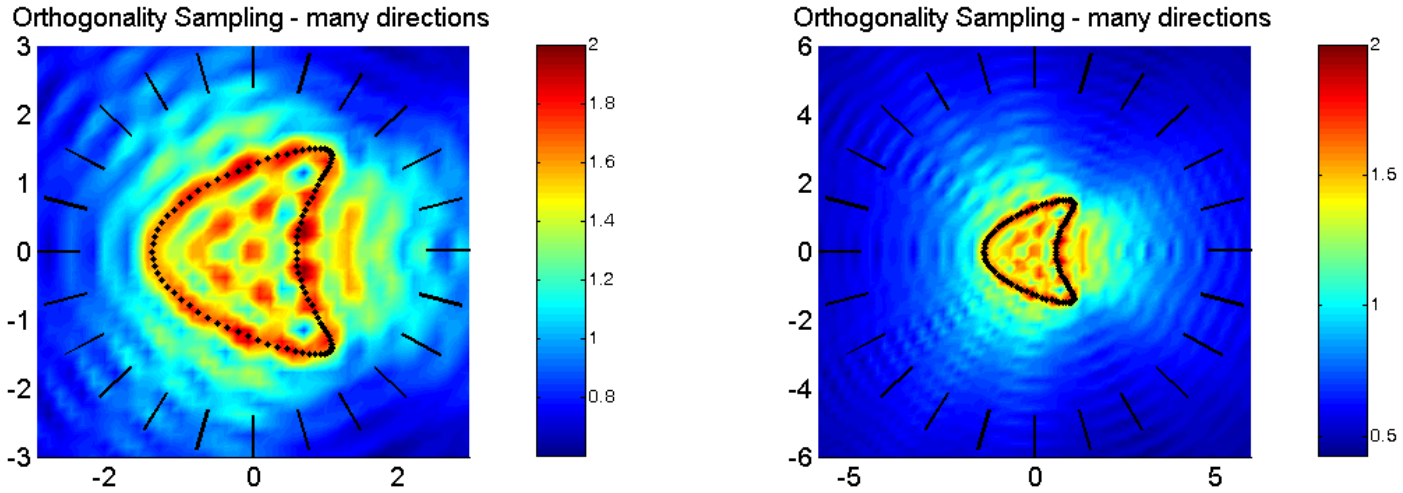
Here also multi-direction multi-frequency sampling is possible by adding the indicator functions for several directions of incidence.

One Wave, one frequency: the simplest setting



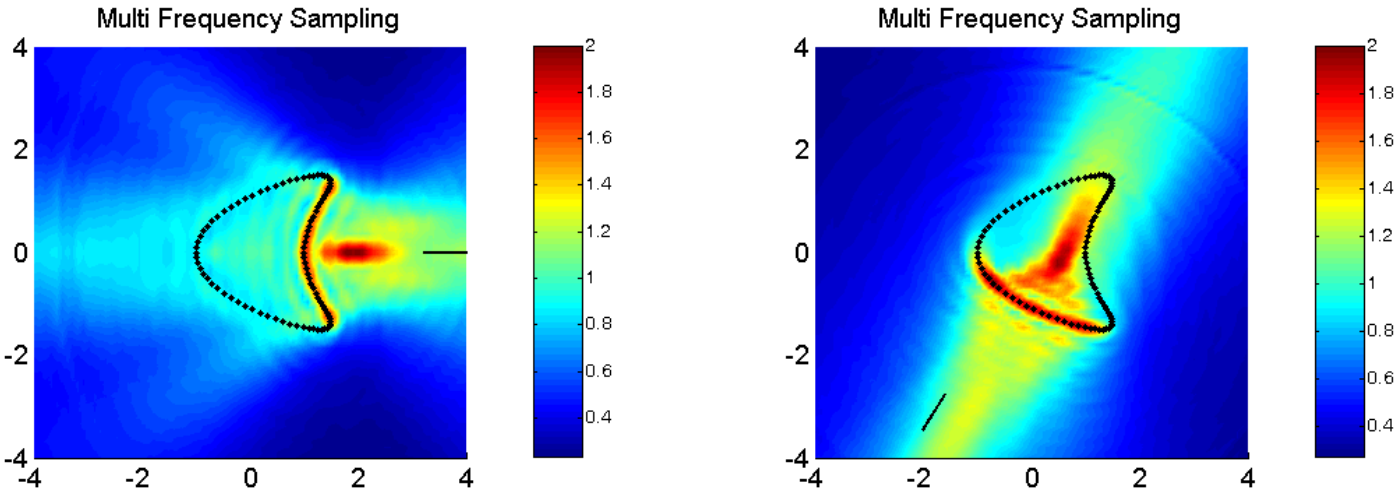
Graphics: Orthogonality sampling with $\kappa = 1$ or $\kappa = 3$ for fixed frequency, one direction of incidence

Multi-direction Ortho Sampling



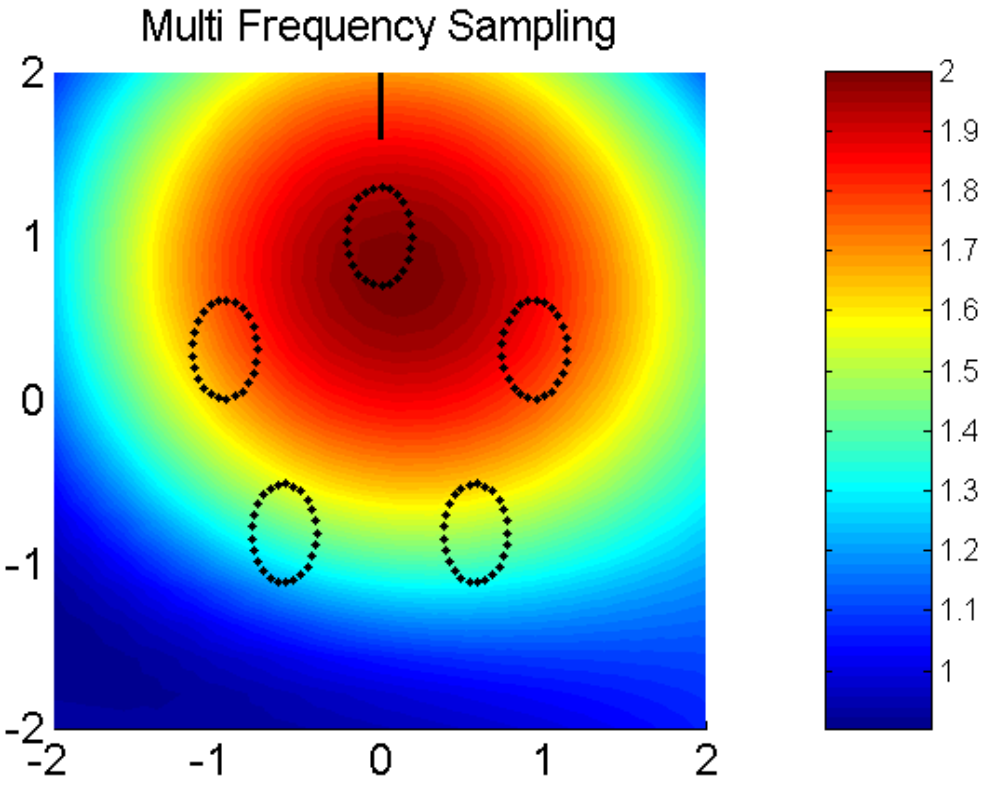
Graphics: Orthogonality sampling, many directions of incidence, fixed frequency

Multi-frequency Ortho Sampling



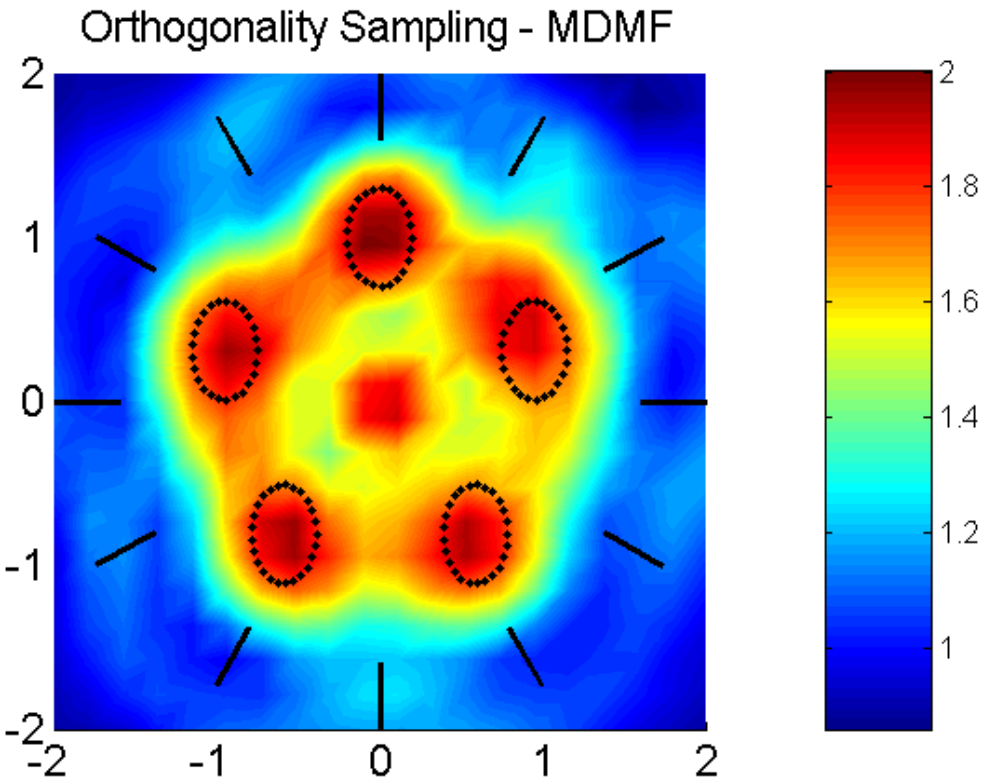
Graphics: Orthogonality sampling, many directions of incidence, fixed frequency

Resolution Study: Large Scale



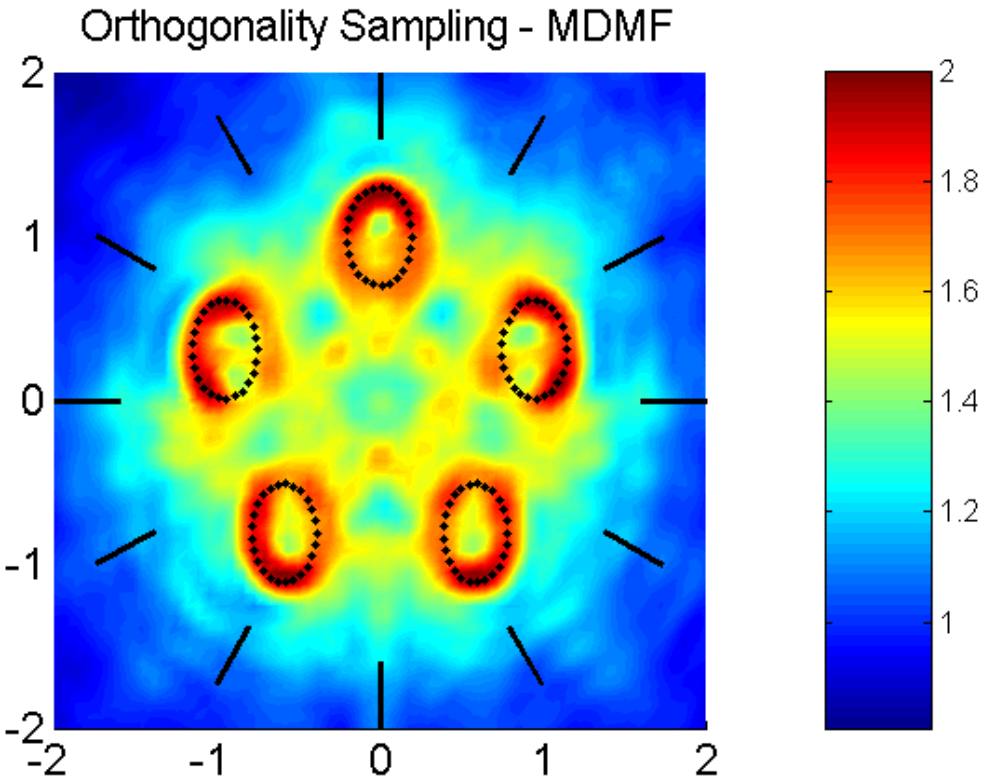
Graphics: Multi-frequency Orthogonality sampling with κ between 0.1 and 1, i.e. with a frequency between $\lambda = 6$ and $\lambda = 60$, one direction of incidence

Resolution Study: Medium Scale



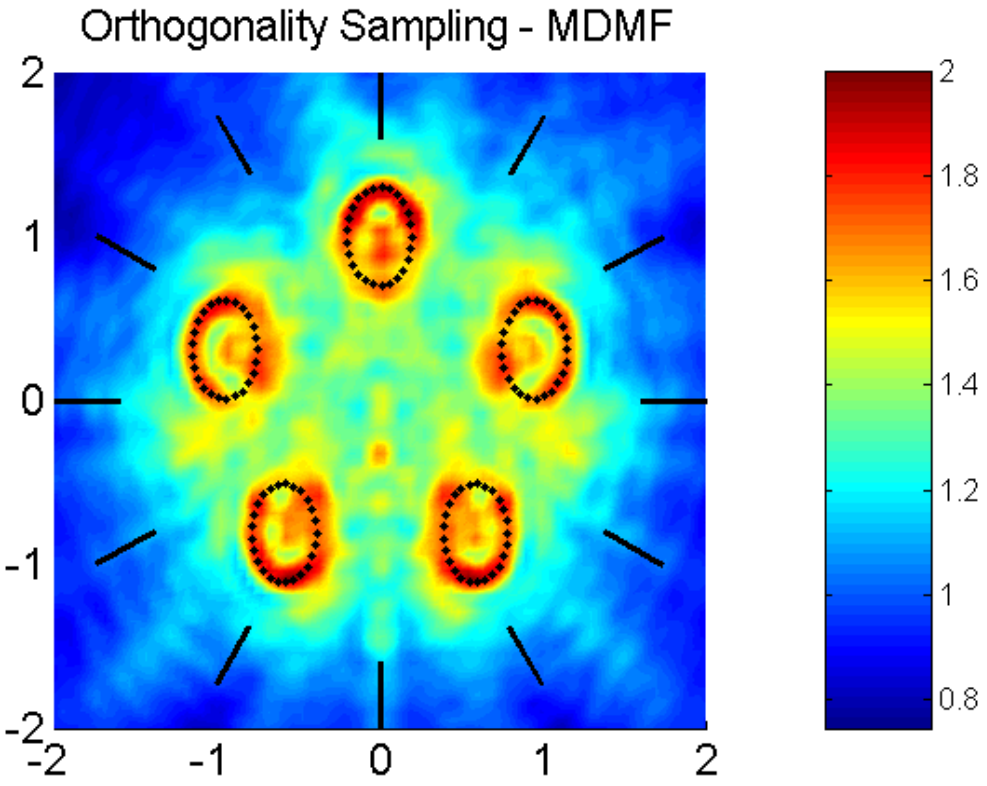
Graphics: MDMF Orthogonality sampling with κ between 3 and 4, i.e. with a frequency between $\lambda = 1.5$ and $\lambda = 2$

Resolution Study: Medium Scale



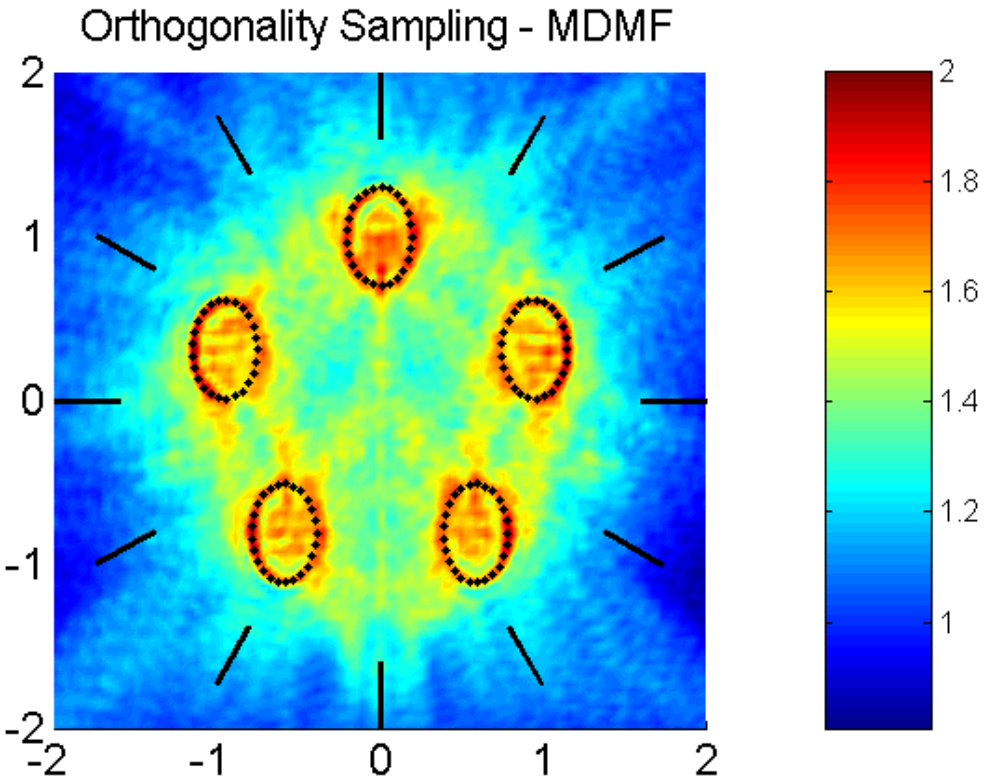
Graphics: MDMF Orthogonality sampling with κ between 6 and 15, i.e. with a frequency between $\lambda = 0.4$ and $\lambda = 1$

Resolution Study: Fine Scale



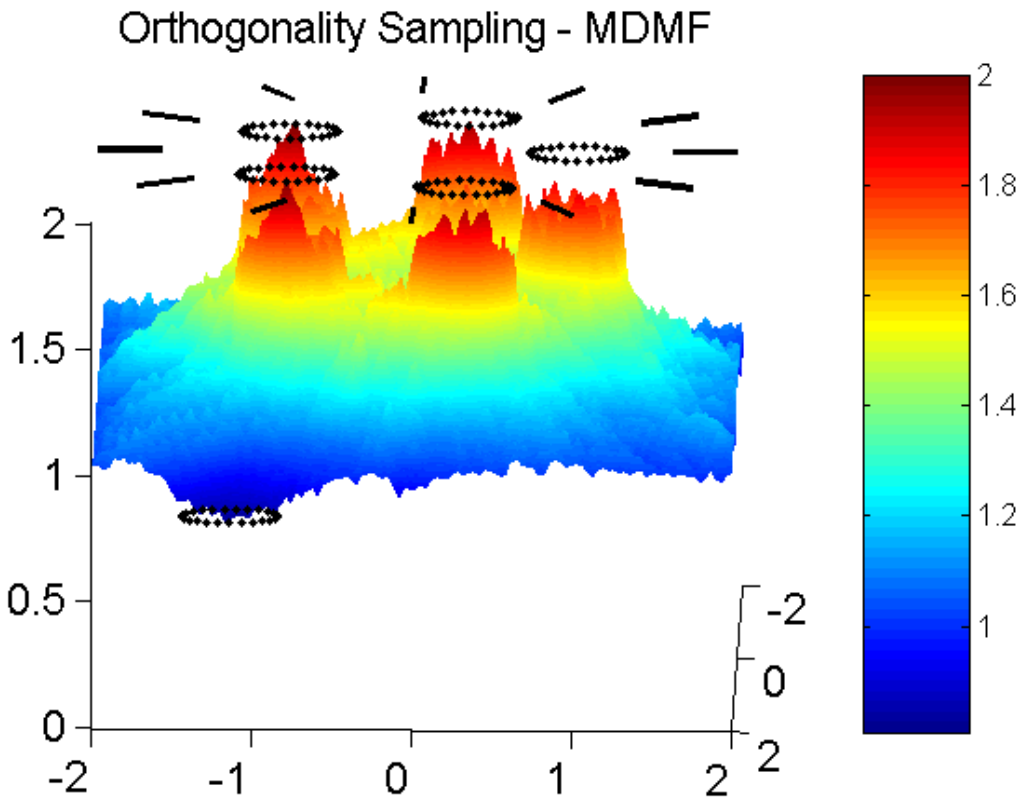
Graphics: MDMF Orthogonality sampling with κ between 10 and 20, i.e. with a frequency between $\lambda = 0.3$ and $\lambda = 0.6$

Resolution Study: Very Fine Scale



Graphics: MDMF Orthogonality sampling with κ between 20 and 40, i.e. with a frequency between $\lambda = 0.15$ and $\lambda = 0.3$

Resolution Study: Very Fine Scale



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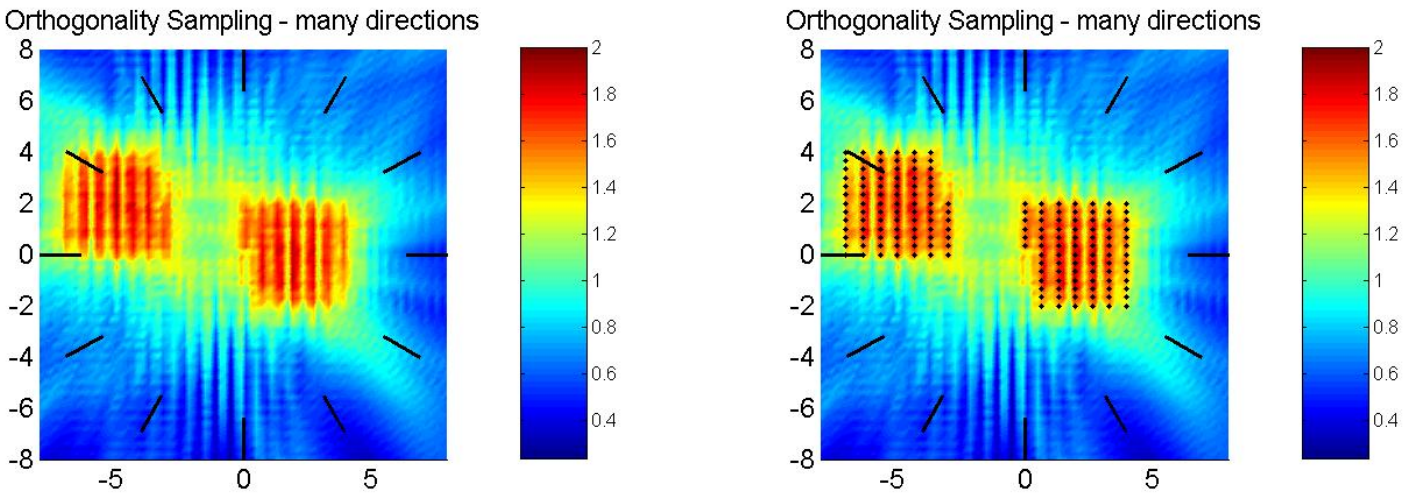
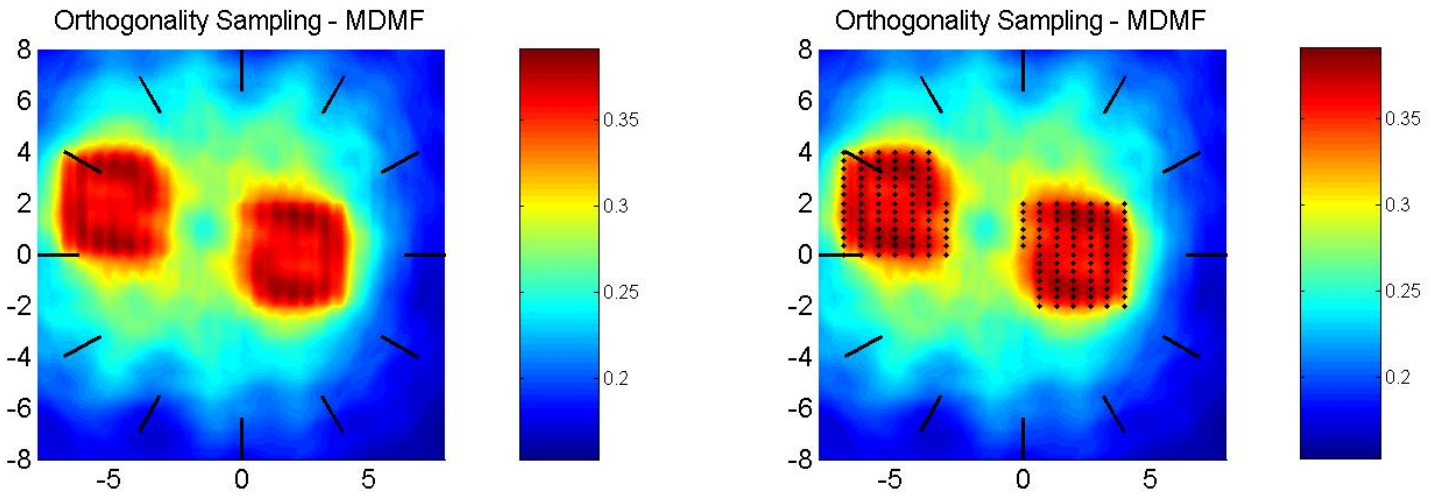
Medium Reconstructions I



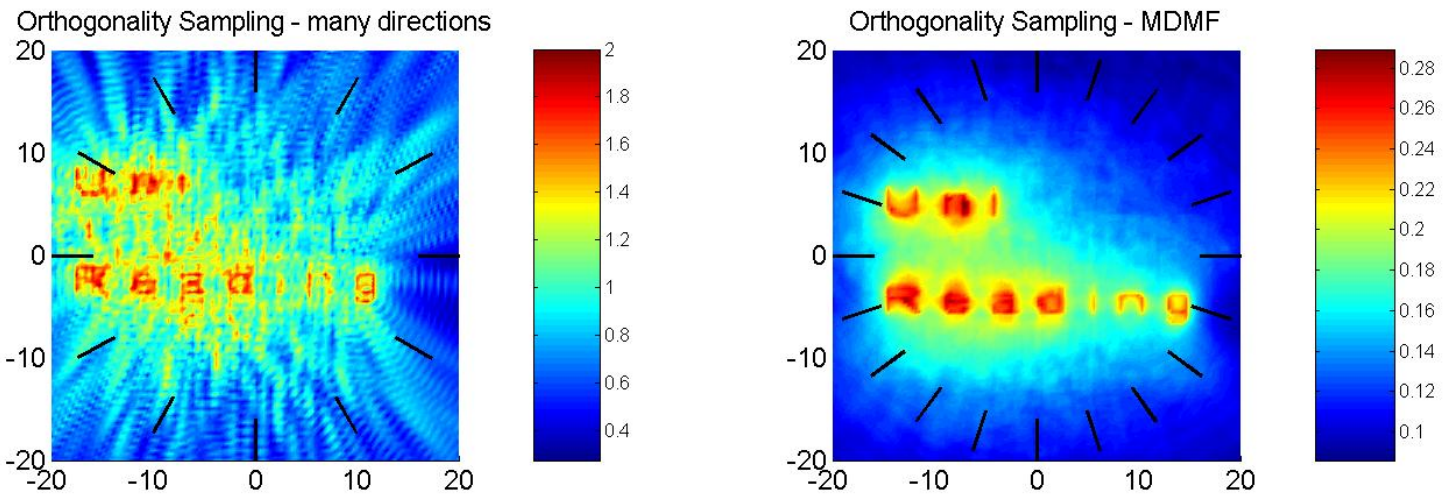
Graphics: Orthogonality sampling for medium reconstruction, MD, fixed frequency $\kappa = 9$.

Medium Reconstructions II



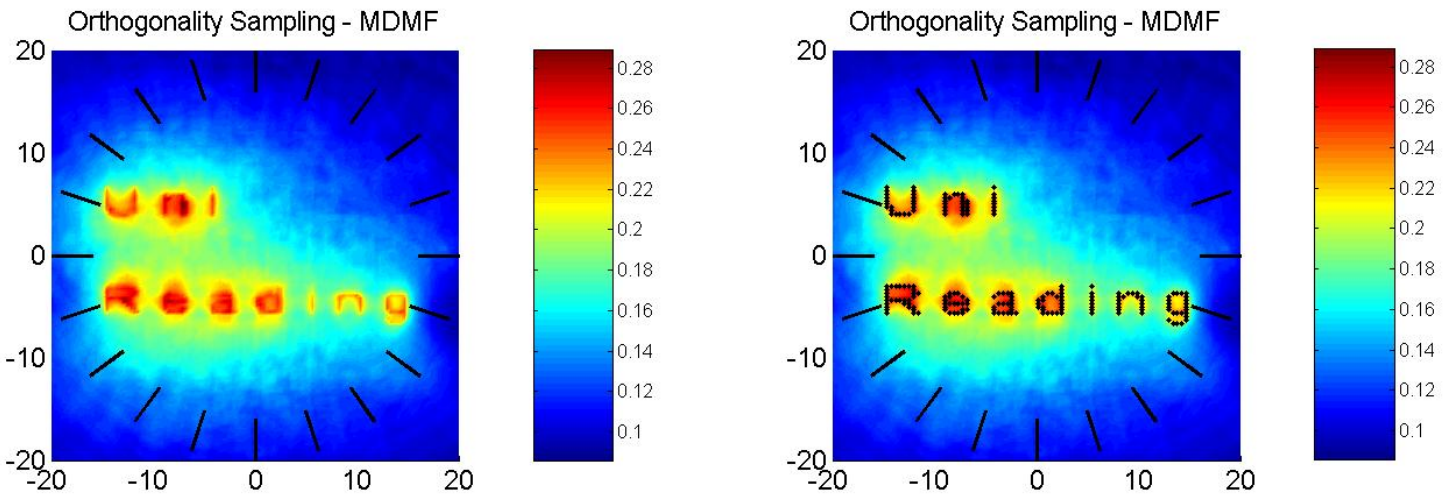
Graphics: Orthogonality sampling for medium reconstruction, MDMF.

Medium Reconstructions III



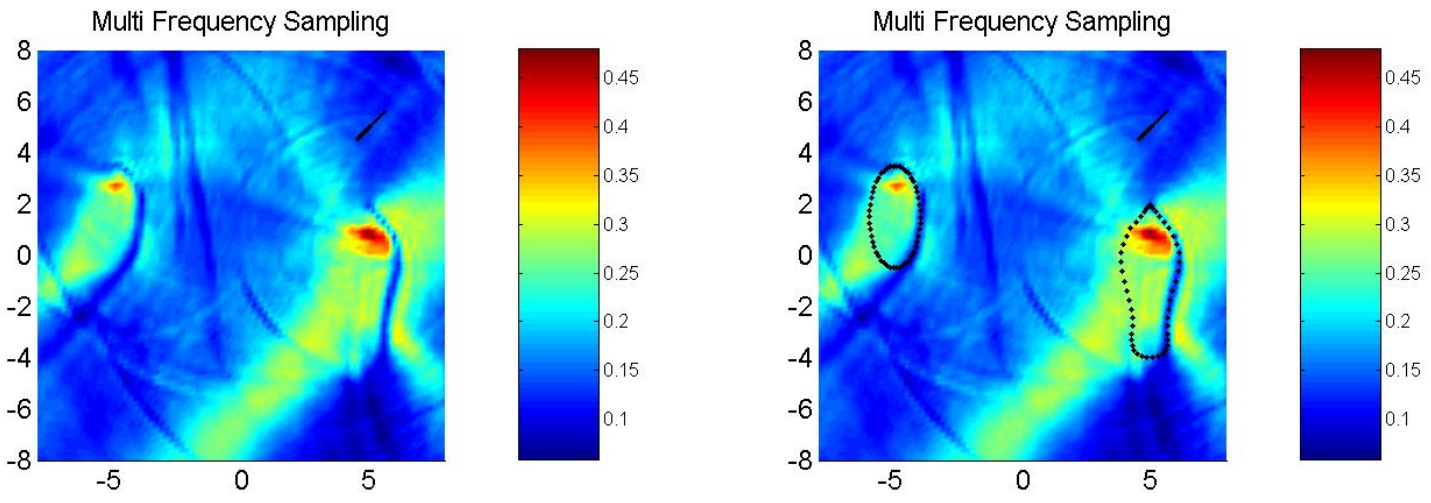
Graphics: Orthogonality sampling for medium reconstruction, MDMF.

Medium Reconstructions IV



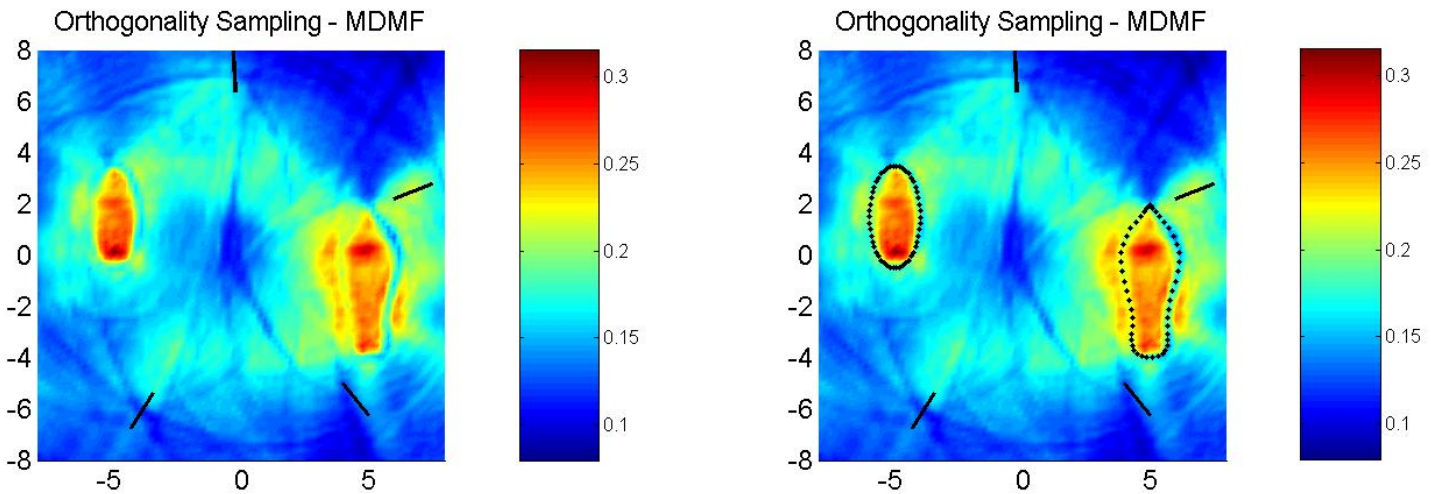
Graphics: Orthogonality sampling for medium reconstruction, MDMF.

Neumann BC I



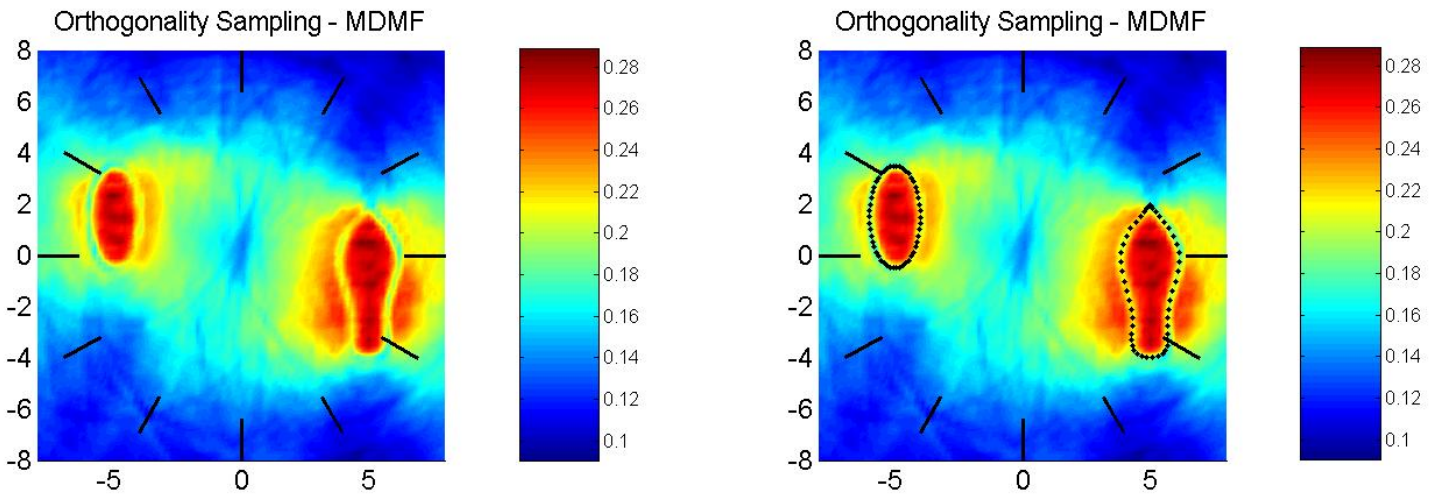
Graphics: Orthogonality sampling for the Neumann BC, MF.

Neumann BC II



Graphics: Orthogonality sampling for the Neumann BC, MDMF.

Neumann BC II



Graphics: Orthogonality sampling for the Neumann BC, MDMF.

Orthogonality Sampling Convergence Dirichlet Case

Theorem (Convergence or Ortho-Sampling, P 2007/08)

The orthogonality sampling algorithm with the Dirichlet boundary condition for one-wave fixed frequency reconstructs the reduced scattered field, i.e.

$$u^s_{red}(x) = \int_{\partial D} j_0(\kappa|x-y|)\frac{\partial u(y)}{\partial \nu(y)}\, ds(y), \quad x \in \mathbb{R}^m. \tag{5}$$

Convergence analysis of the method can be based on the Funk-Hecke formula.

Literature

Potthast, R.: Acoustic Tomography by Orthogonality Sampling, Institute of Acoustics Spring Conference, Reading, UK 2008.

Potthast, R: Orthogonality Sampling for Object Visualization, Inverse Problems 2010.

Griesmaier, R: Multi-frequency orthogonality sampling for inverse obstacle scattering problems, Inverse Problems (2011)

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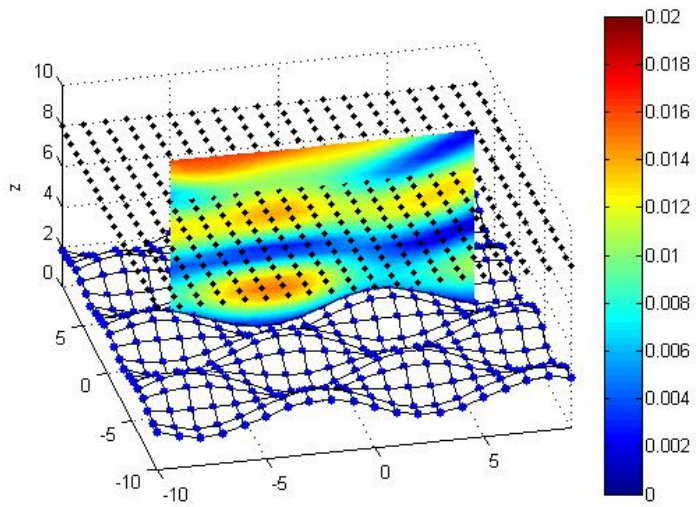
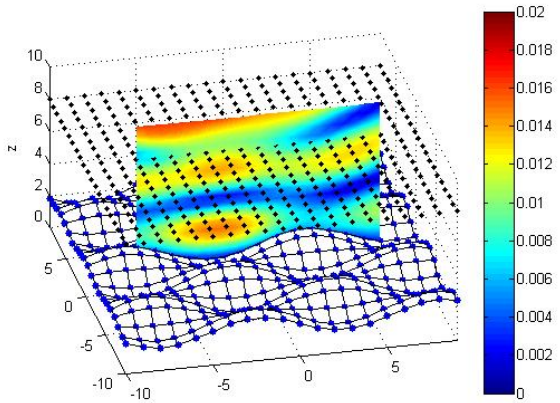
Setup of Inverse Rough Surface Scattering



- Measurements on some surface $\Gamma_{h,A}$
- Unknown surface Γ below measurement surface and above zero-surface. Dirichlet boundary condition.
- Measure total scattered field v from one time-harmonic incident field $G(\cdot, z)$ with source point z above or on $\Gamma_{h,A}$.

Tasks of Inverse Rough Surface Scattering

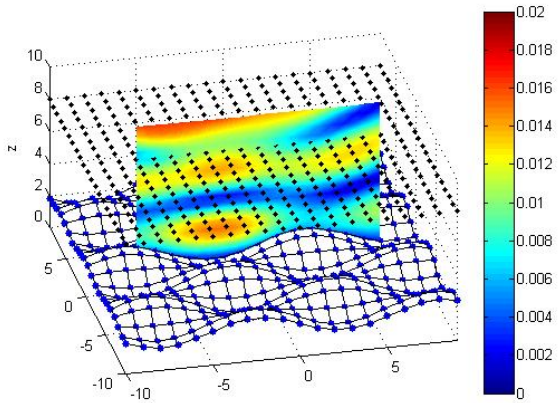


Tasks:

1. Reconstruct the total field u or scattered field u^s. Since the incident field $u^i = G(\cdot, z)$ is known, these tasks are equivalent.
2. Reconstruct the scattering surface Γ or any surface which generates the data for the given incident field $u^i = G(\cdot, z)$.

Remark. If we do this for sufficiently many incident waves simultaneously, we have uniqueness of the reconstruction.

Tasks of Inverse Rough Surface Scattering



Tasks:

1. Reconstruct the total field u or scattered field u^s. Since the incident field $u^i = G(\cdot, z)$ is known, these tasks are equivalent.
2. Reconstruct the scattering surface Γ or any surface which generates the data for the given incident field $u^i = G(\cdot, z)$.

Remark. If we do this for sufficiently many incident waves simultaneously, we have uniqueness of the reconstruction.

Inverse Rough Surface Scattering

Method of Kirsch-Kress or Potential Method I

The idea of the Kirsch-Kress method is to calculate an approximation of the scattered field by a single-layer approach $S\varphi$ defined on a subset of an auxiliary surface Γ_t.

It is carried out by minimization of the Tikhonov functional

$$J_{\alpha,B} = \underbrace{\|SP_B\varphi - v\|^2_{L^2(\Gamma_{h,A})}}_{=\text{measured data}} + \alpha \underbrace{\|P_B\varphi\|^2}_{=\text{regularization}} . \tag{6}$$

The unknown scattering surface Γ by the minimization of the approximated total field

$$\|u\|_{L^2(\Gamma)} = \underbrace{\|G(\cdot, z) + SP_B\varphi\|_{L^2(\Gamma)}}_{=\text{field on surface}} \tag{7}$$

over some suitable set U of admissible surfaces Γ.

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Inverse Rough Surface Scattering

Method of Kirsch-Kress or Potential Method II

- Burkard, C. and Potthast, R.: A multi-section approach for rough surface reconstruction via the Kirsch-Kress scheme, Inverse Problems Vol. 26, No. 4, 2010.
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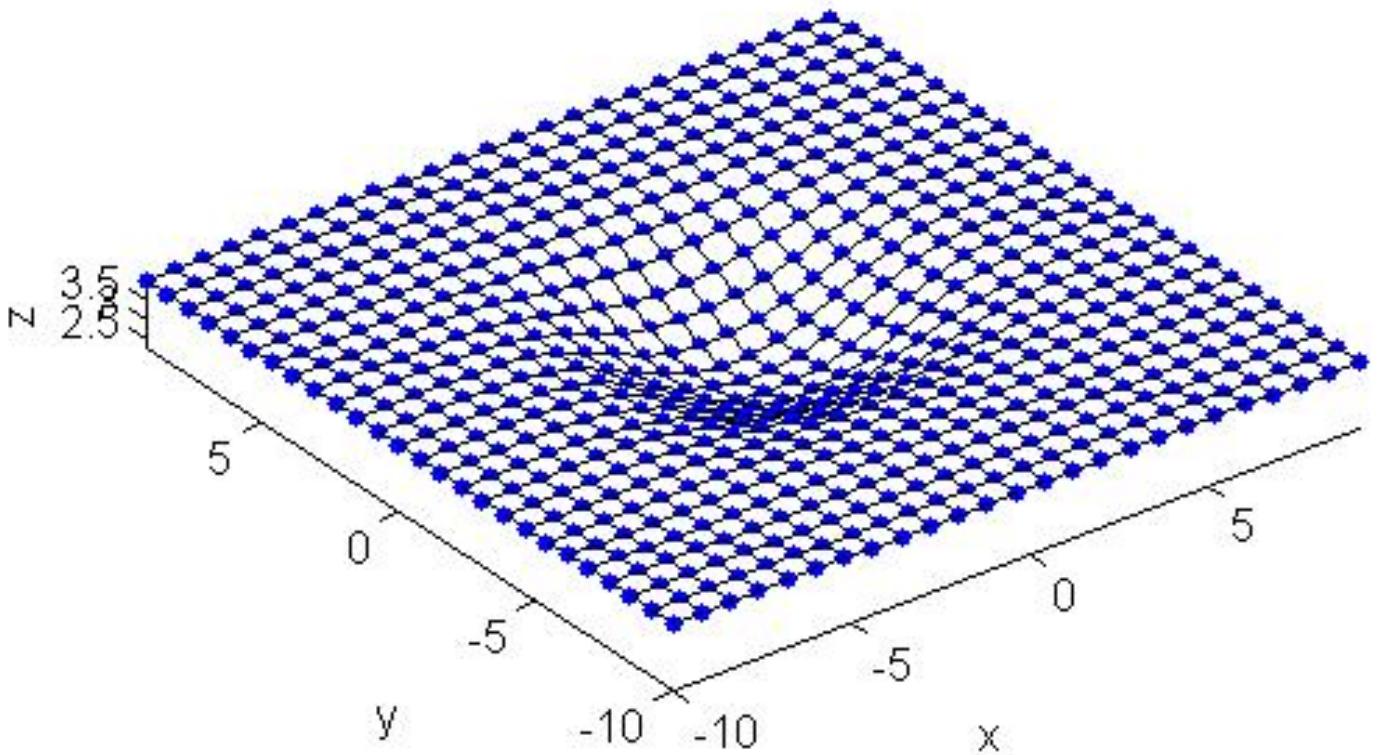
Inverse Rough Surface Scattering

Numerical Examples I



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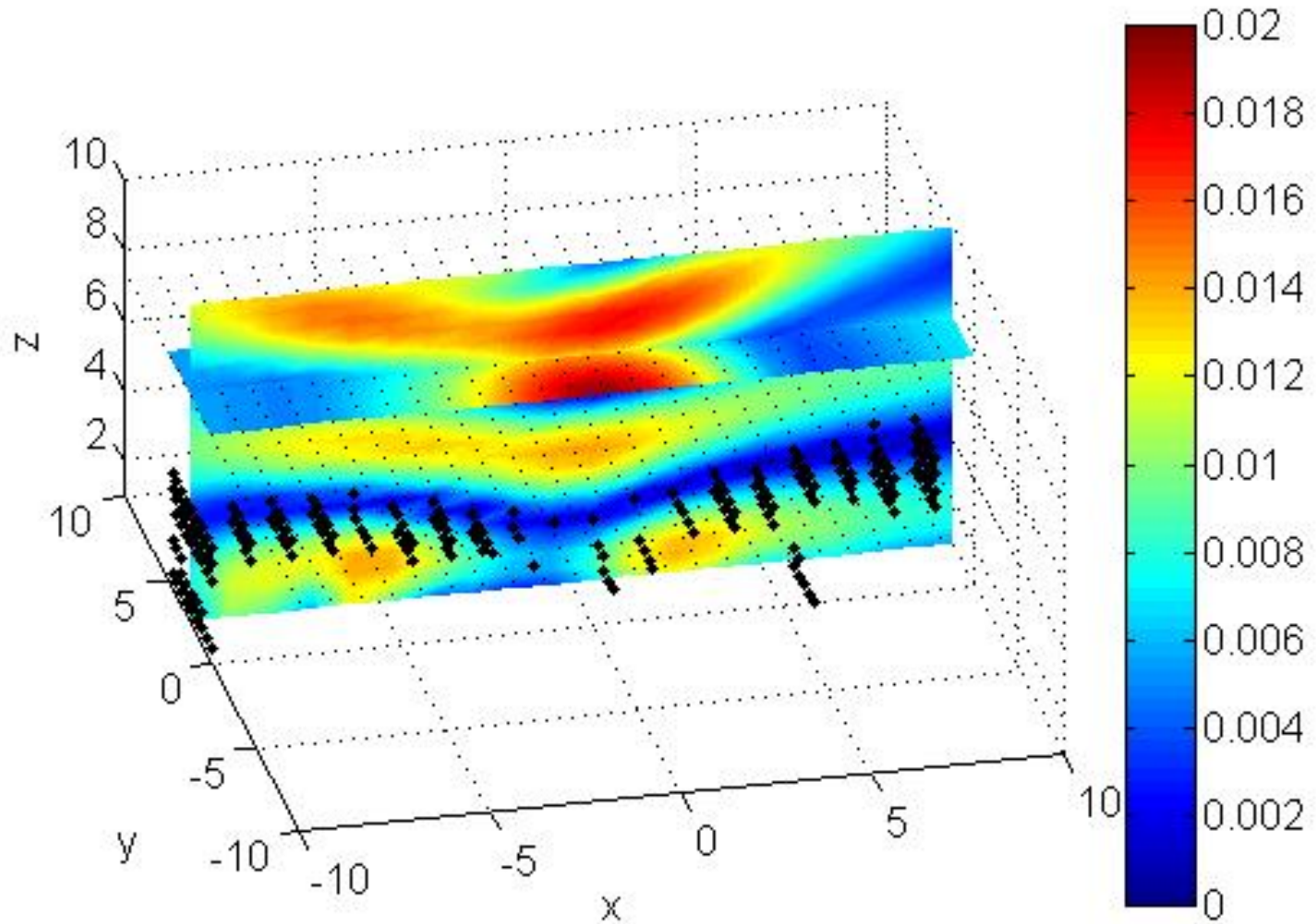

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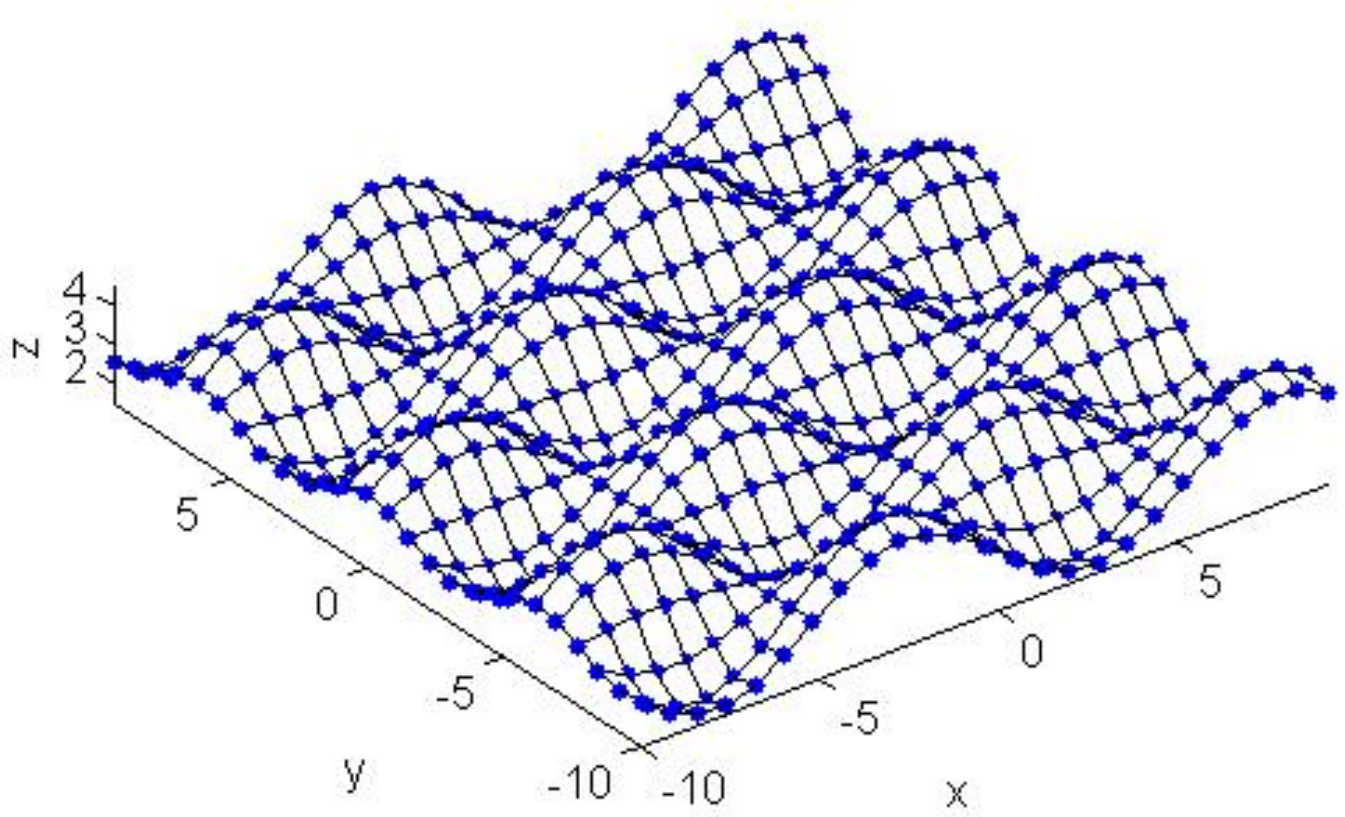

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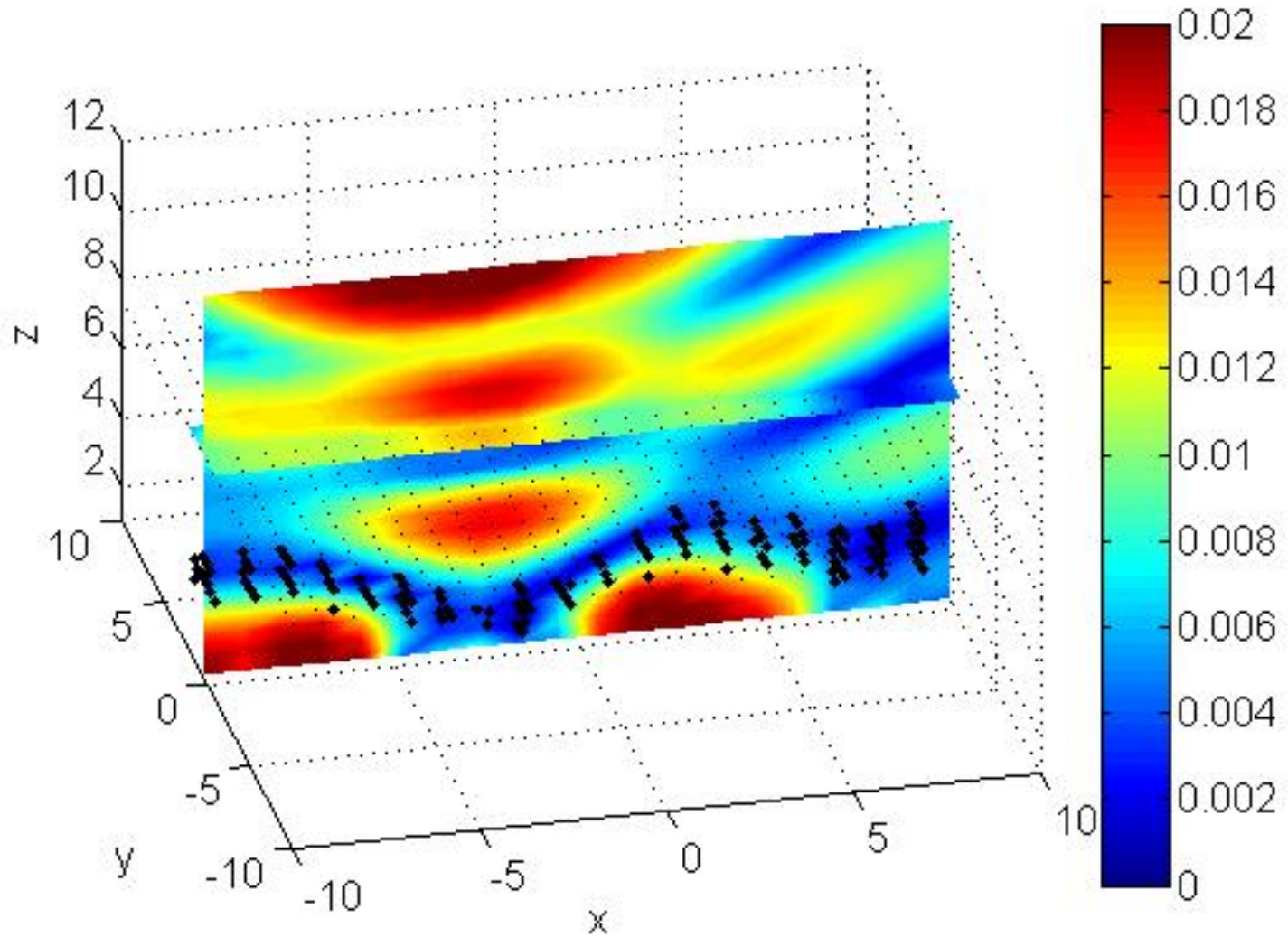

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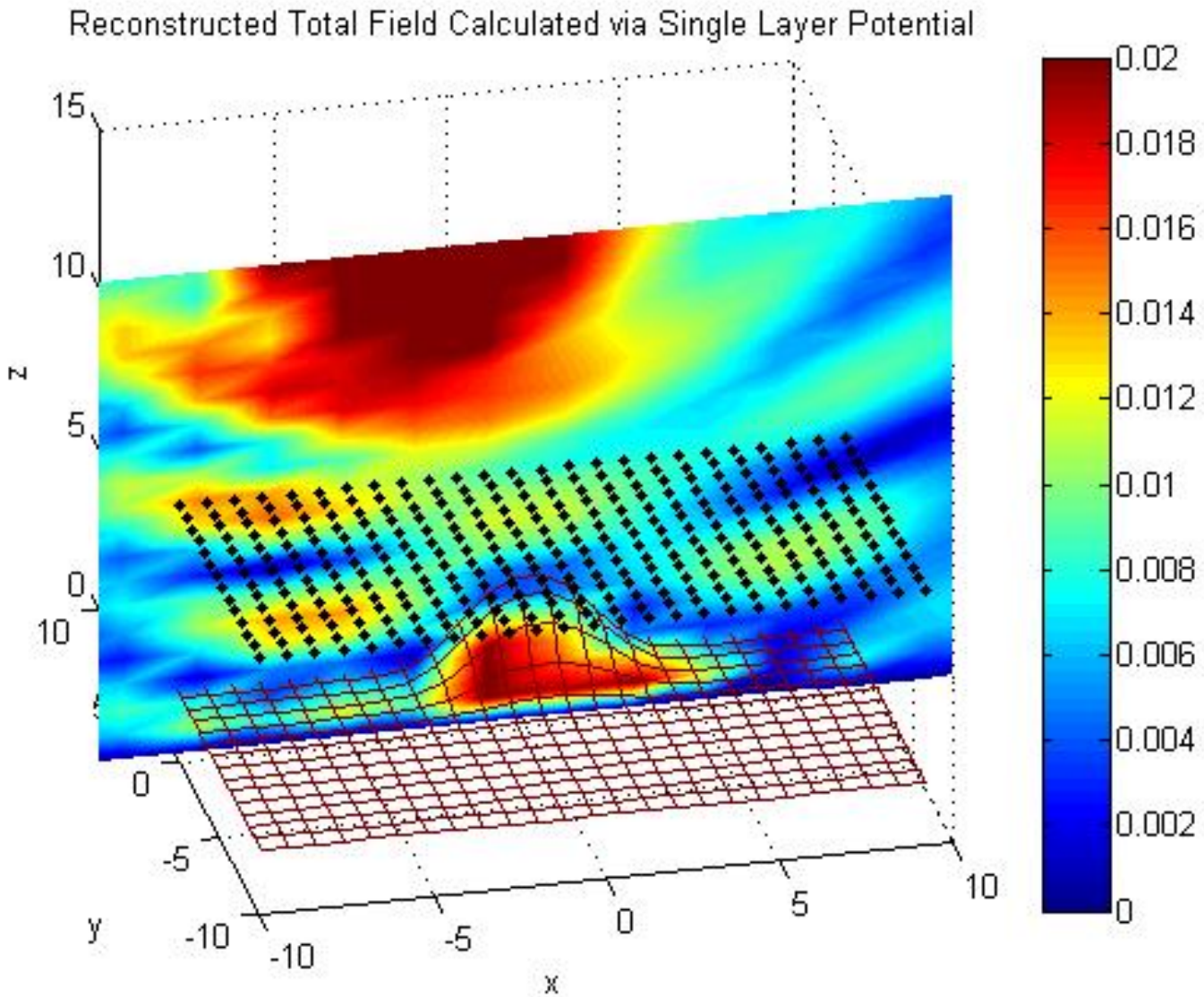
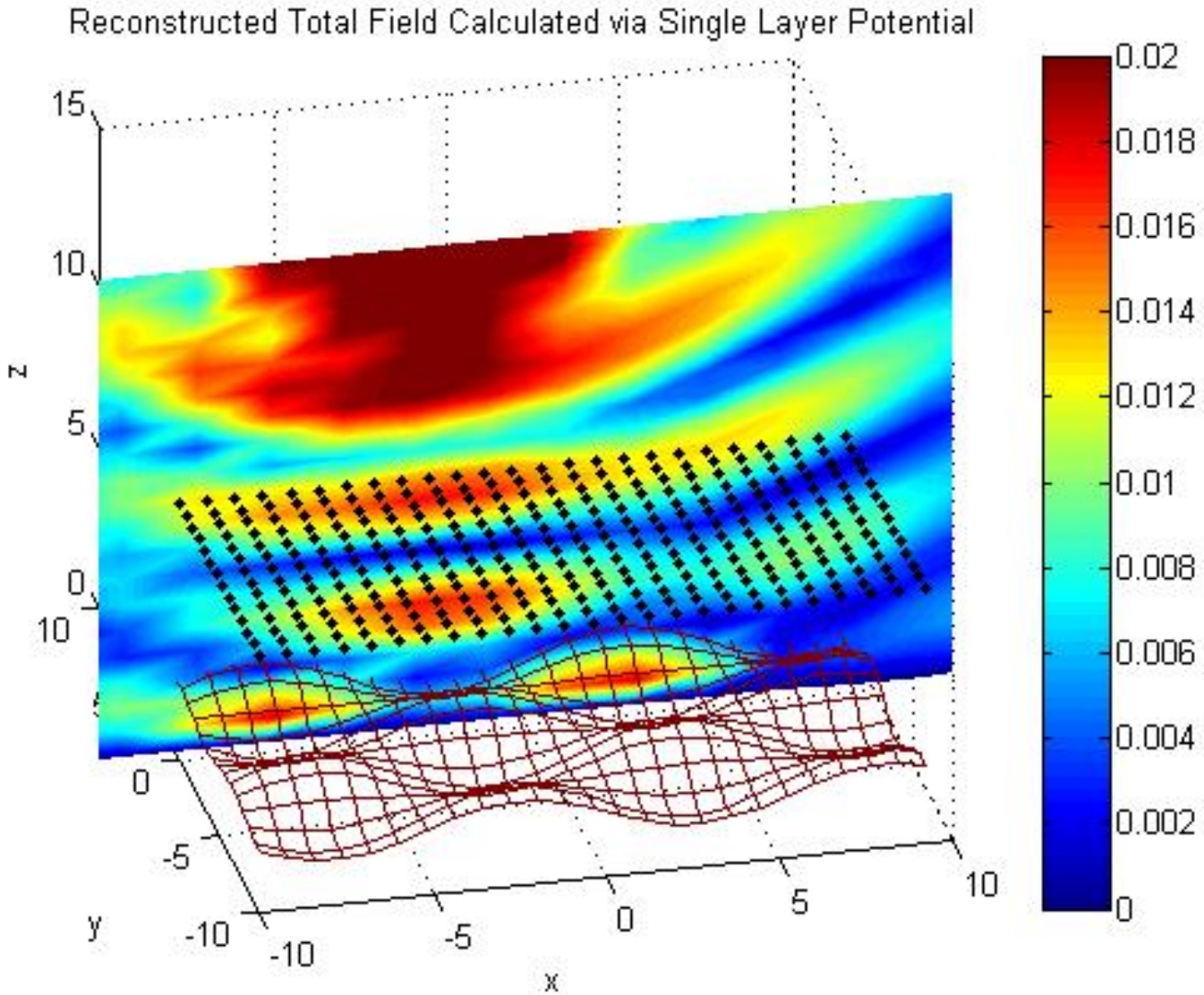


Inverse Rough Surface Scattering

Numerical Examples IV



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Orthogonality Sampling

Time-Domain Probe Method

- Field & Shape Reconstruction
- Time-Domain Probe

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- 3dVar
- 4dVar
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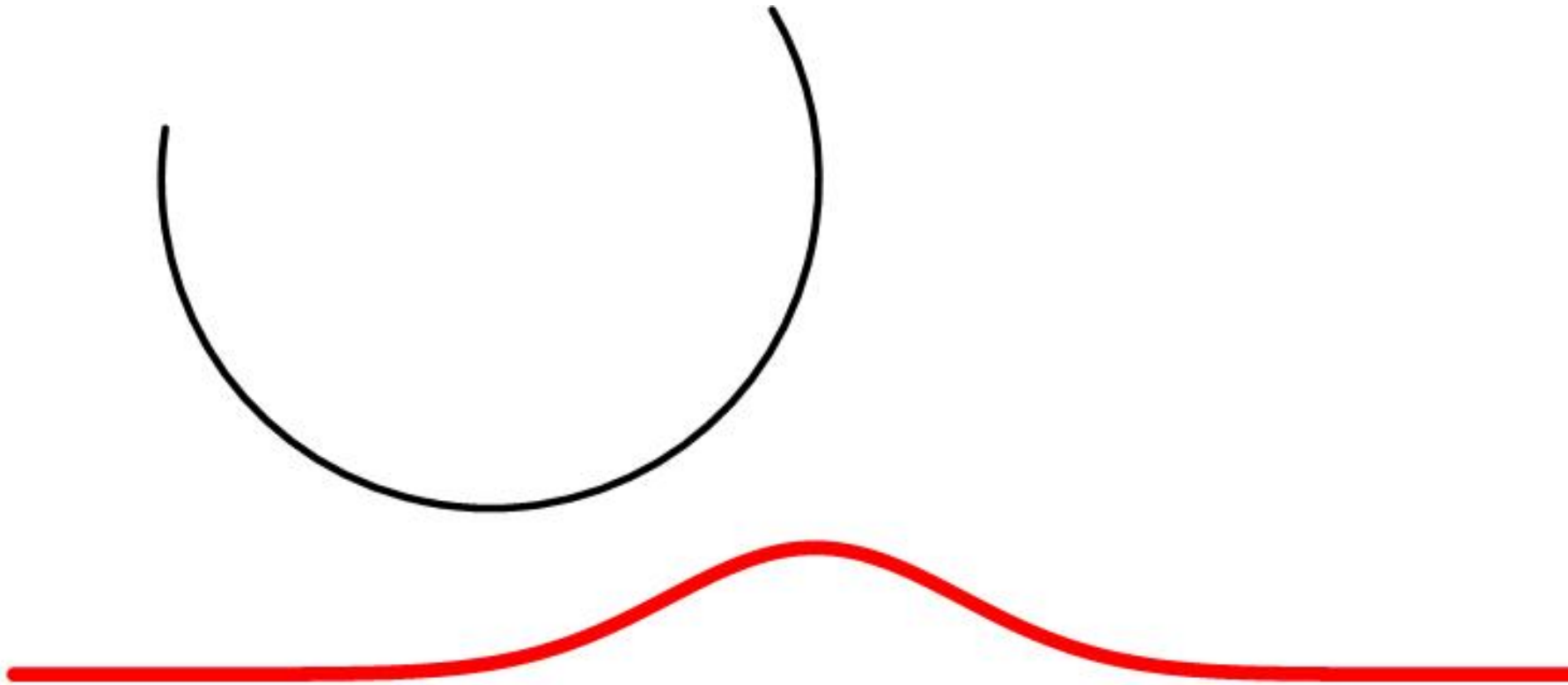
Time-Domain Probe Method: the Idea

- Incident time-dependent pulse coming from some point $z \in D$.
- When the pulse reaches some point of the scattering surface, a scattered field starts to evolve.
- By reconstructing the time-dependent field we can probe the region and determine those points where a scattered field evolves right at the moment when the incident pulse first reaches a particular point.
- Use the potential method of Kirsch-Kress or the point-source method of the author to reconstruct $U^s(x,t)$ for $x \in \Omega$, $t \in \mathbb{R}$.

Inverse Rough Surface Scattering

Time-domain probe method, Idea



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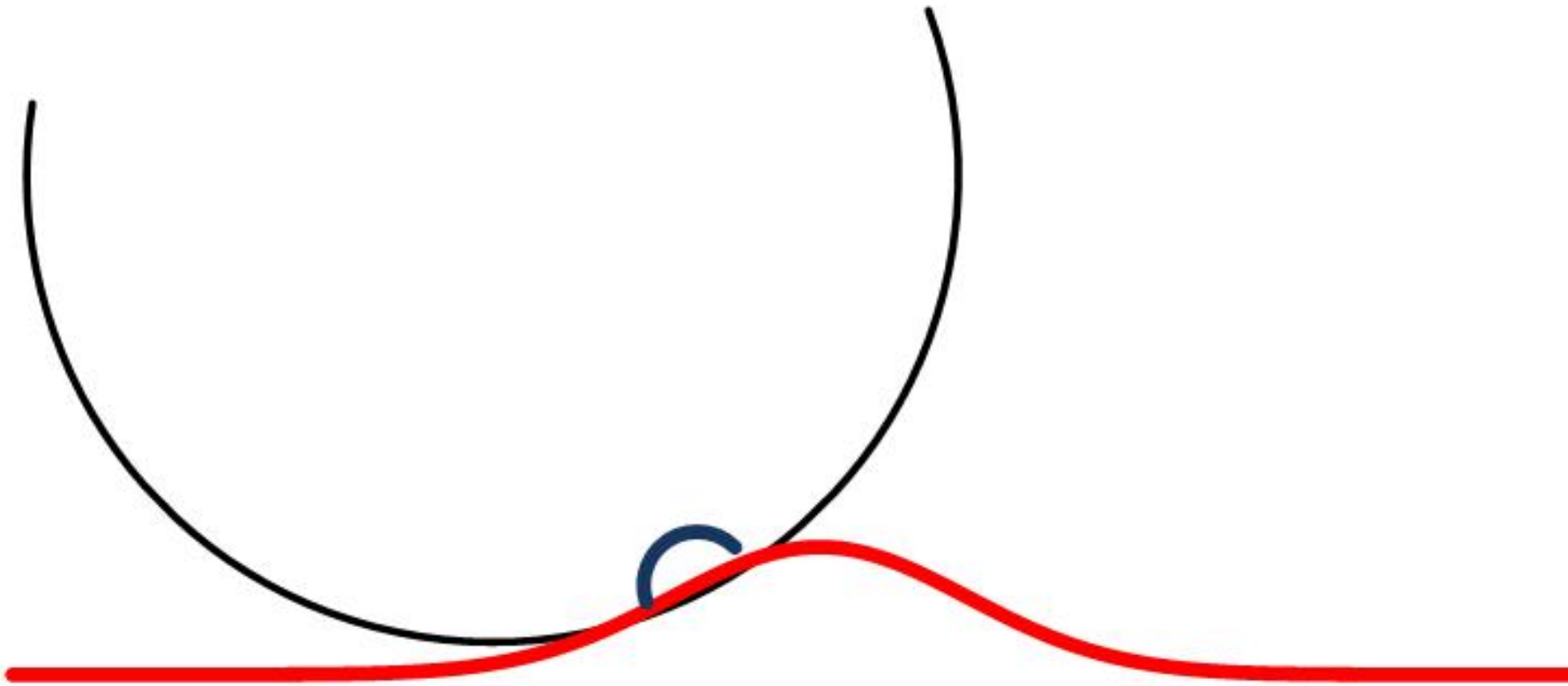

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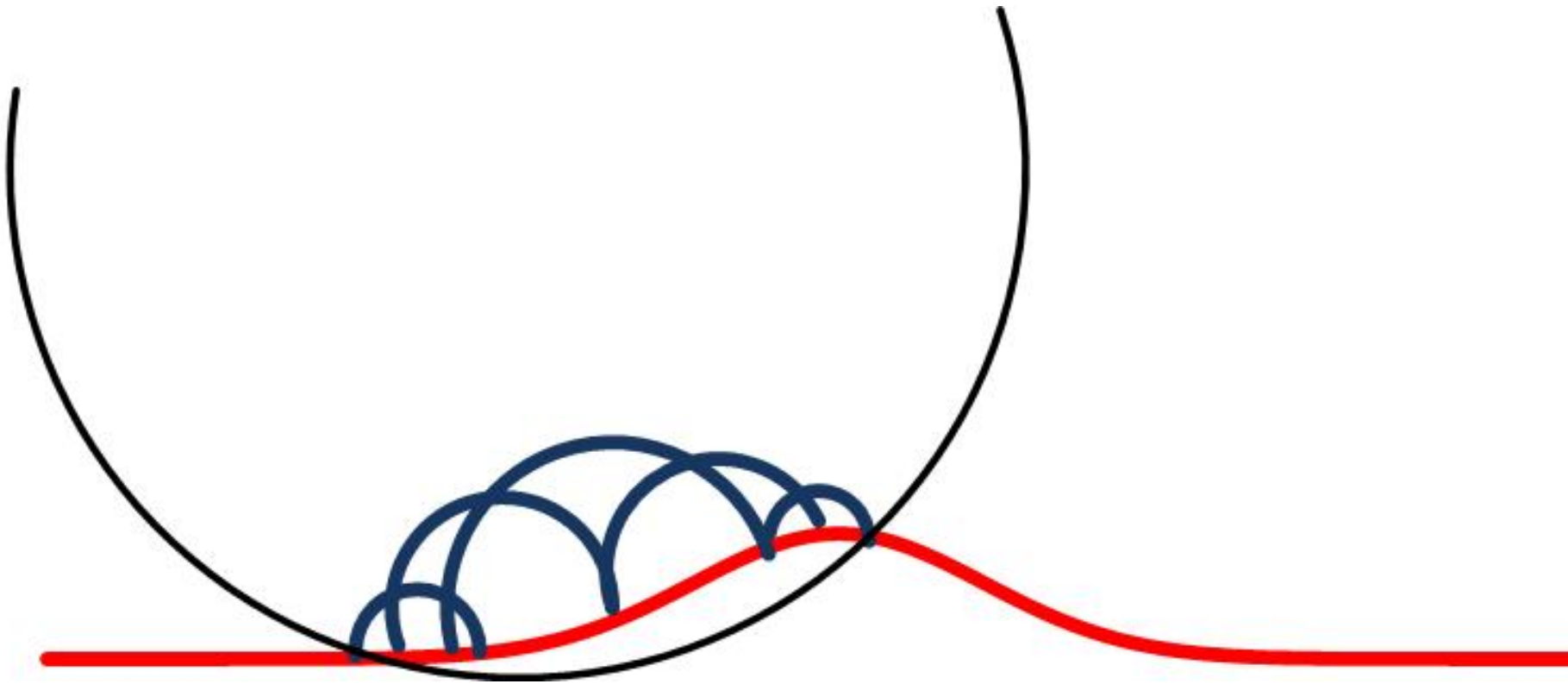

Inverse Rough Surface Scattering

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- Chandler-Wilde, S. and Lines, C.: Inverse Scattering by Rough Surfaces in the Time Domain, Waves 2003
- Chandler-Wilde, S. and Lines, C.: A Time Domain Point Source Method for Inverse Scattering by Rough Surfaces, Computing, Volume 75, Numbers 2-3, (2005), 157-180
- Luke, D.R. and Potthast, R.: The point source method for inverse scattering in the time domain. Math. Meth. Appl. Sci. Volume 29, Issue 13 (2006) 1501-1521
- Burkard, C. and Potthast, R.: A Time-Domain Probe Method for Three-dimensional Rough Surface Reconstructions, Inverse Problems and Imaging, Volume 3, No. 2 (2009)

Inverse Rough Surface Scattering

Time-domain probe method, Characteristics

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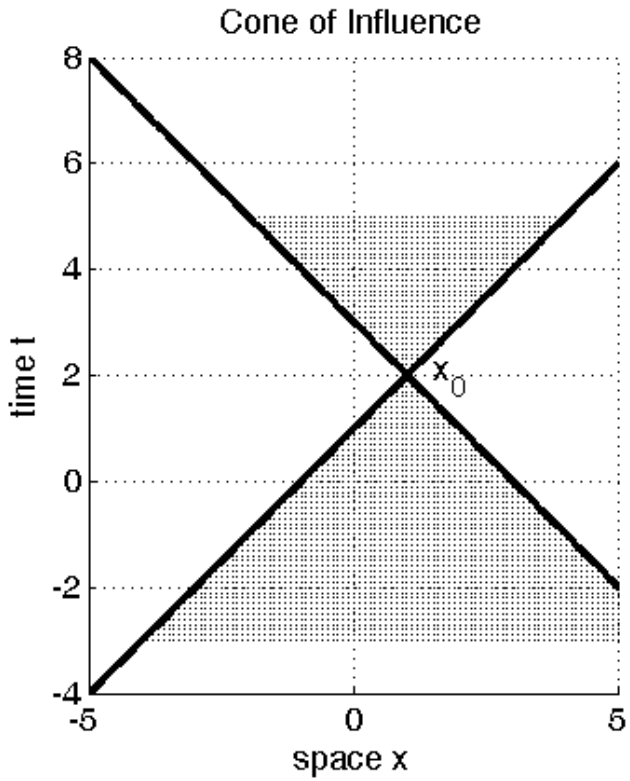

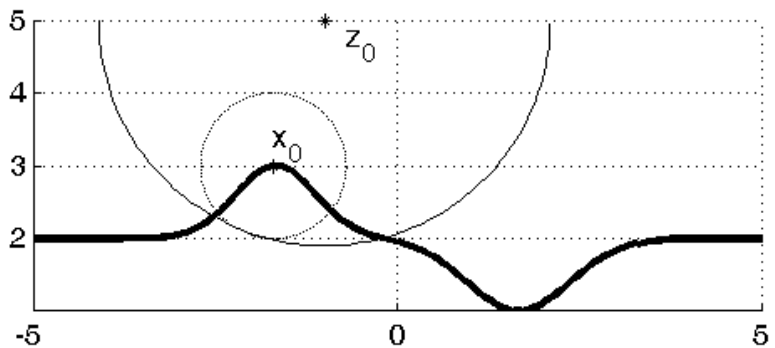

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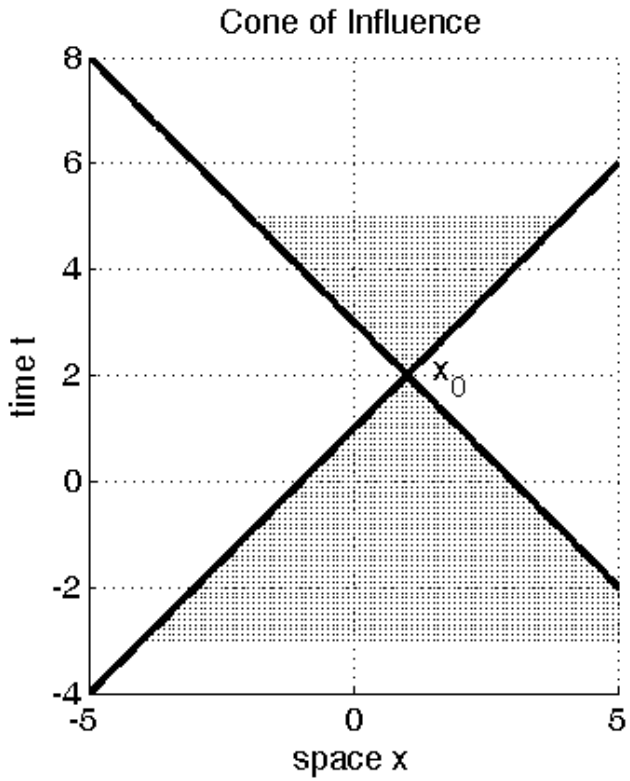

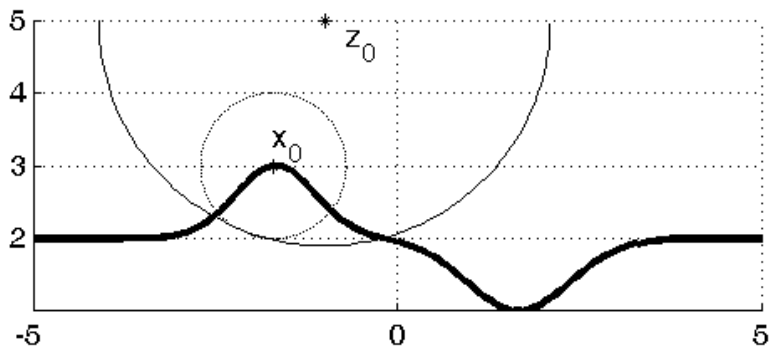

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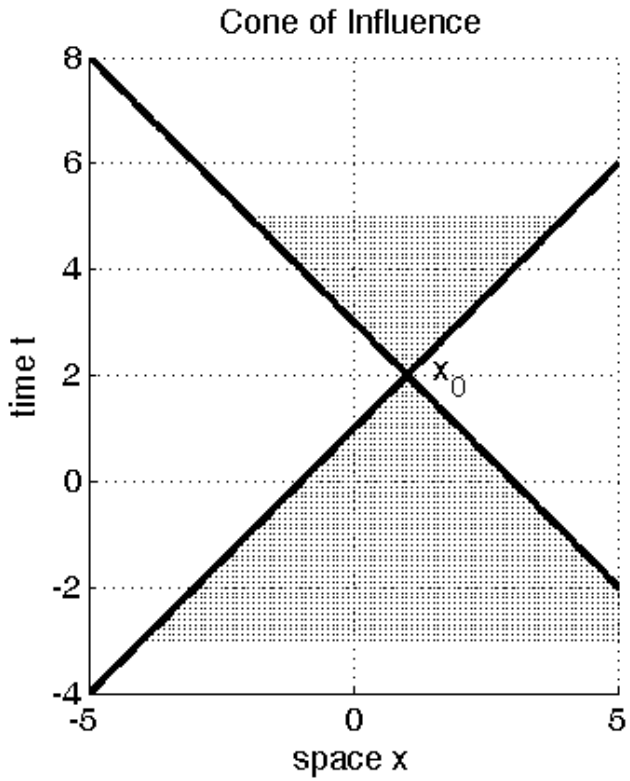
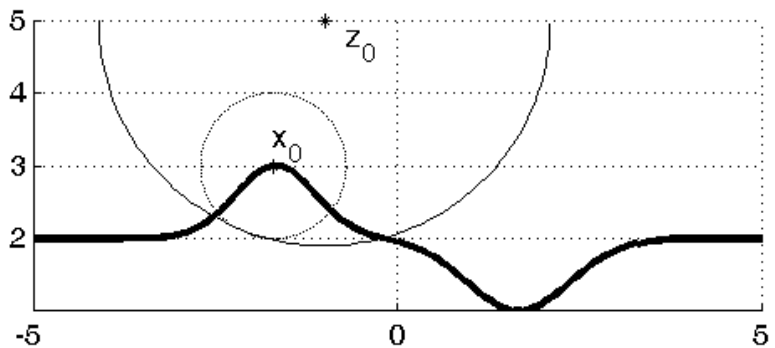
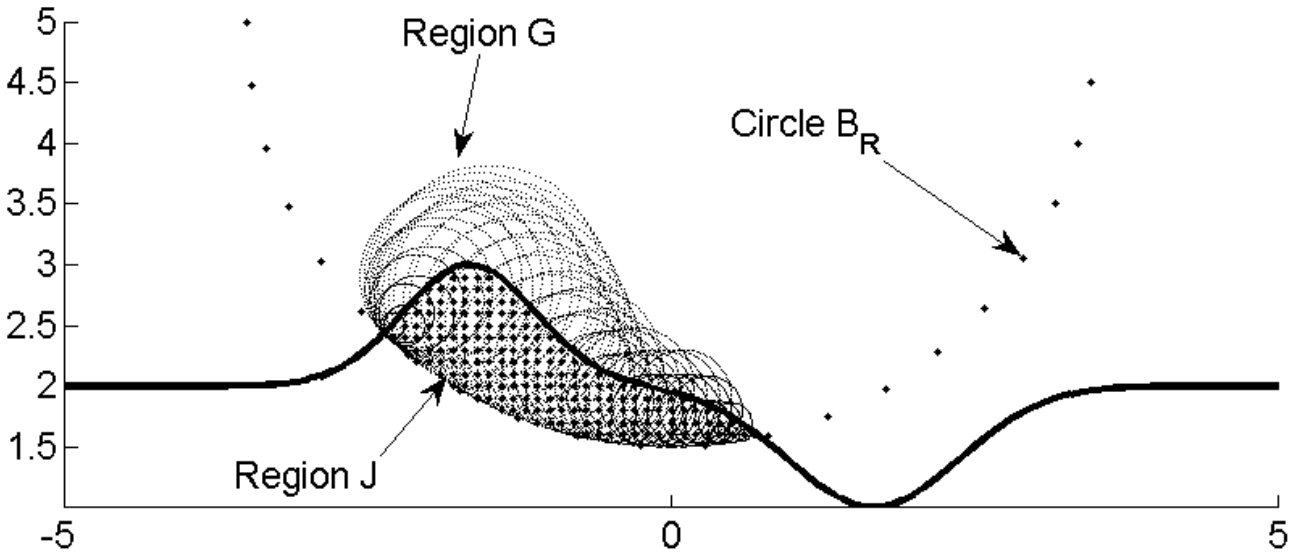




Inverse Rough Surface Scattering

Time-domain probe method, Convergence I

The basic idea behind a convergence proof:



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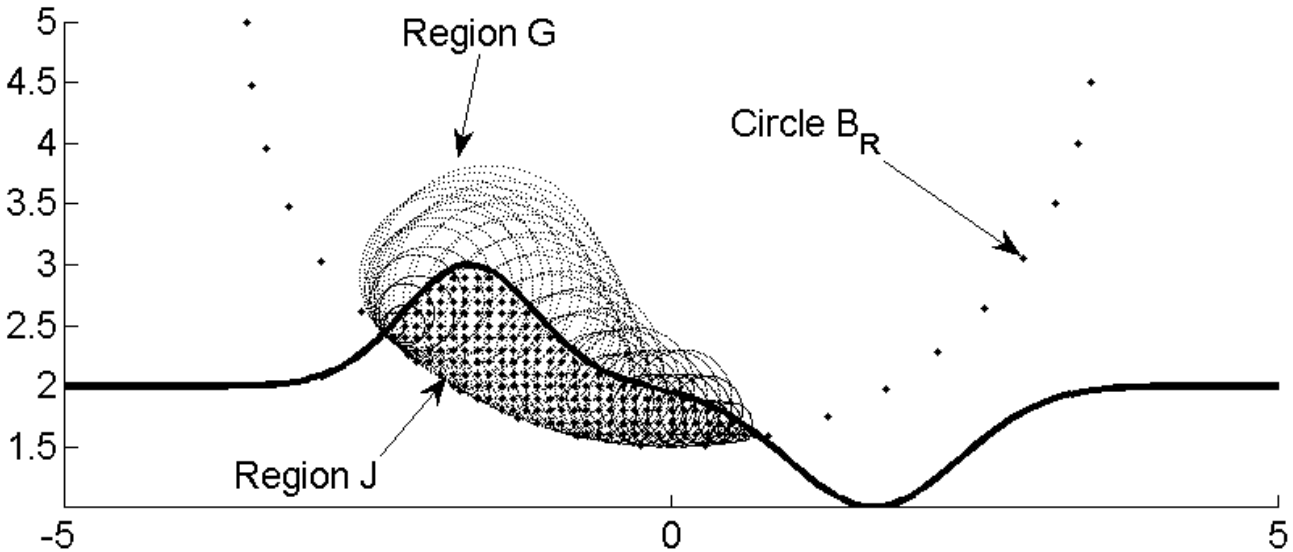

Inverse Rough Surface Scattering

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For a point $x \in \Omega$ we define the first hitting time with respect to the incident field U^i by

$$T(x) := \inf_{t \geq 0} \left| U^i(x,t) \right| > \rho, \tag{8}$$

where we usually employ $\rho = 0$ or small $\rho > 0$ in dependence of the particular choice of the incident field.

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Let U^i be an incident spherical pulse. For every point $x \in \Omega$ we have that

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Since $U^s(x,t) = -U^i(x,t)$ according to the Dirichlet boundary condition, we know that

$$|U^s(x,t)| > \rho \geq 0, \qquad T(x) < t < T(x) + \epsilon, \quad \text{for } x \in \partial\Omega,$$
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for $\epsilon > 0$ sufficiently small. This can be used to detect the boundary $\partial\Omega$.

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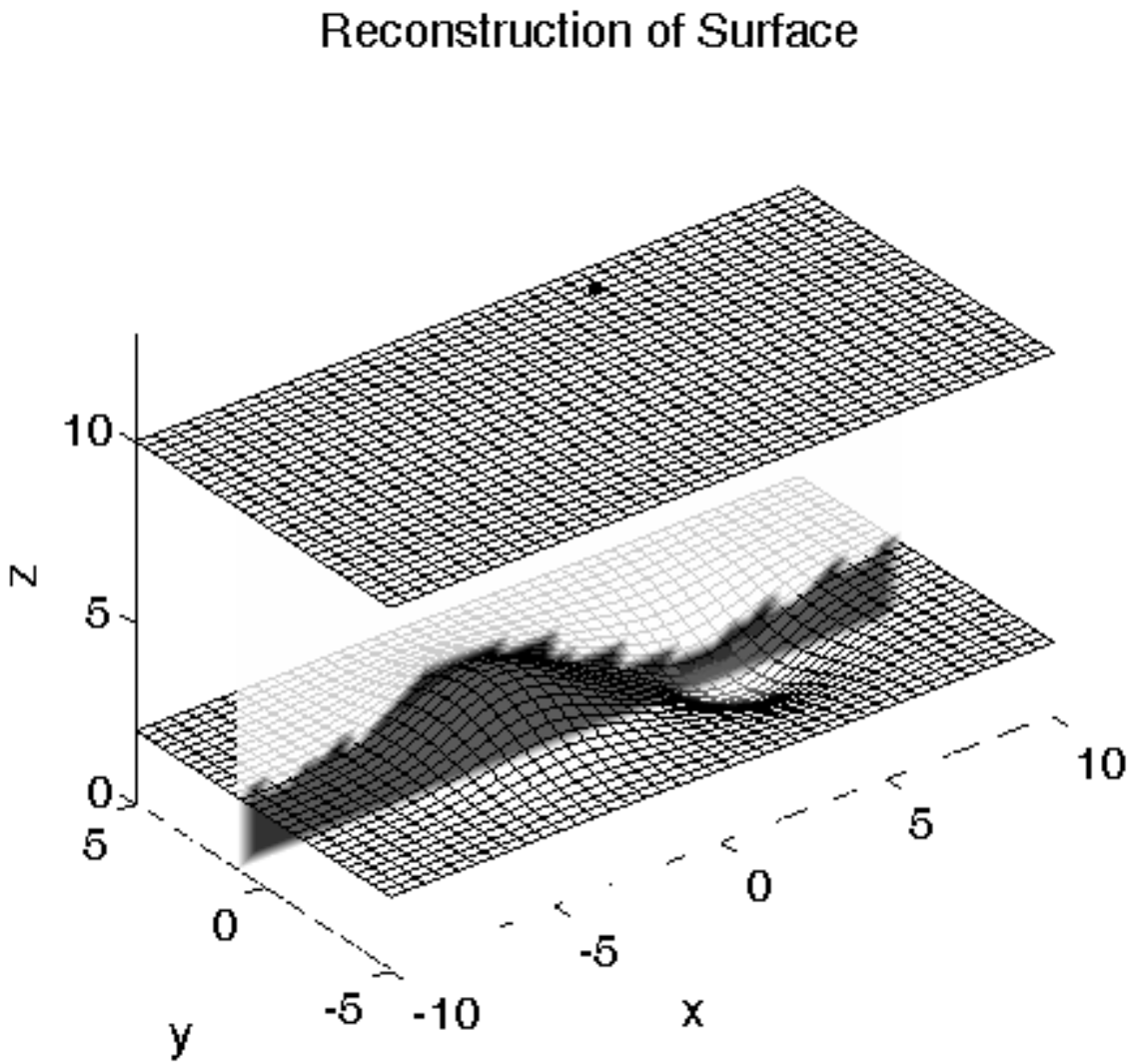

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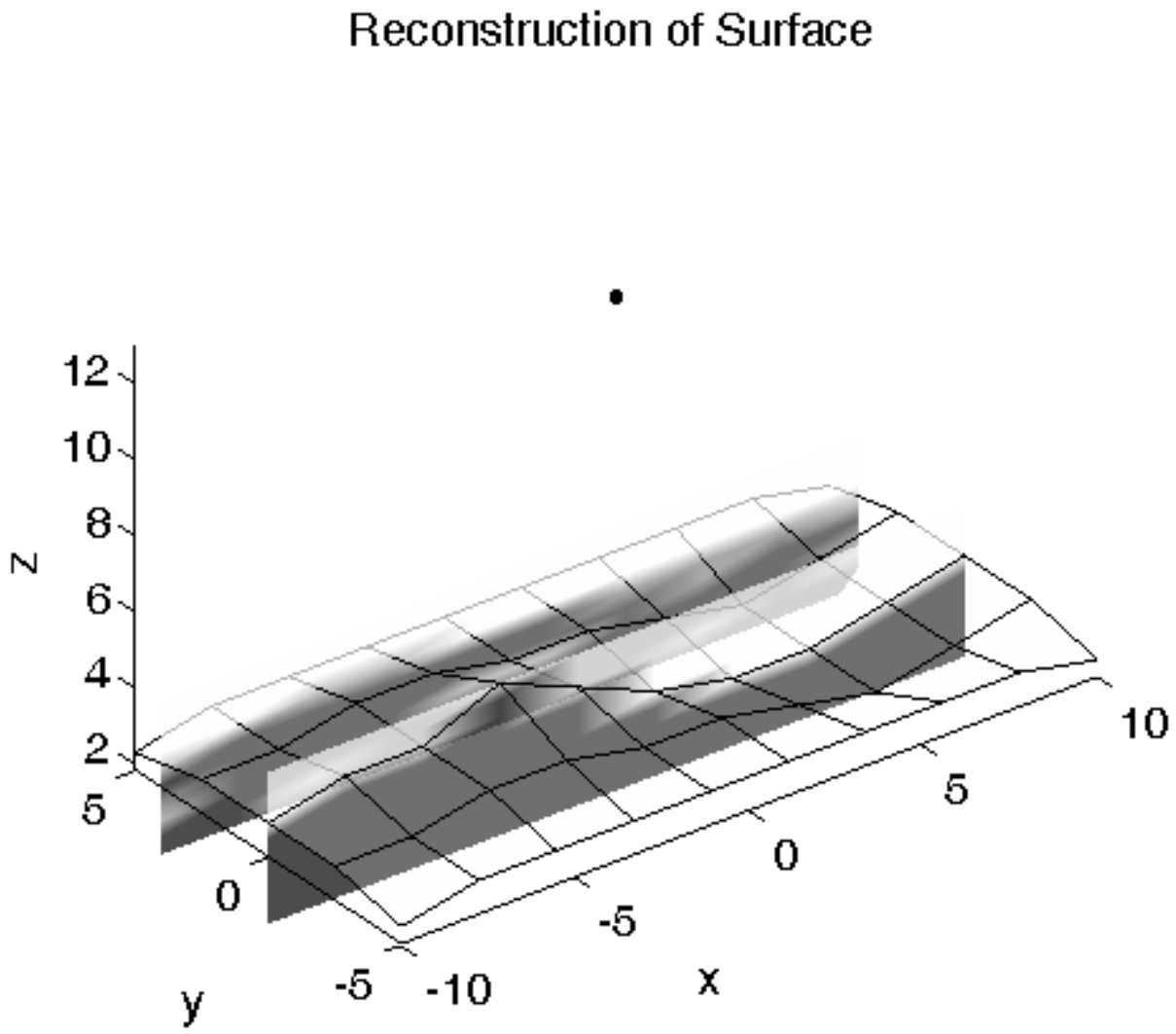

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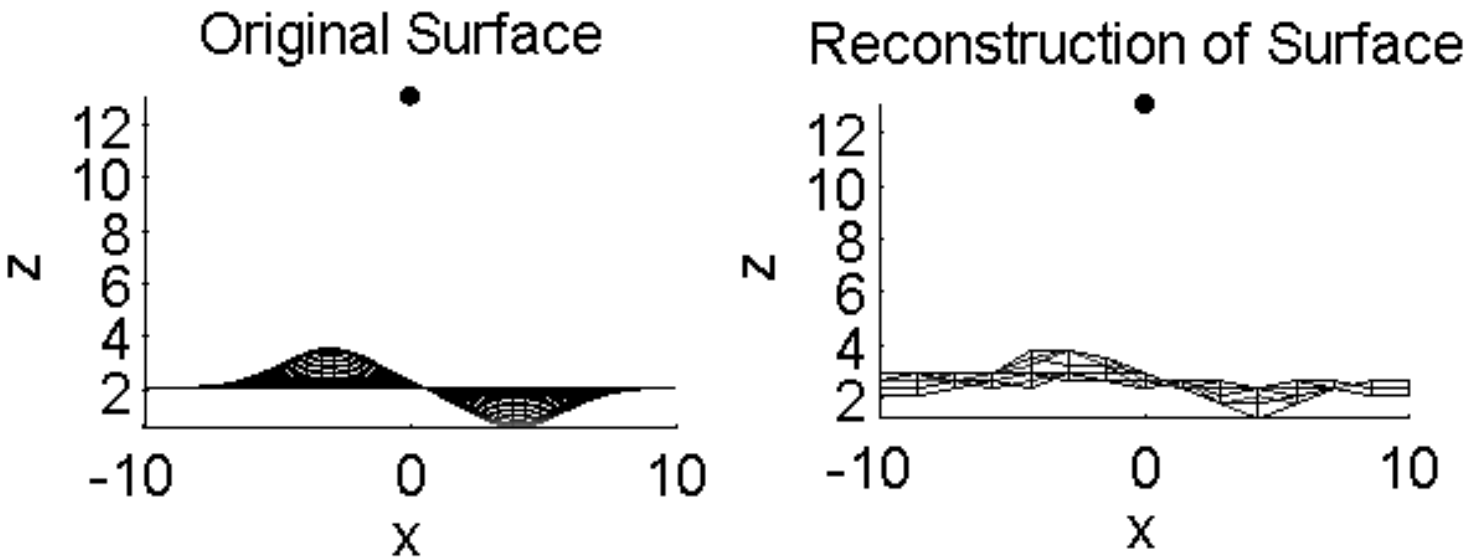

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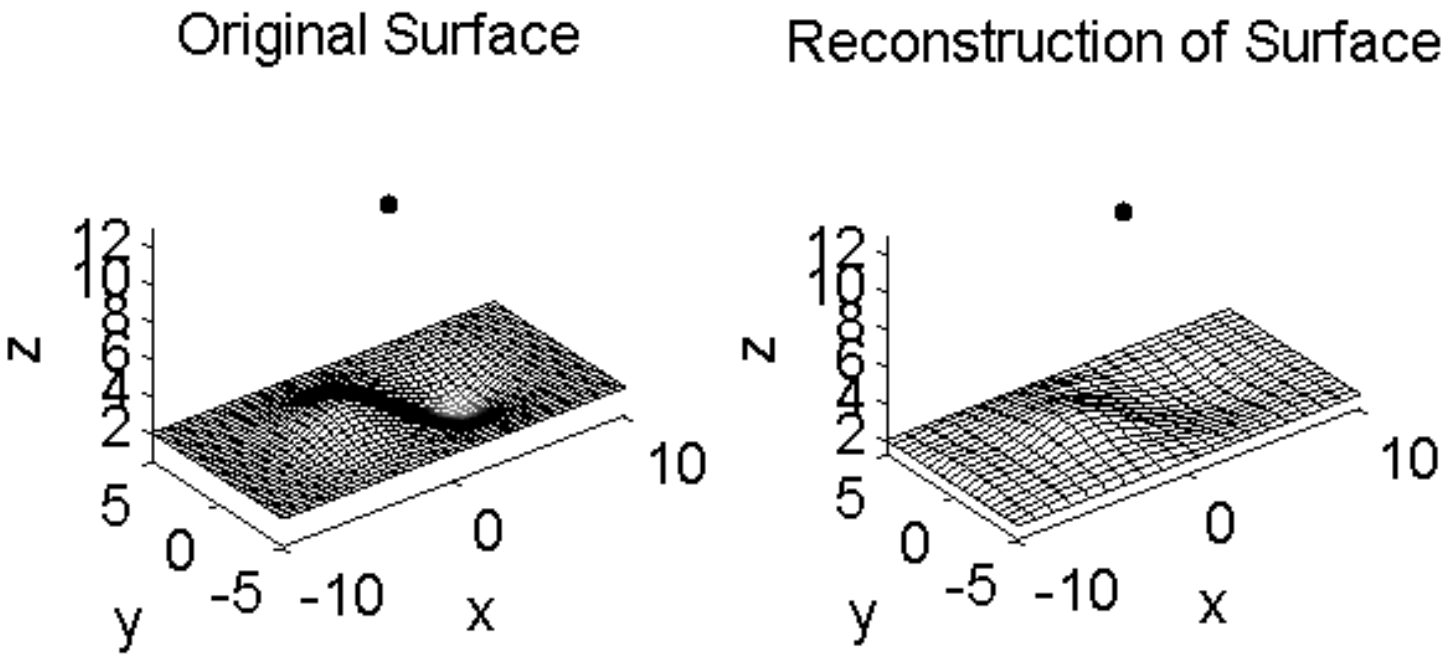

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Data Assimilation Task

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- Measurement Operator $H : \varphi_k \mapsto f_k$ with measurements f_k at time t_k
- Reconstruct φ_k using the knowledge of M and of f_k at t_k!

Basic Notation:

- We call the reconstruction at time t_k the analysis $\varphi_k^{(a)}$.
- The propagated state $\varphi_{k+1}^{(b)} := M(\varphi_k^{(a)})$ is called background.

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Consider an equation

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where H^{-1} is unstable or unbounded.

$$\begin{aligned} & H\varphi = f \\ \Rightarrow \quad & H^* H\varphi = H^* f \\ \Rightarrow \quad & (\alpha I + H^* H)\varphi = H^* f. \end{aligned} \tag{14}$$

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where H^{-1} is unstable or unbounded.

$$\begin{aligned} & H\varphi = f \\ \Rightarrow \quad & H^*H\varphi = H^*f \\ \Rightarrow \quad & (\alpha I + H^*H)\varphi = H^*f. \end{aligned} \tag{14}$$

where $(\alpha I + H^*H)$ has a stable inverse!

Tikhonov Regularization: Replace H^{-1} by the stable version

$$R_\alpha := (\alpha I + H^*H)^{-1}H^* \tag{15}$$

with regularization parameter $\alpha > 0$.

Regularization 2: Least Squares

Tikhonov regularization is equivalent to the minimization of

$$J(\varphi) := \left(\alpha\|\varphi\|^2 + \|H\varphi - f\|^2\right) \tag{16}$$

The **normal equations** are obtained from *first order optimality conditions*

$$\nabla_\varphi J \stackrel{!}{=} 0. \tag{17}$$

Differentiation leads to

$$\begin{aligned} 0 &= 2\alpha\varphi + 2H^*(H\varphi - f) \\ \Rightarrow \quad 0 &= (\alpha I + H^*H)\varphi - H^*f, \end{aligned} \tag{18}$$

which is our well-known *Tikhonov equation*

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Covariances and Weighted Norms

Usually, the relation between variables at different points is incorporated by using **covariances / weighted norms**:

$$J(x) := \left(\|\varphi - \varphi^{(b)}\|^2_{B^{-1}} + \|H\varphi - f\|^2_{R^{-1}} \right) \tag{19}$$

The **update formula** is now

$$\varphi = \varphi^{(b)} + \left(B^{-1} + H^* R^{-1} H\right)^{-1} H^* R^{-1} \left(f - H\varphi^{(b)}\right) \tag{20}$$

or

$$\varphi = \varphi^{(b)} + BH^* \left(R + HBH^*\right)^{-1} \left(f - H\varphi^{(b)}\right). \tag{21}$$

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- Field & Shape Reconstruction
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- 3dVar
- 4dVar
- Dynamic Inverse Scattering

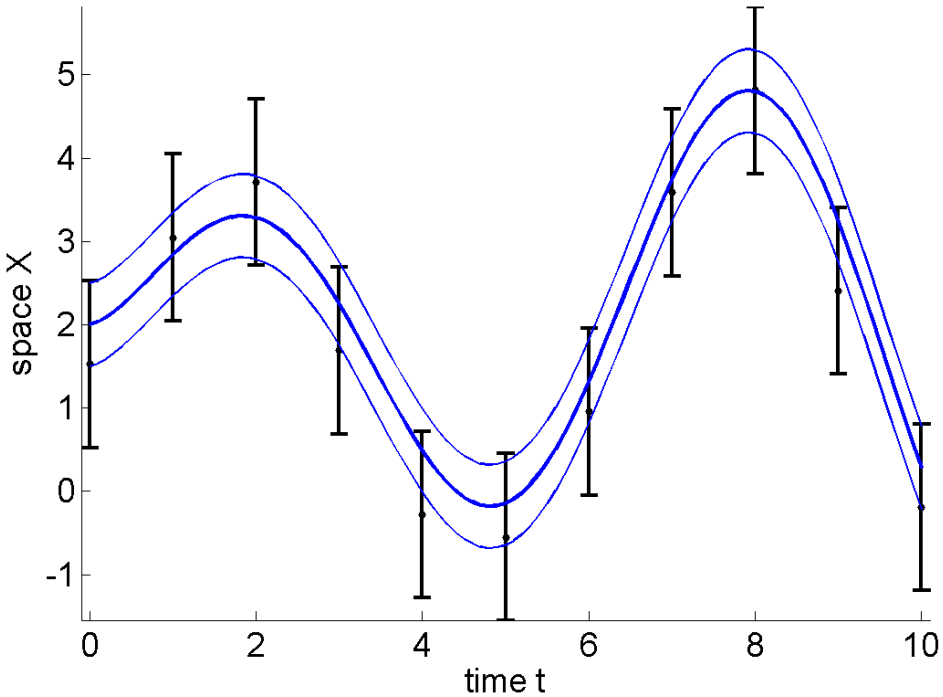
4dVar: Use System Dynamics

So far we have not used the system $M : \varphi_0 \mapsto \varphi(t)$.

Consider some regular grid in time:

$$t_k = \frac{k}{n} T, \tag{22}$$

$$\varphi_k := \varphi(t_k) = M(t_k)\varphi_0, \quad k = 0, \ldots \tag{23}$$



The **4dVar functional** is given by:

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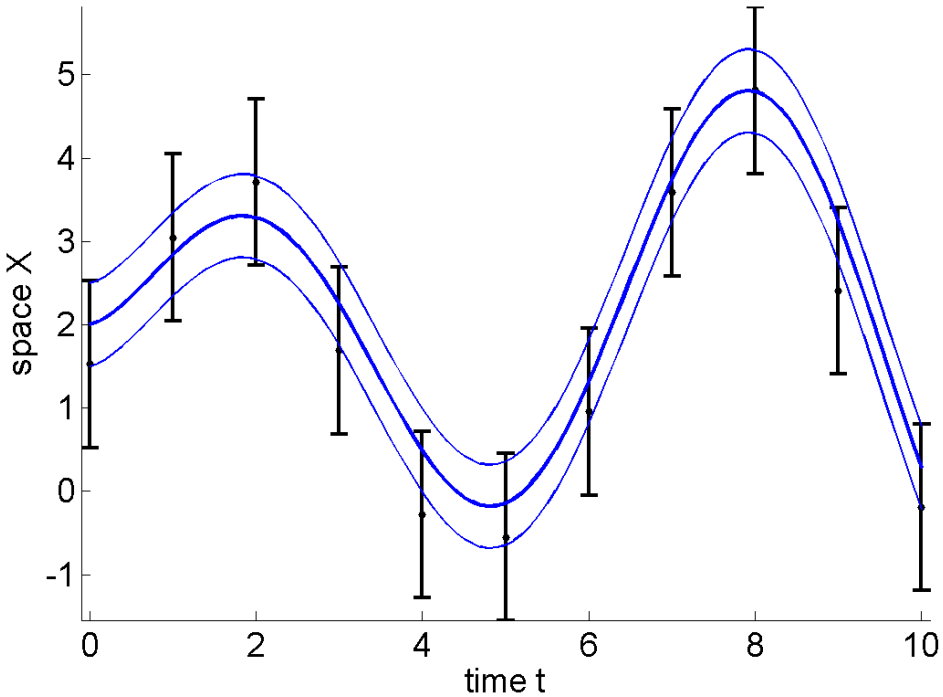

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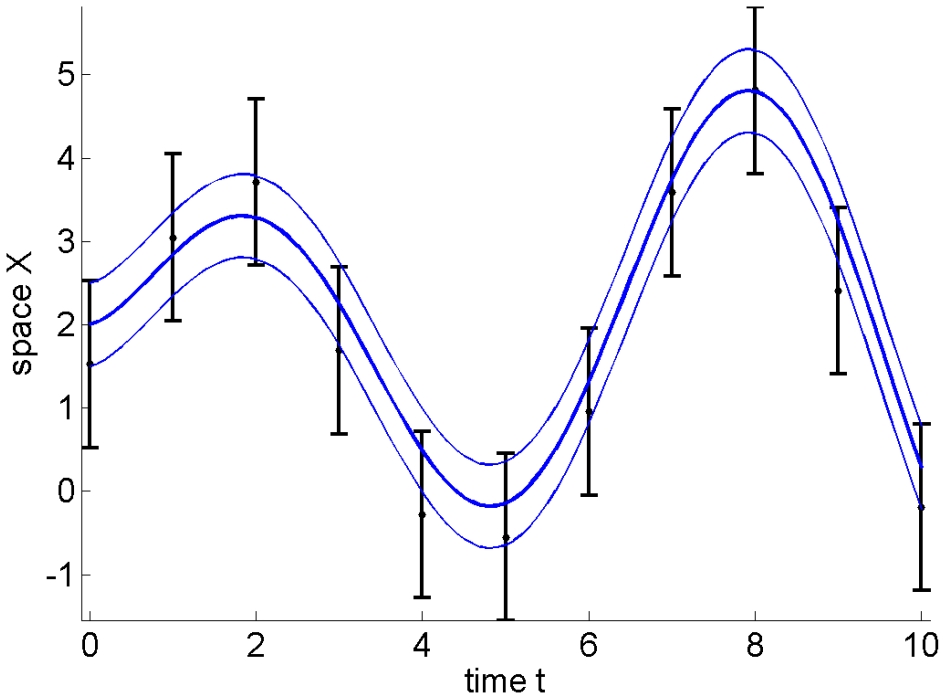

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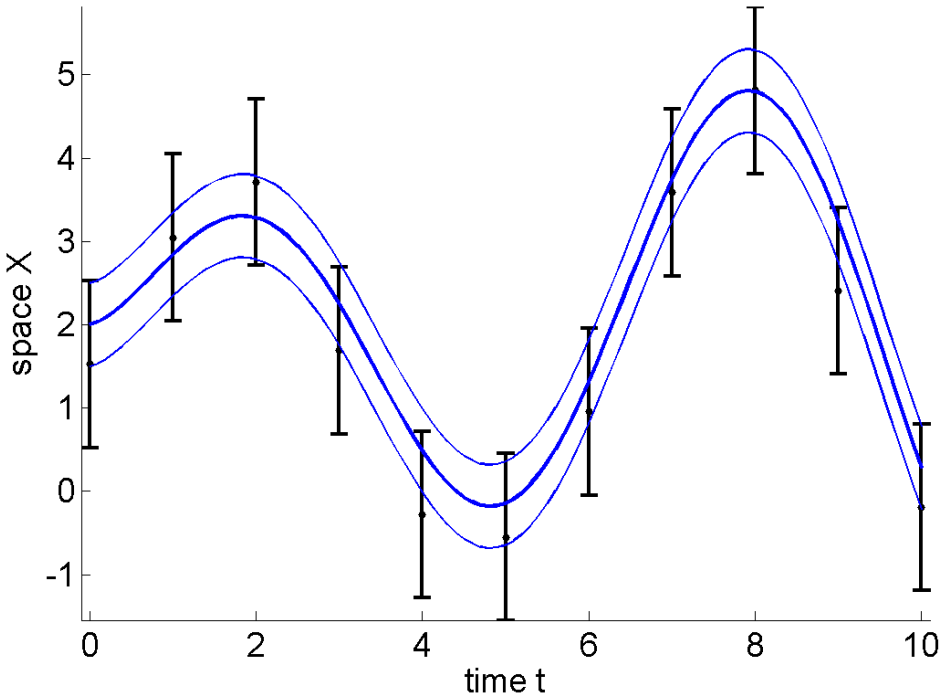

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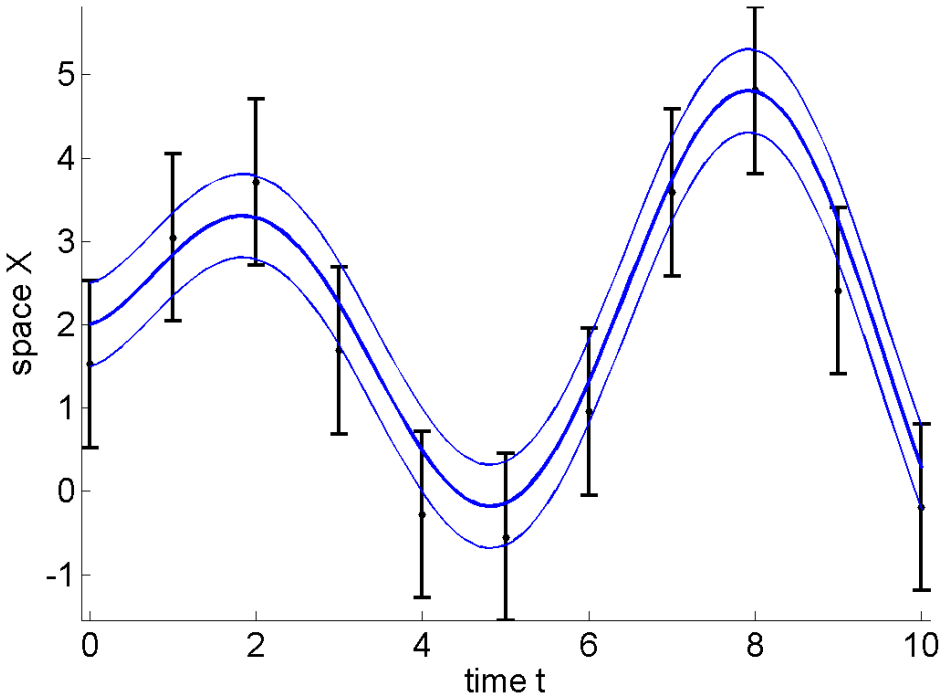

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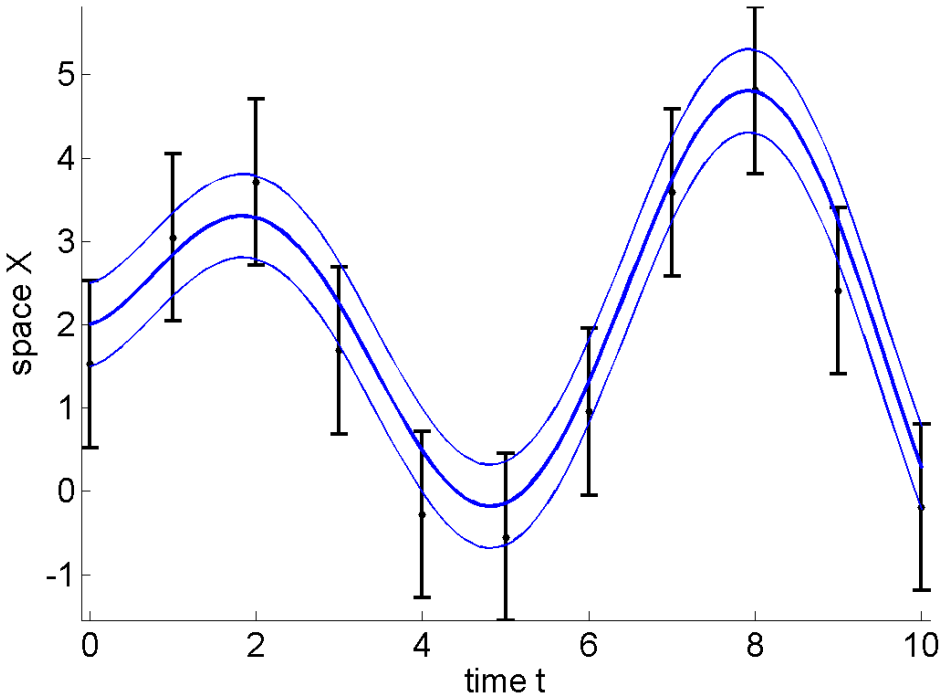

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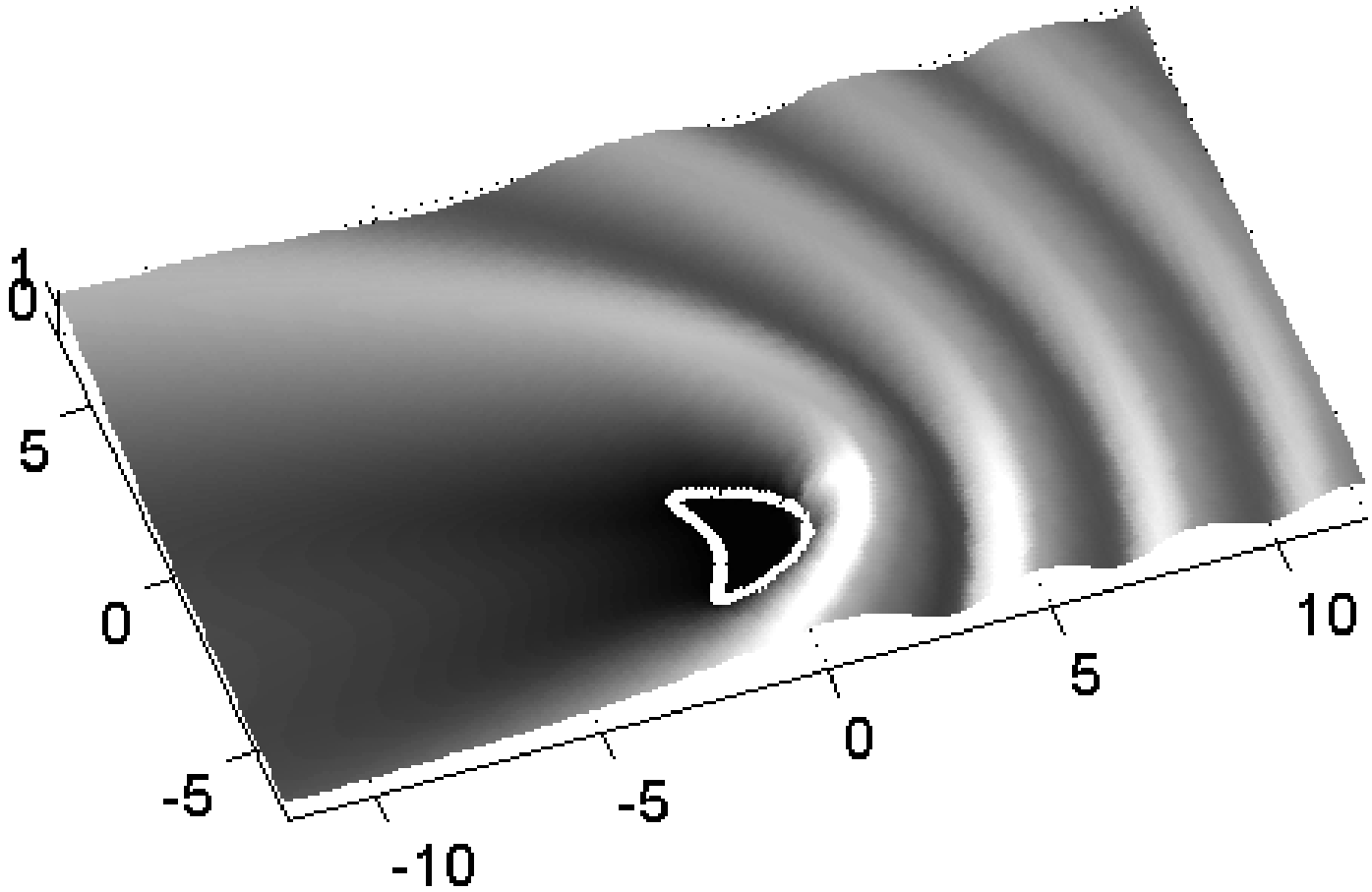
Scattering



Dynamic Inverse Problem: Moving Scatterer

- Moving Scatterer
- Wave scattering at times t_k, $k = 1, 2, 3, \ldots$, temporal scales separated!
- Measurements of the far field patterns u_k^∞ at time t_k.
- Task: Track Location of the Scatterer
- Systems M: dynamics is movement to the right with unknown v_2-component of the speed v, only known approximately!
- For numerical example: form of scatterer known, local inversion using the point source method (P. 1996) or Kirsch-Kress method (1986)

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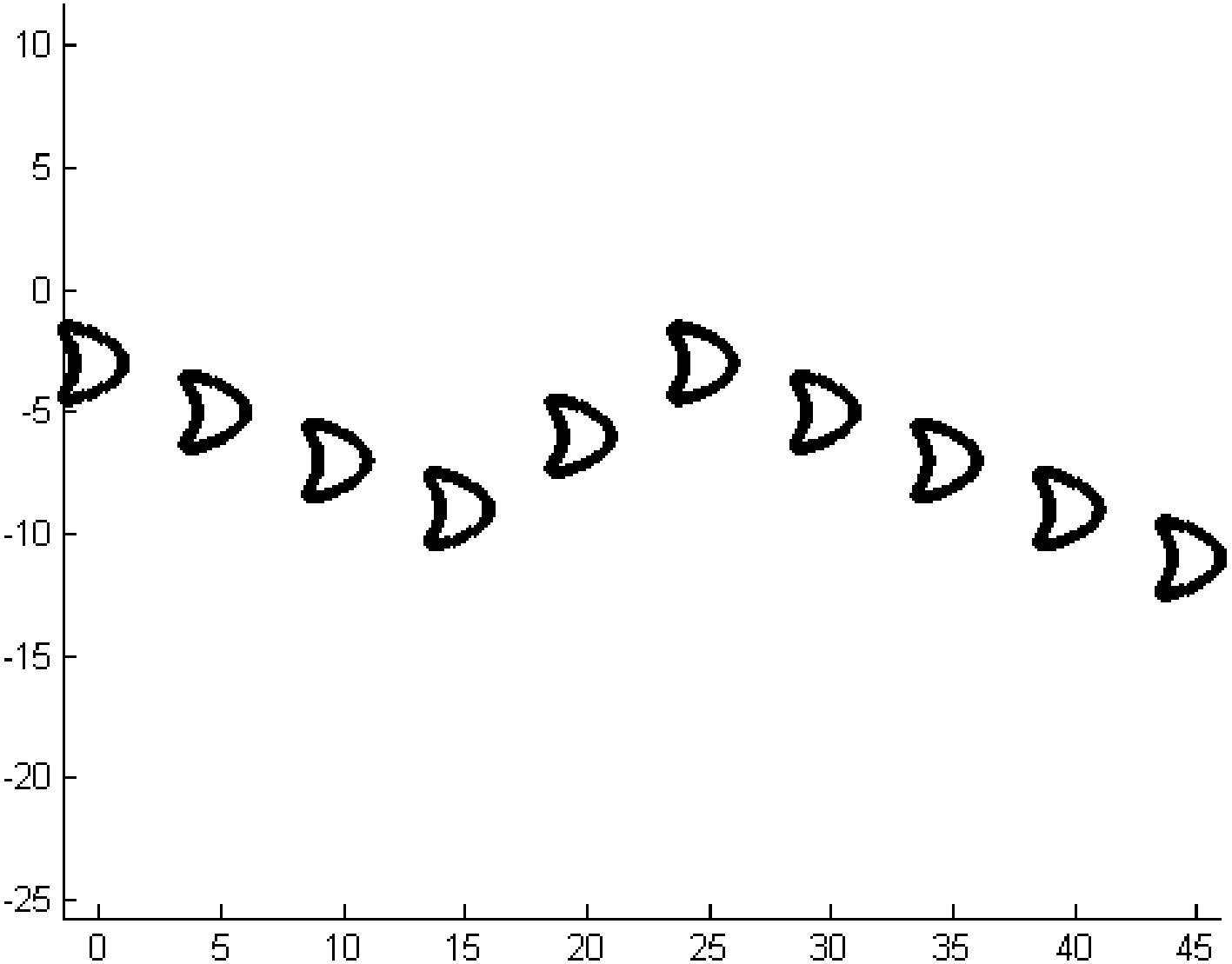

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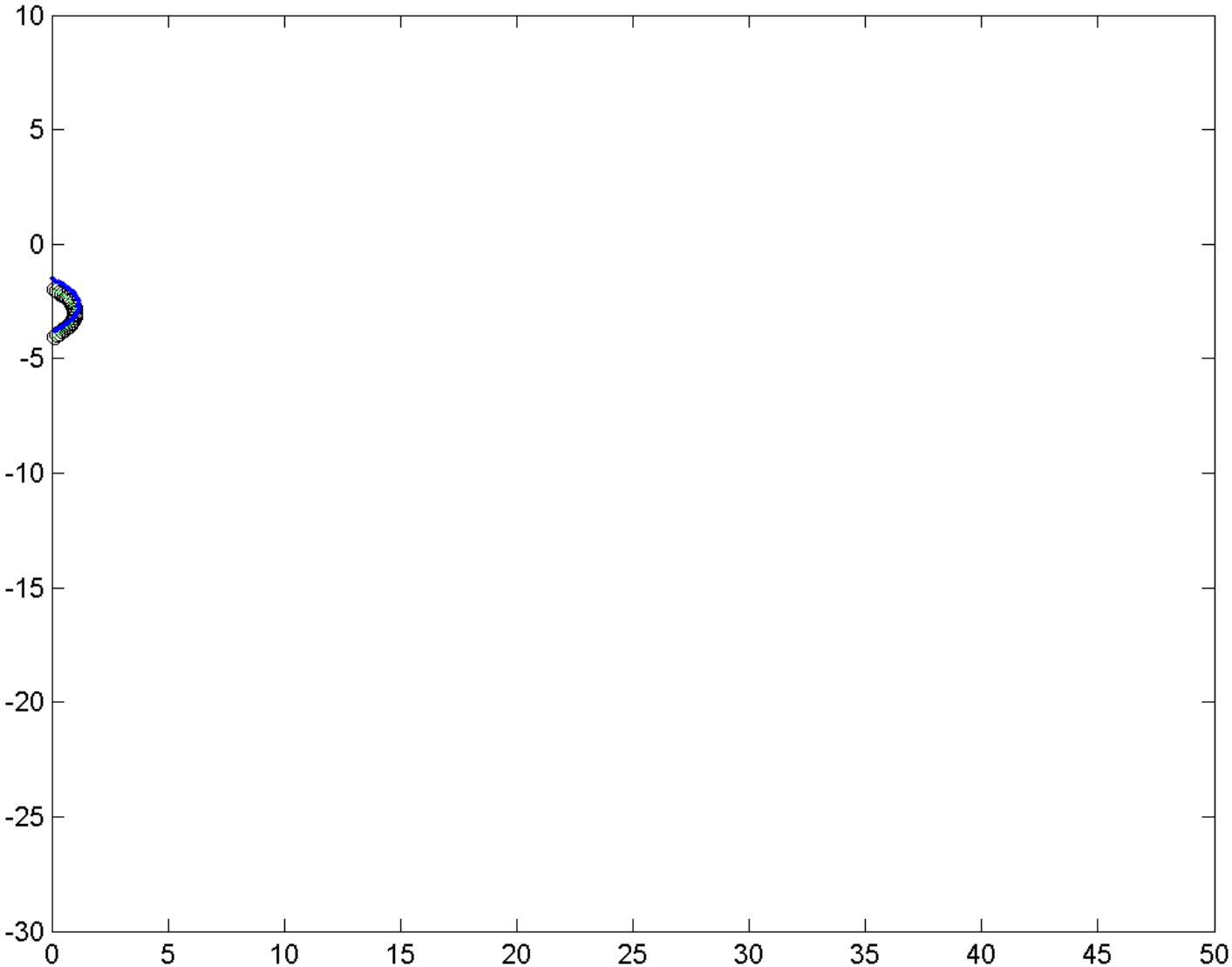

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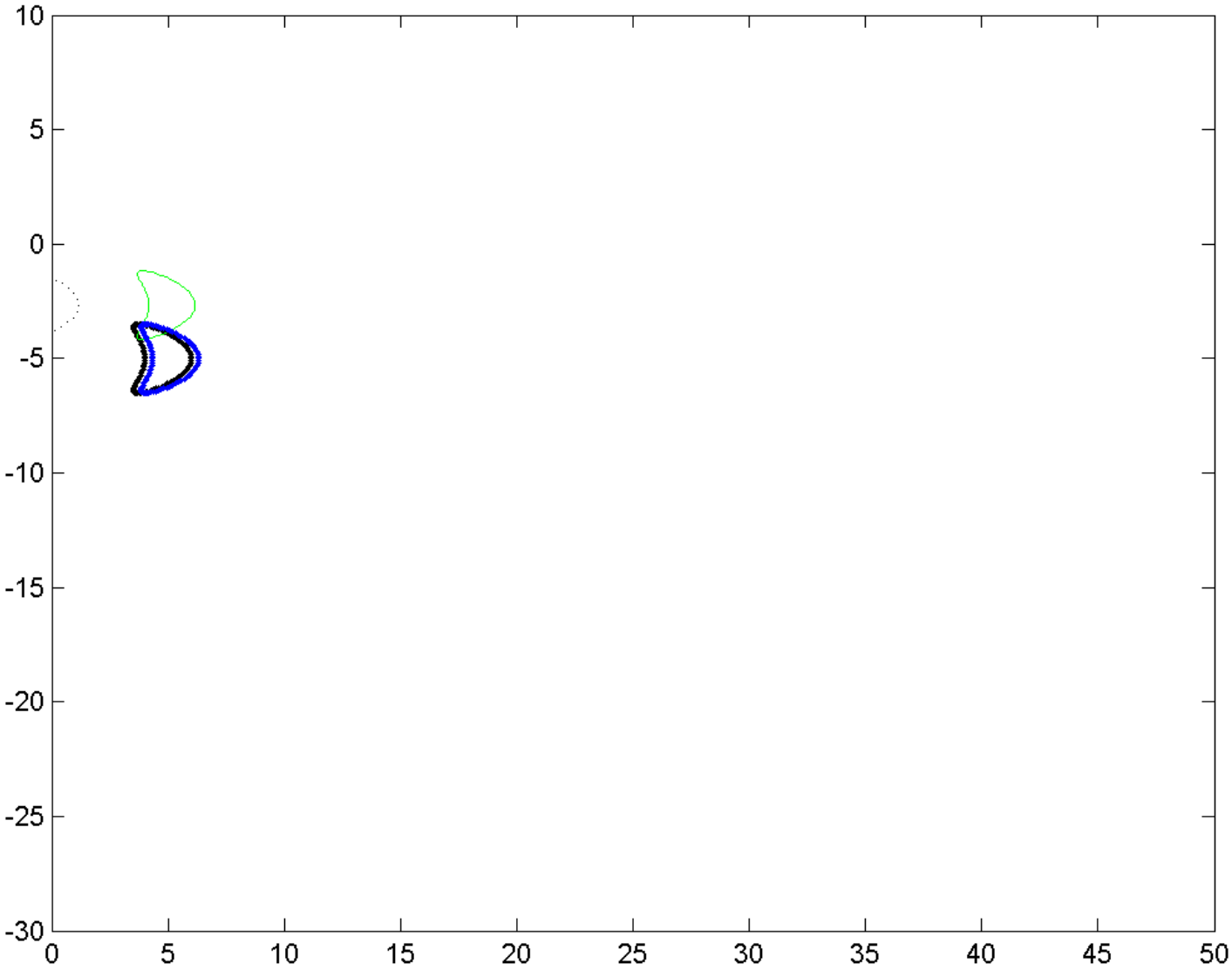

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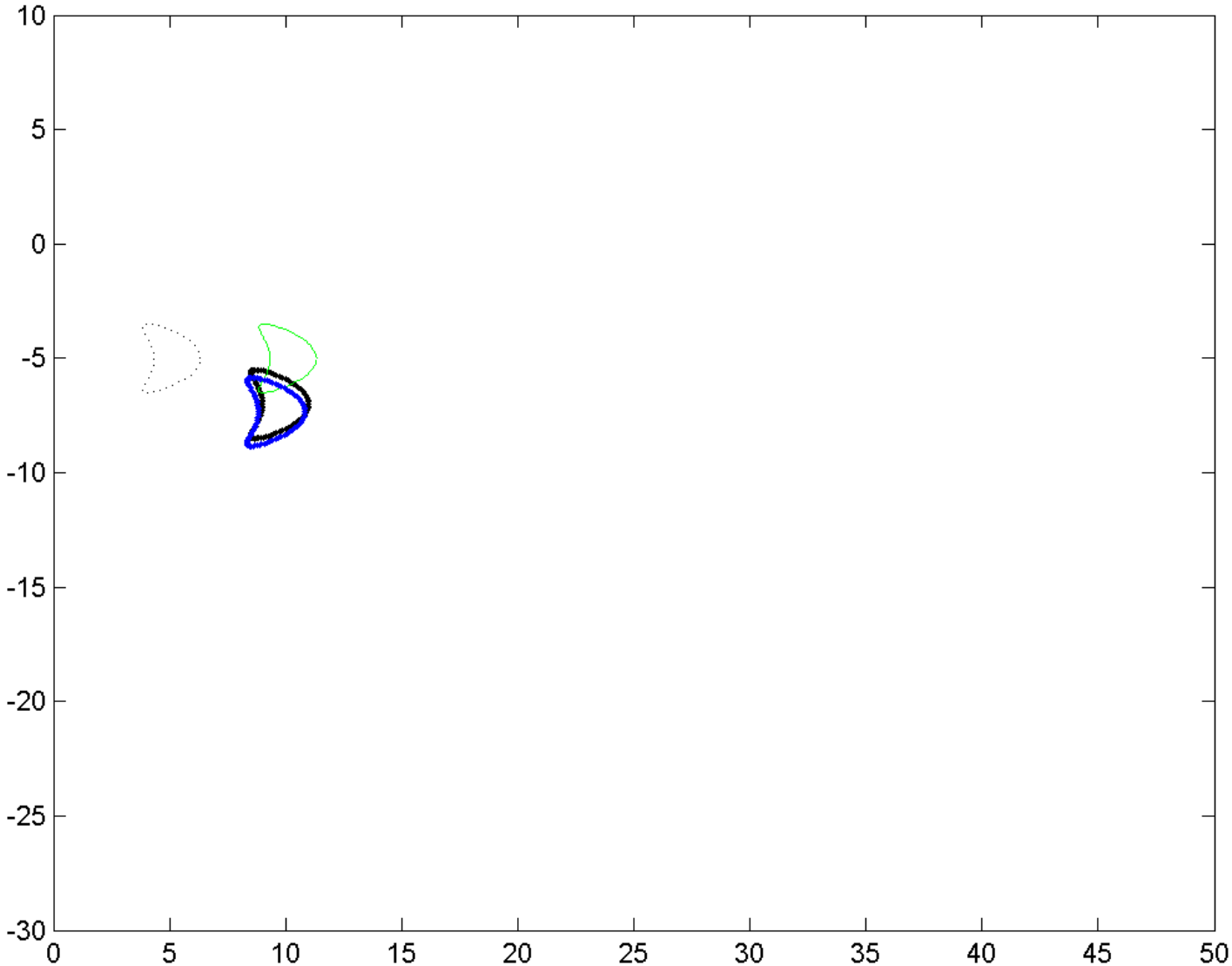

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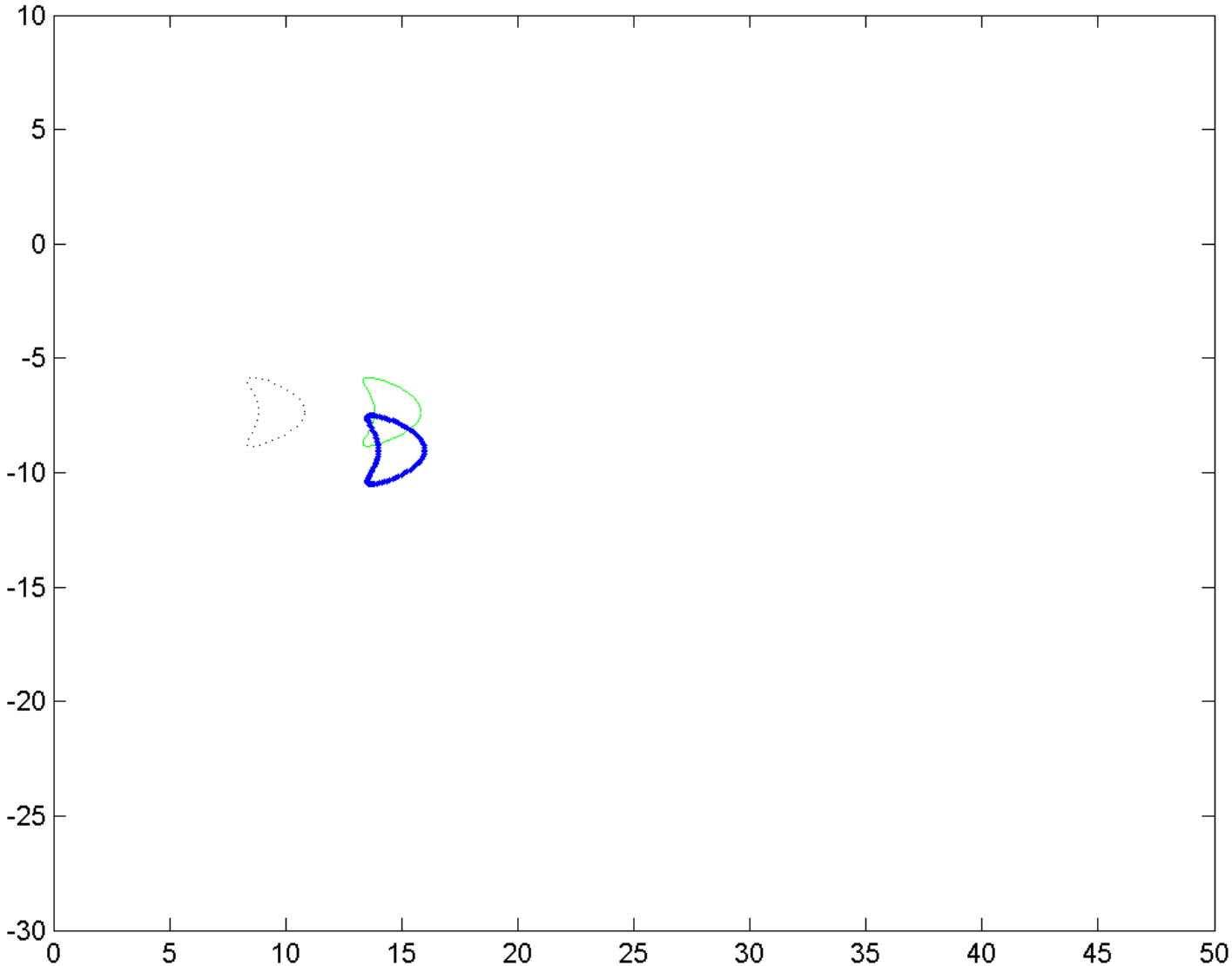

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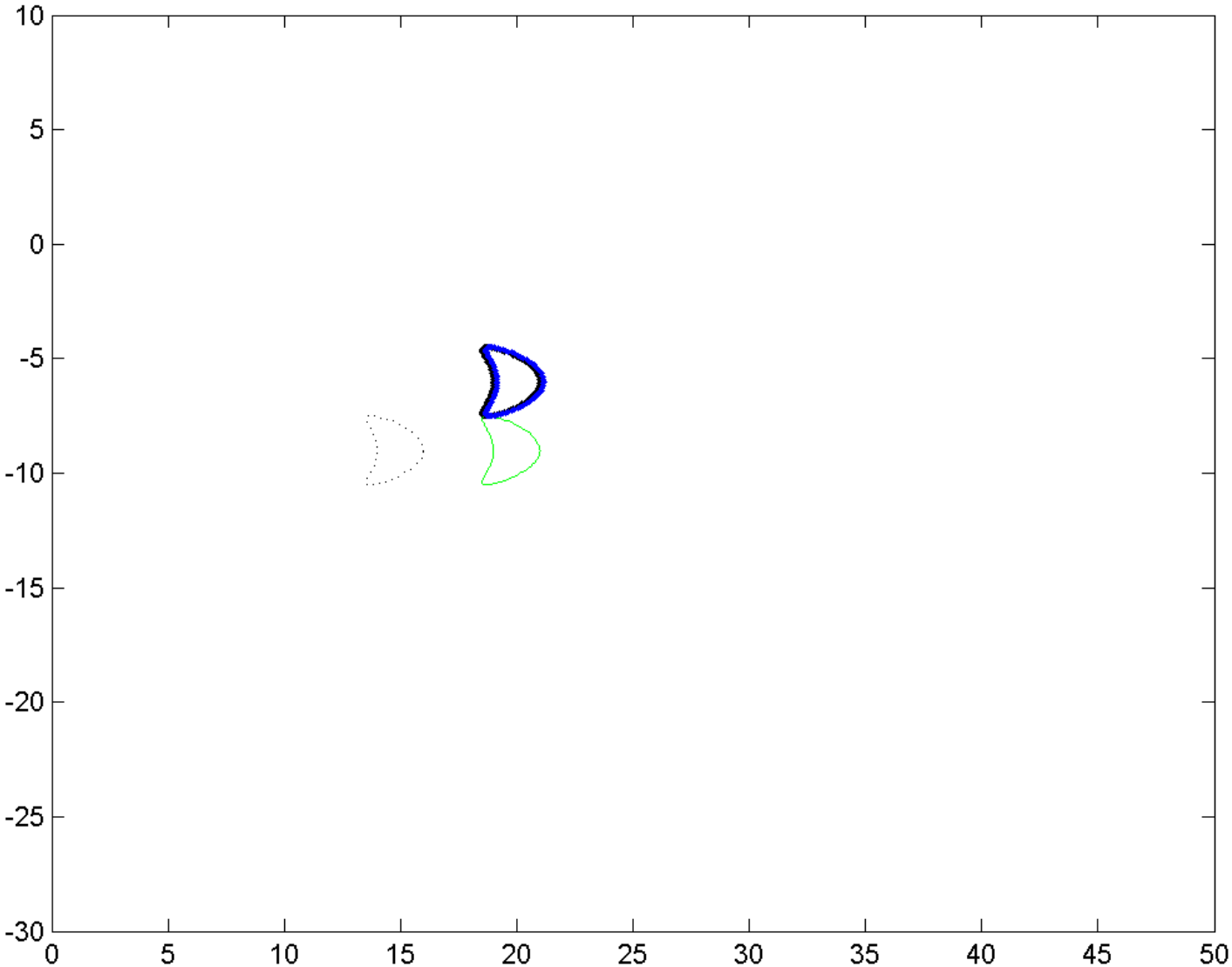

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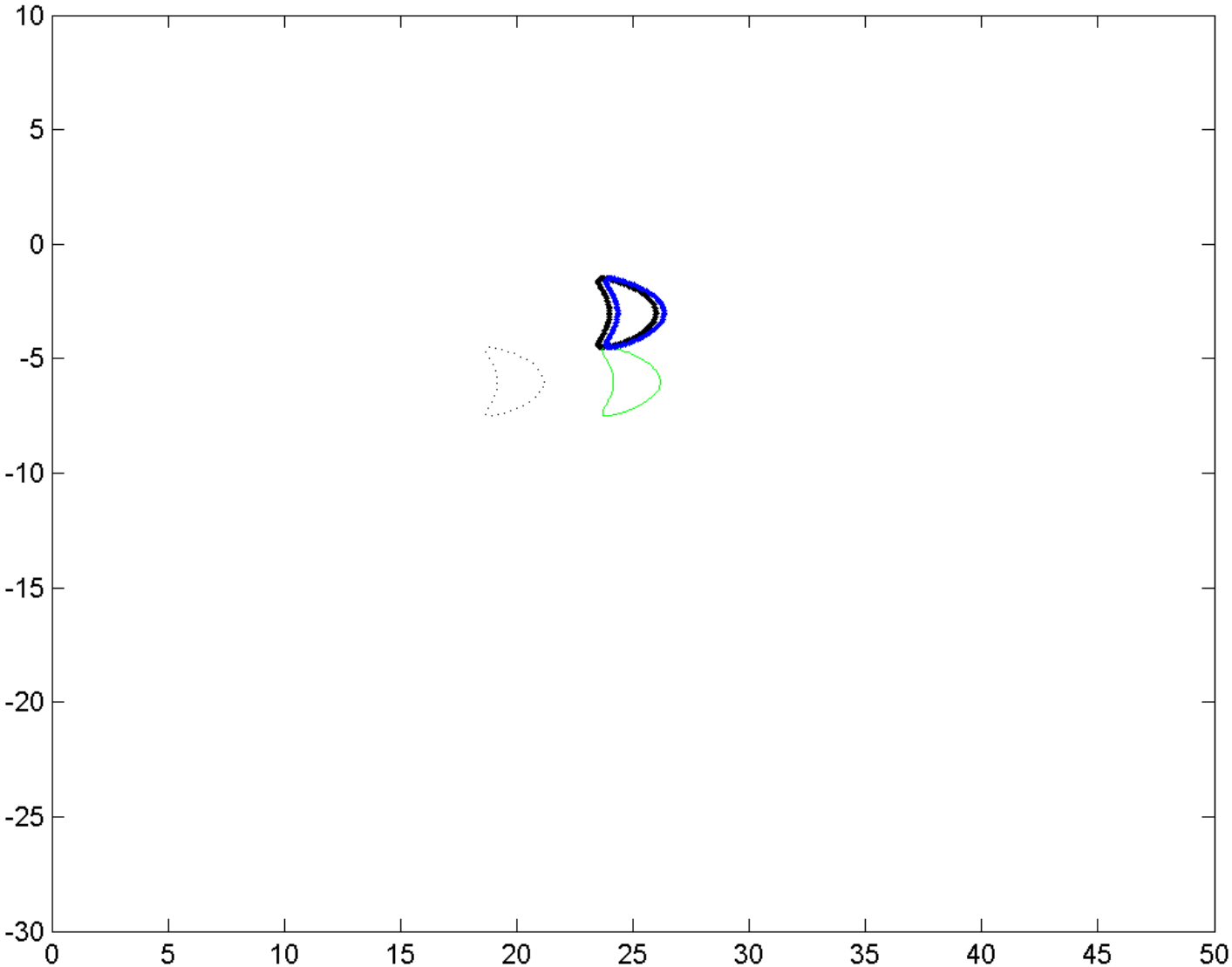

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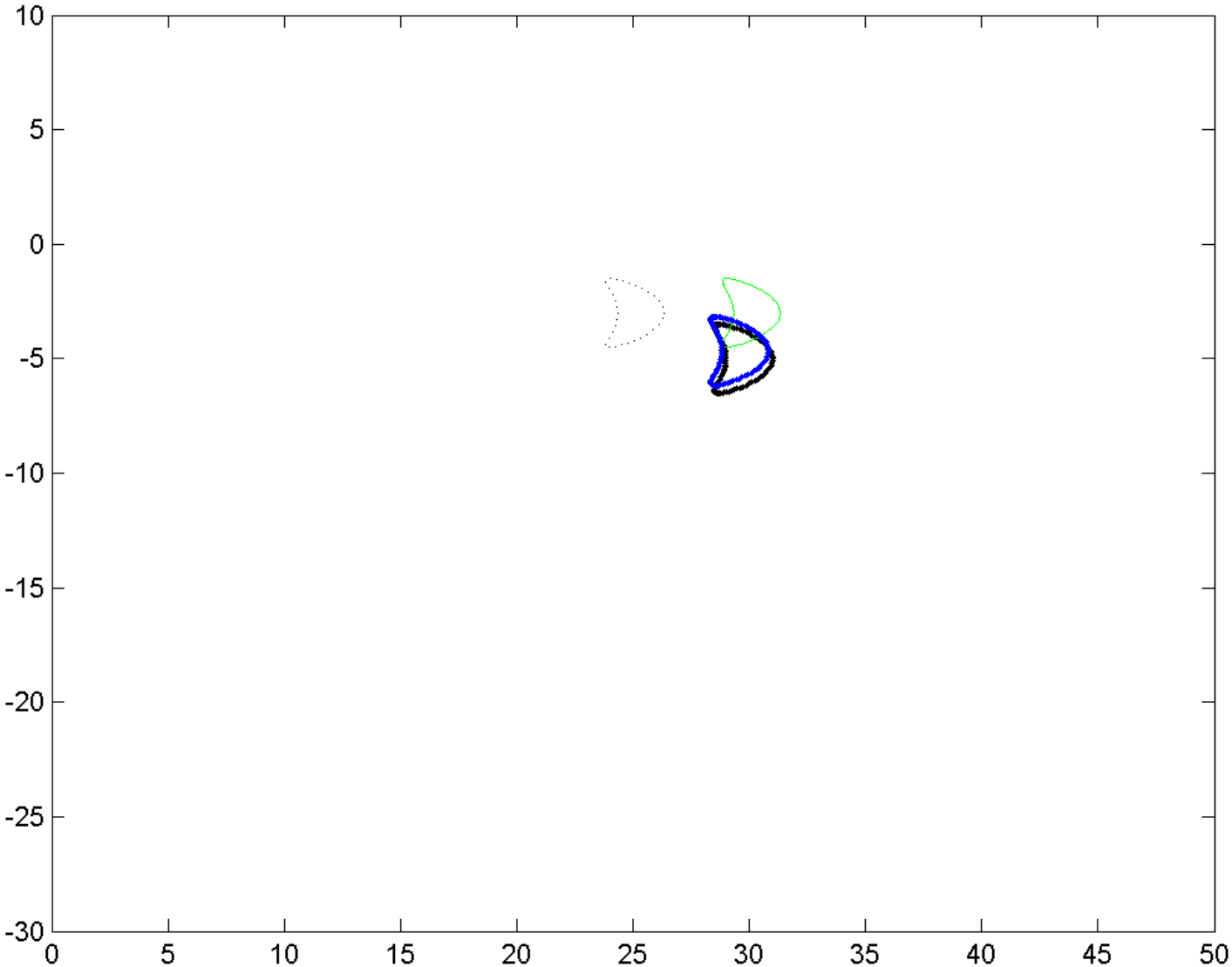

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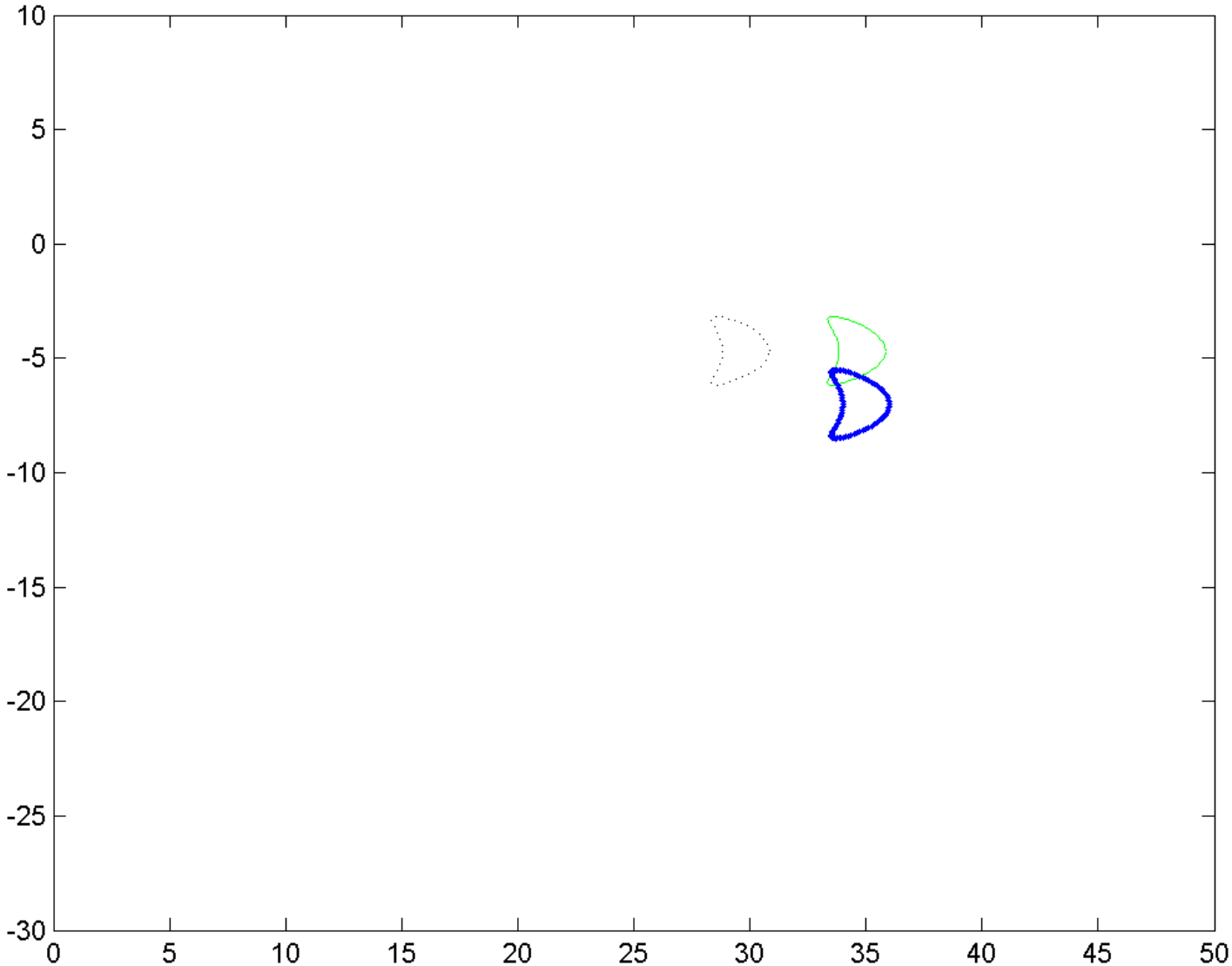

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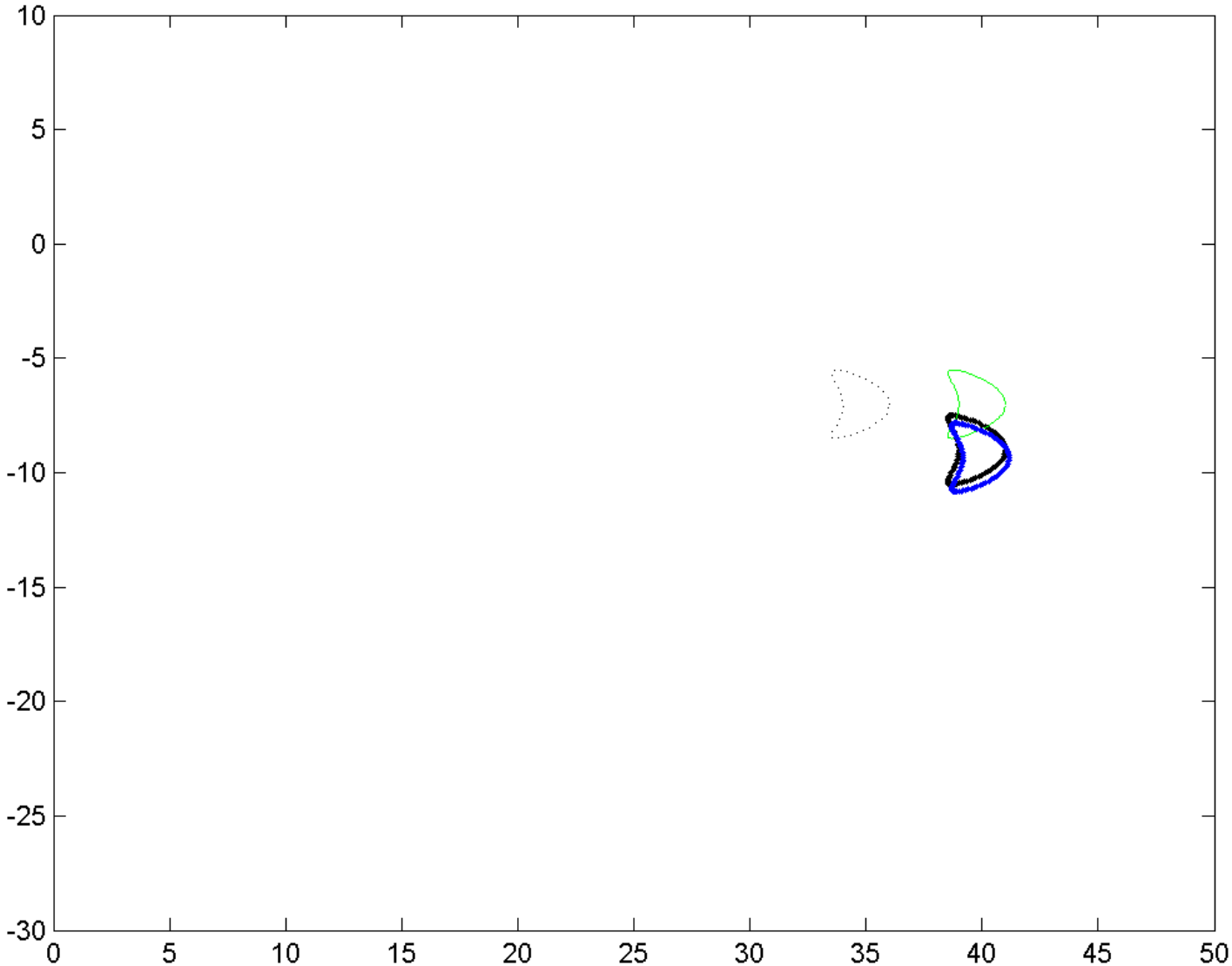

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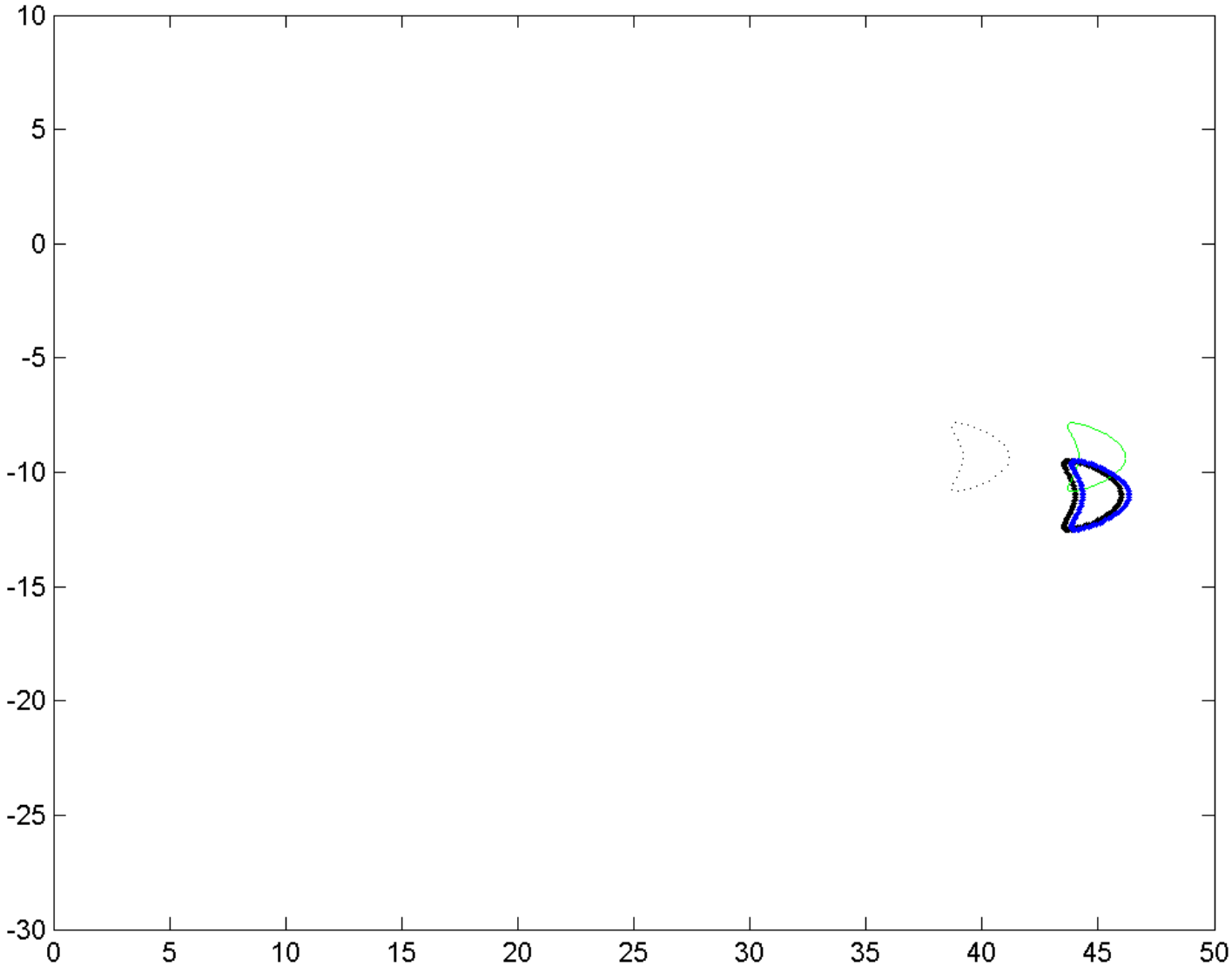
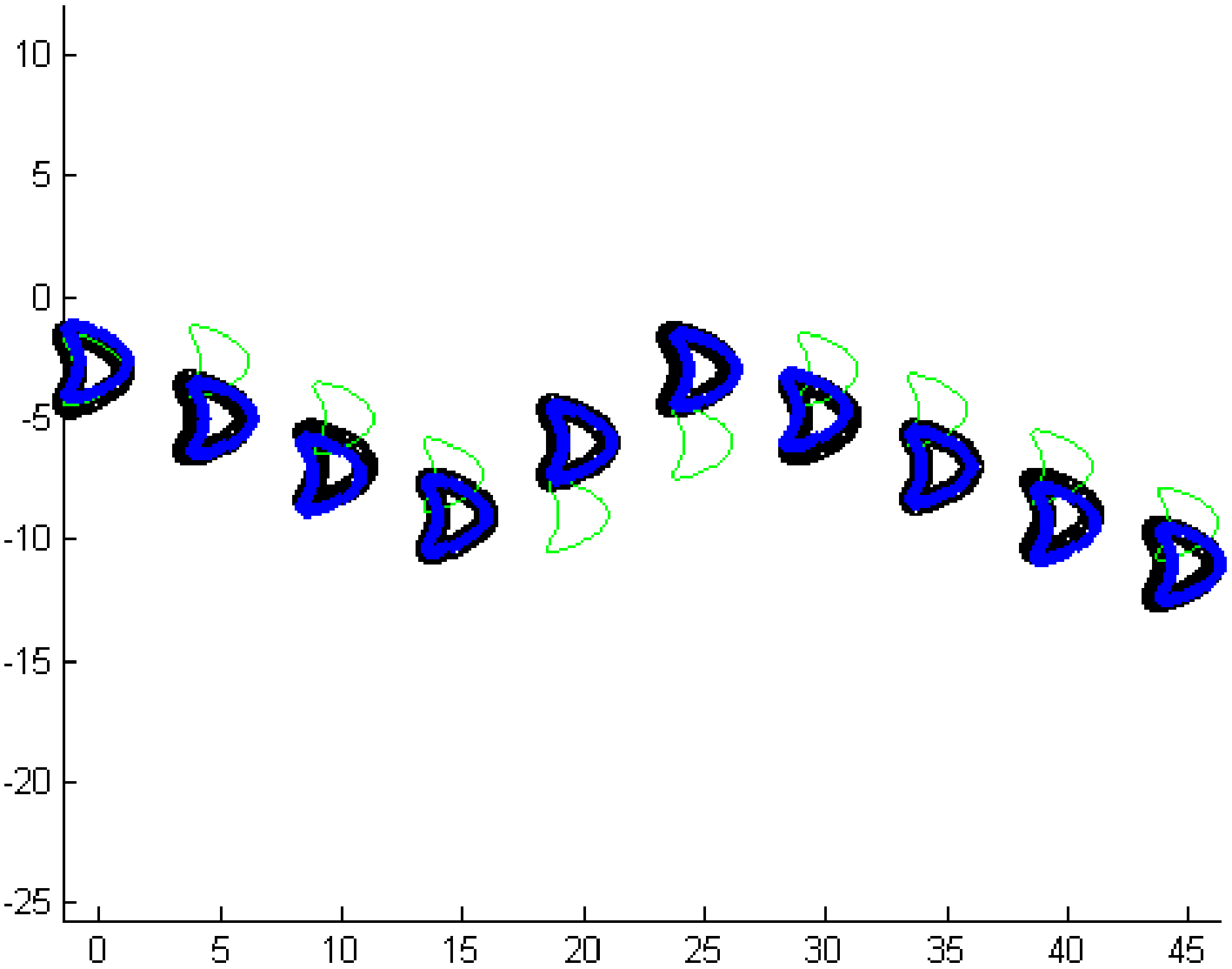
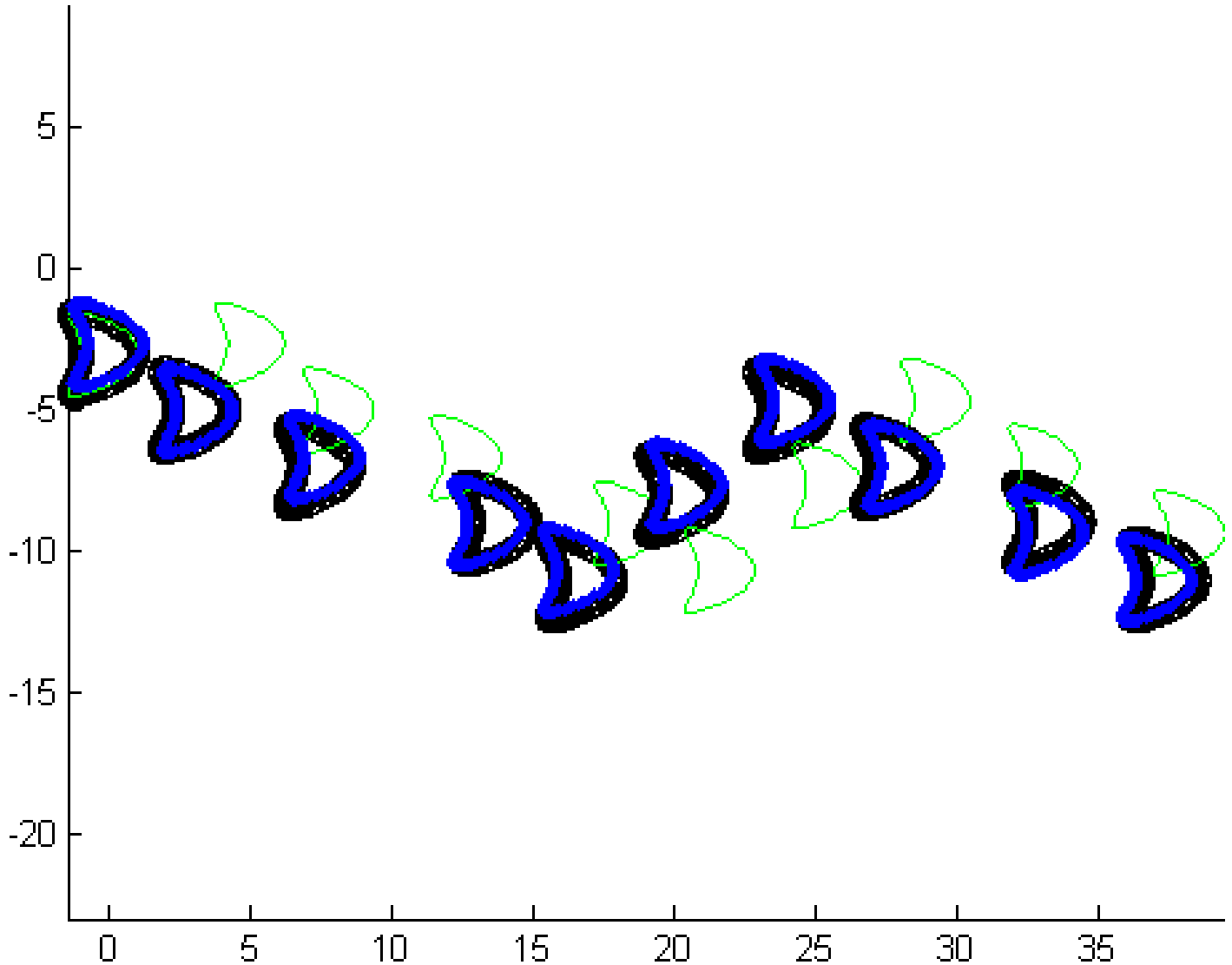


Reconstructed Movement



Reconstructed Movement with random speed



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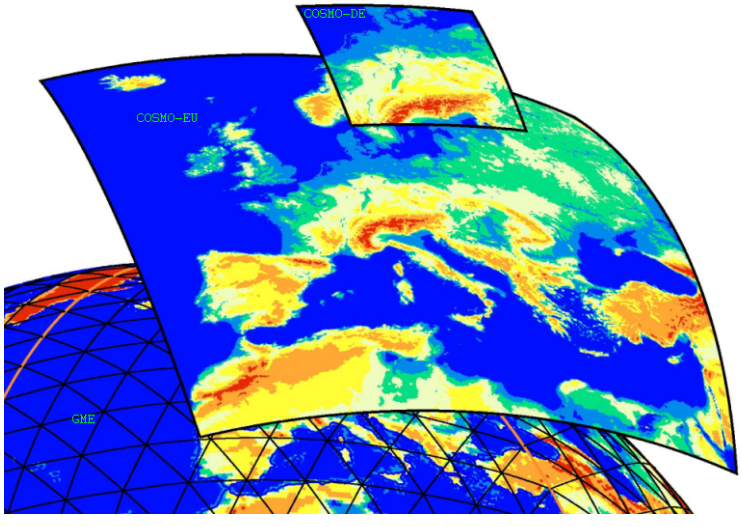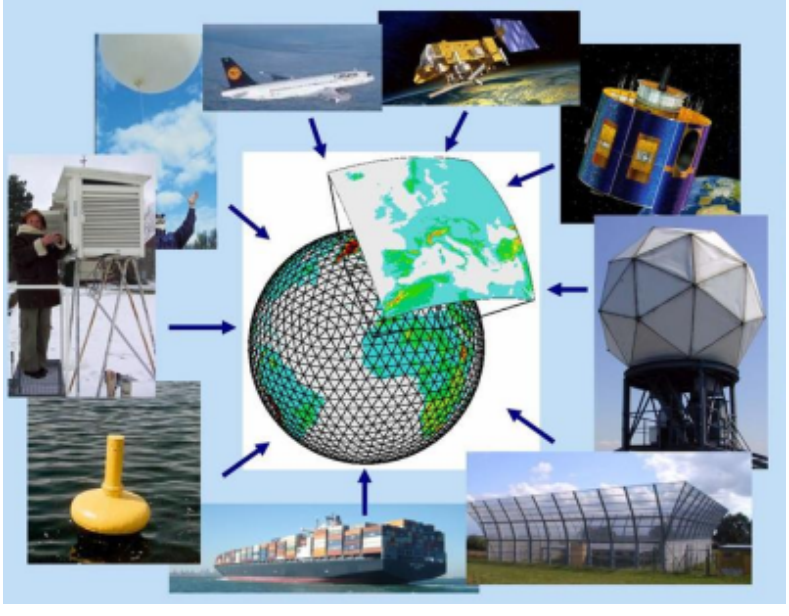
Thank You