Dynamical Inverse Scattering

Roland Potthast

Deutscher Wetterdienst
University of Reading
(Uni Göttingen)

Linz
24.11.2011
Outline

Introduction

Orthogonality Sampling

Time-Domain Probe Method
  Field & Shape Reconstruction
  Time-Domain Probe

Data Assimilation
  3dVar
  4dVar
  Dynamic Inverse Scattering
Introduction

Orthogonality Sampling
Time-Domain Probe Method
Data Assimilation
Setup

Partial Differential Equation (Acoustic, Electromagnetic, Elastic)

Boundary Condition on Object

Remote Measurements

Incident Wave
Dynamical Inverse Scattering, Survey

1. **Static** scatterer and wave, i.e. one frequency time-harmonic wave

2. **Multi-Frequency** scattering, static scatterer

3. **Dynamical wave field**, i.e. time-dependent pulse

4. **Moving Scatterer**, i.e. constant speed, accelerating, rotating

5. Scatterer is **evolving**, i.e. changing its location or shape, we get repeated measurements for various time-slices

6. Fully coupled **time-space-wave dynamic problem**, where the time-dependent wave and the time-dependent scatterer are interacting
Dynamical Inverse Scattering, Survey

1. **Static** scatterer and wave, i.e. one frequency time-harmonic wave

2. **Multi-Frequency** scattering, static scatterer

3. **Dynamical wave field**, i.e. time-dependent pulse

4. **Moving Scatterer**, i.e. constant speed, accelerating, rotating

5. Scatterer is **evolving**, i.e. changing its location or shape, we get repeated measurements for various time-slices

6. Fully coupled **time-space-wave dynamic problem**, where the time-dependent wave and the time-dependent scatterer are interacting
1. Static scatterer and wave, i.e. one frequency time-harmonic wave

2. Multi-Frequency scattering, static scatterer

3. Dynamical wave field, i.e. time-dependent pulse

4. Moving Scatterer, i.e. constant speed, accelerating, rotating

5. Scatterer is evolving, i.e. changing its location or shape, we get repeated measurements for various time-slices

6. Fully coupled time-space-wave dynamic problem, where the time-dependent wave and the time-dependent scatterer are interacting
Dynamical Inverse Scattering, Survey

1. **Static** scatterer and wave, i.e. one frequency time-harmonic wave

2. **Multi-Frequency** scattering, static scatterer

3. **Dynamical wave field**, i.e. time-dependent pulse

4. **Moving Scatterer**, i.e. constant speed, accelerating, rotating

5. Scatterer is **evolving**, i.e. changing its location or shape, we get repeated measurements for various time-slices

6. Fully coupled *time-space-wave dynamic problem*, where the time-dependent wave and the time-dependent scatterer are interacting
Dynamical Inverse Scattering, Survey

1. **Static** scatterer and wave, i.e. one frequency time-harmonic wave

2. **Multi-Frequency** scattering, static scatterer

3. **Dynamical wave field**, i.e. time-dependent pulse

4. **Moving Scatterer**, i.e. constant speed, accelerating, rotating

5. Scatterer is **evolving**, i.e. changing its location or shape, we get repeated measurements for various time-slices

6. Fully coupled **time-space-wave dynamic problem**, where the time-dependent wave and the time-dependent scatterer are interacting
Dynamical Inverse Scattering, Survey

1. **Static** scatterer and wave, i.e. one frequency time-harmonic wave

2. **Multi-Frequency** scattering, static scatterer

3. **Dynamical wave field**, i.e. time-dependent pulse

4. **Moving Scatterer**, i.e. constant speed, accelerating, rotating

5. Scatterer is **evolving**, i.e. changing its location or shape, we get repeated measurements for various time-slices

6. Fully coupled **time-space-wave dynamic problem**, where the time-dependent wave and the time-dependent scatterer are interacting
2. Multi-Frequency scattering, static scatterer
   Orthogonality Sampling (P. 2010)

3. Dynamical wave field, i.e. time-dependent pulse
   Time-Domain Probe Method (Burkard, P. 2009)

4. Moving Scatterer, i.e. constant speed, accelerating, rotating
   Doppler Effect (Standard)

5. Scatterer is evolving, i.e. changing its shape, we get repeated
   measurements for various time-slices Variational Methods (3dVar/4dVar)
   or Ensemble Filter (Sini, P., in preparation)
Dynamical Inverse Scattering, Selection

2. Multi-Frequency scattering, static scatterer
   Orthogonality Sampling (P. 2010)

3. Dynamical wave field, i.e. time-dependent pulse
   Time-Domain Probe Method (Burkard, P. 2009)

4. Moving Scatterer, i.e. constant speed, accelerating, rotating
   Doppler Effect (Standard)

5. Scatterer is evolving, i.e. changing its shape, we get repeated measurements for various time-slices
   Variational Methods (3dVar/4dVar) or Ensemble Filter (Sini, P., in preparation)
2. **Multi-Frequency** scattering, static scatterer
   Orthogonality Sampling (P. 2010)

3. **Dynamical wave field**, i.e. time-dependent pulse
   Time-Domain Probe Method (Burkard, P. 2009)

4. **Moving Scatterer**, i.e. constant speed, accelerating, rotating
   Doppler Effect (Standard)

5. Scatterer is evolving, i.e. changing its shape, we get repeated measurements for various time-slices Variational Methods (3dVar/4dVar) or Ensemble Filter (Sini, P., in preparation)
Dynamical Inverse Scattering, Selection

2. **Multi-Frequency** scattering, static scatterer
   Orthogonality Sampling (P. 2010)

3. **Dynamical wave field**, i.e. time-dependent pulse
   Time-Domain Probe Method (Burkard, P. 2009)

4. **Moving Scatterer**, i.e. constant speed, accelerating, rotating
   Doppler Effect (Standard)

5. Scatterer is **evolving**, i.e. changing its shape, we get repeated measurements for various time-slices
   Variational Methods (3dVar/4dVar) or Ensemble Filter (Sini, P., in preparation)
Outline

**Introduction**

**Orthogonality Sampling**

**Time-Domain Probe Method**
- Field & Shape Reconstruction
- Time-Domain Probe

**Data Assimilation**
- 3dVar
- 4dVar
- Dynamic Inverse Scattering
We assume that we take a windowed Fast Fourier Transform of our available time-domain data.

This leads to a large set of wave numbers for which data in the frequency domain is available.

Usually we will have a
- low number of directions of incidence (sources)
- larger number of measurement points for the scattered or total wave.
- many wavenumbers

\[ u^\infty(\hat{x}_j, d_\ell, \kappa_\xi), \quad j = 1, \ldots, N, \quad \ell = 1, \ldots, M, \quad \xi = 1, \ldots, Q. \] (1)
We assume that we take a windowed Fast Fourier Transform of our available time-domain data.

This leads to a large set of wave numbers for which data in the frequency domain is available.

Usually we will have a

- low number of directions of incidence (sources)
- larger number of measurement points for the scattered or total wave.
- many wavenumbers

\[
u^\infty(\hat{x}_j, d_\ell, \kappa_\xi), \quad j = 1, \ldots, N, \quad \ell = 1, \ldots, M, \quad \xi = 1, \ldots, Q. \quad (1)\]
We assume that we take a **windowed Fast Fourier Transform** of our available time-domain data.

This leads to a **large set of wave numbers** for which data in the frequency domain is available.

Usually we will have a

- low number of directions of incidence (sources)
- larger number of measurement points for the scattered or total wave.
- many wavenumbers

\[
u^\infty(\hat{x}_j, d_\ell, \kappa_\xi), \quad j = 1, \ldots, N, \quad \ell = 1, \ldots, M, \quad \xi = 1, \ldots, Q. \quad (1)\]
We assume that we take a windowed Fast Fourier Transform of our available time-domain data.

This leads to a large set of wave numbers for which data in the frequency domain is available.

Usually we will have a

- low number of directions of incidence (sources)
- larger number of measurement points for the scattered or total wave.
- many wavenumbers

\[
u^{\infty}(\hat{x}_j, d_{\ell}, \kappa_{\xi}), \quad j = 1, \ldots, N, \quad \ell = 1, \ldots, M, \quad \xi = 1, \ldots, Q.
\] (1)
We assume that we take a windowed Fast Fourier Transform of our available time-domain data.

This leads to a large set of wave numbers for which data in the frequency domain is available.

Usually we will have a

- low number of directions of incidence (sources)
- larger number of measurement points for the scattered or total wave.
- many wavenumbers

\[ u^\infty(\hat{x}_j, d_\ell, \kappa_\xi), \quad j = 1, \ldots, N, \quad \ell = 1, \ldots, M, \quad \xi = 1, \ldots, Q. \tag{1} \]
We assume that we take a windowed Fast Fourier Transform of our available time-domain data.

This leads to a large set of wave numbers for which data in the frequency domain is available.

Usually we will have a

- low number of directions of incidence (sources)
- larger number of measurement points for the scattered or total wave.
- many wavenumbers

\[ u^\infty(\hat{x}_j, d_\ell, \kappa_\xi), \quad j = 1, \ldots, N, \quad \ell = 1, \ldots, M, \quad \xi = 1, \ldots, Q. \quad (1) \]
We assume that we take a windowed Fast Fourier Transform of our available time-domain data.

This leads to a large set of wave numbers for which data in the frequency domain is available.

Usually we will have a

- low number of directions of incidence (sources)
- larger number of measurement points for the scattered or total wave
- many wavenumbers

\[
\begin{align*}
    u^\infty(\hat{x}_j, d_\ell, \kappa_\xi), & \quad j = 1, \ldots, N, \quad \ell = 1, \ldots, M, \quad \xi = 1, \ldots, Q. 
\end{align*}
\]
We assume that we take a windowed Fast Fourier Transform of our available time-domain data.

This leads to a large set of wave numbers for which data in the frequency domain is available.

Usually we will have a

- low number of directions of incidence (sources)
- larger number of measurement points for the scattered or total wave.
- many wavenumbers

\[ u^\infty(\hat{x}_j, d_\ell, \kappa_\xi), \quad j = 1, \ldots, N, \quad \ell = 1, \ldots, M, \quad \xi = 1, \ldots, Q. \]  \quad (1)
Orthogonality Sampling Method

**Algorithm (One-Wave OS, Multi-Wave OS)**

For fixed wave number $\kappa$, one-wave orthogonality sampling calculates

$$
\mu(y, \kappa) = \left| \int_{S} e^{i\kappa \hat{x} \cdot y} u^{\infty}(\hat{x}) \ ds(\hat{x}) \right|
$$

(2)

on a grid $\mathcal{G}$ of points $\tilde{y} \in \mathbb{R}^m$ from the knowledge of the far field pattern $u^{\infty}$ on the unit sphere $S$.

For fixed wave number $\kappa$, multi-direction orthogonality sampling calculates

$$
\mu(y, \kappa) = \int_{S} \left| \int_{S} e^{i\kappa \hat{x} \cdot y} u^{\infty}(\hat{x}, \theta) \ ds(\hat{x}) \right| ds(\theta)
$$

(3)

on a grid $\mathcal{G}$ of points $\tilde{y} \in \mathbb{R}^m$ from the knowledge of the far field pattern $u^{\infty}(\hat{x}, \theta)$ for $\hat{x}, \theta \in S$. 

Orthogonality Sampling Method

**Algorithm (One-Wave OS, Multi-Wave OS)**

*For fixed wave number $\kappa$ one-wave orthogonality sampling calculates*

$$
\mu(y, \kappa) = \left| \int_{S} e^{i\kappa \hat{x} \cdot y} u^\infty(\hat{x}) \, ds(\hat{x}) \right|
$$

(2)

on a grid $\mathcal{G}$ of points $\tilde{y} \in \mathbb{R}^m$ from the knowledge of the far field pattern $u^\infty$ on the unit sphere $S$.

*For fixed wave number $\kappa$ multi-direction orthogonality sampling calculates*

$$
\mu(y, \kappa) = \int_{S} \left| \int_{S} e^{i\kappa \hat{x} \cdot y} u^\infty(\hat{x}, \theta) \, ds(\hat{x}) \right| \, ds(\theta)
$$

(3)

on a grid $\mathcal{G}$ of points $\tilde{y} \in \mathbb{R}^m$ from the knowledge of the far field pattern $u^\infty(\hat{x}, \theta)$ for $\hat{x}, \theta \in S$. 

---

**Introduction**

**Orthogonality Sampling**

**Time-Domain Probe Method**

**Data Assimilation**
Orthogonality Sampling Method

**Algorithm (One-wave OS, Multi-wave OS)**

For fixed wave number $\kappa$, one-wave orthogonality sampling calculates

$$
\mu(y, \kappa) = \left| \int_{S} e^{i\kappa \hat{x} \cdot y} u^\infty(\hat{x}) \, ds(\hat{x}) \right|
$$

(2)

on a grid $\mathcal{G}$ of points $\tilde{y} \in \mathbb{R}^m$ from the knowledge of the far field pattern $u^\infty$ on the unit sphere $S$.

For fixed wave number $\kappa$, multi-direction orthogonality sampling calculates

$$
\mu(y, \kappa) = \int_{S} \left| \int_{S} e^{i\kappa \hat{x} \cdot y} u^\infty(\hat{x}, \theta) \, ds(\hat{x}) \right| \, ds(\theta)
$$

(3)

on a grid $\mathcal{G}$ of points $\tilde{y} \in \mathbb{R}^m$ from the knowledge of the far field pattern $u^\infty(\hat{x}, \theta)$ for $\hat{x}, \theta \in S$. 

Illumination
Orthogonality Sampling
Time-Domain Probe Method
Data Assimilation
Multi-frequency Orthogonality Sampling

**Algorithm (Multi-Frequency)**

The multi-frequency orthogonality sampling calculates

\[
\mu(y, \theta) = \int_{\kappa_0}^{\kappa_1} \left| \int_{\mathbb{S}} e^{i\kappa \hat{x} \cdot y} u_{\kappa}^{\infty}(\hat{x}, \theta) \ ds(\hat{x}) \right| \ d\kappa
\]

(4)

on a grid \( \mathcal{G} \) of points \( \tilde{y} \in \mathbb{R}^m \) from the knowledge of the far field pattern \( u_{\kappa}^{\infty}(\hat{x}) \) for \( \hat{x} \in \mathbb{S} \) and \( \kappa \in [\kappa_0, \kappa_1] \).

Here also multi-direction multi-frequency sampling is possible by adding the indicator functions for several directions of incidence.
One Wave, one frequency: the simplest setting

Graphics: Orthogonality sampling with $\kappa = 1$ or $\kappa = 3$ for fixed frequency, one direction of incidence
Graphics: Orthogonality sampling, many directions of incidence, fixed frequency
Multi-frequency Ortho Sampling

Graphics: Orthogonality sampling, many directions of incidence, fixed frequency
Resolution Study: Large Scale

Graphics: Multi-frequency Orthogonality sampling with $\kappa$ between 0.1 and 1, i.e. with a frequency between $\lambda = 6$ and $\lambda = 60$, one direction of incidence
Graphics: MDMF Orthogonality sampling with $\kappa$ between 3 and 4, i.e. with a frequency between $\lambda = 1.5$ and $\lambda = 2$
Graphics: MDMF Orthogonality sampling with $\kappa$ between 6 and 15, i.e. with a frequency between $\lambda = 0.4$ and $\lambda = 1$
Graphics: MDMF Orthogonality sampling with $\kappa$ between 10 and 20, i.e. with a frequency between $\lambda = 0.3$ and $\lambda = 0.6$
Resolution Study: Very Fine Scale

Graphics: MDMF Orthogonality sampling with $\kappa$ between 20 and 40, i.e. with a frequency between $\lambda = 0.15$ and $\lambda = 0.3$
Graphics: MDMF Orthogonality sampling with $\kappa$ between 20 and 40, i.e. with a frequency between $\lambda = 0.15$ and $\lambda = 0.3$
Medium Reconstructions I

Graphics: Orthogonality sampling for medium reconstruction, MD, fixed frequency $\kappa = 9$. 
Medium Reconstructions II

Graphics: Orthogonality sampling for medium reconstruction, MDMF.
Introduction
Orthogonality Sampling
Time-Domain Probe Method
Data Assimilation

Medium Reconstructions III

Graphics: Orthogonality sampling for medium reconstruction, MDMF.
Medium Reconstructions IV

Graphics: Orthogonality sampling for medium reconstruction, MDMF.
Neumann BC I

Graphics: Orthogonality sampling for the Neumann BC, MF.
Neumann BC II

Graphics: Orthogonality sampling for the Neumann BC, MDMF.
Neumann BC II

Graphics: Orthogonality sampling for the Neumann BC, MDMF.
The orthogonality sampling algorithm with the Dirichlet boundary condition for one-wave fixed frequency reconstructs the reduced scattered field, i.e.

\[ u_{\text{red}}^s(x) = \int_{\partial D} j_0(\kappa |x - y|) \frac{\partial u(y)}{\partial \nu(y)} \, ds(y), \quad x \in \mathbb{R}^m. \]  

(5)

Convergence analysis of the method can be based on the Funk-Hecke formula.
Literature


Potthast, R: Orthogonality Sampling for Object Visualization, Inverse Problems 2010.

Griesmaier, R: Multi-frequency orthogonality sampling for inverse obstacle scattering problems, Inverse Problems (2011)
Outline

Introduction

Orthogonality Sampling

**Time-Domain Probe Method**
- Field & Shape Reconstruction
- Time-Domain Probe

Data Assimilation
- 3dVar
- 4dVar
- Dynamic Inverse Scattering
Outline

Introduction

Orthogonality Sampling

Time-Domain Probe Method
  Field & Shape Reconstruction
  Time-Domain Probe

Data Assimilation
  3dVar
  4dVar
  Dynamic Inverse Scattering
Setup of Inverse Rough Surface Scattering

- Measurements on some surface $\Gamma_{h,A}$
- Unknown surface $\Gamma$ below measurement surface and above zero-surface. Dirichlet boundary condition.
- Measure total scattered field $v$ from one time-harmonic incident field $G(\cdot, z)$ with source point $z$ above or on $\Gamma_{h,A}$.
Tasks of Inverse Rough Surface Scattering

Tasks:

1. Reconstruct the total field \( u \) or scattered field \( u^s \). Since the incident field \( u^i = G(\cdot, z) \) is known, these tasks are equivalent.

2. Reconstruct the scattering surface \( \Gamma \) or any surface which generates the data for the given incident field \( u^i = G(\cdot, z) \).

Remark. If we do this for sufficiently many incident waves simultaneously, we have uniqueness of the reconstruction.
Tasks of Inverse Rough Surface Scattering

Tasks:

1. Reconstruct the total field \( u \) or scattered field \( u^S \). Since the incident field \( u^i = G(\cdot, z) \) is known, these tasks are equivalent.

2. Reconstruct the scattering surface \( \Gamma \) or any surface which generates the data for the given incident field \( u^i = G(\cdot, z) \).

Remark. If we do this for sufficiently many incident waves simultaneously, we have uniqueness of the reconstruction.
Inverse Rough Surface Scattering
Method of Kirsch-Kress or Potential Method I

The idea of the Kirsch-Kress method is to calculate an approximation of the scattered field by a single-layer approach $S\varphi$ defined on a subset of an auxiliary surface $\Gamma_t$.

It is carried out by minimization of the Tikhonov functional

$$J_{\alpha,B} = \left\| SP_B \varphi - v \right\|_{L^2(\Gamma_{h,A})}^2 + \alpha \left\| P_B \varphi \right\|_{L^2(\Gamma)}^2. \quad (6)$$

The unknown scattering surface $\Gamma$ by the minimization of the approximated total field

$$\| u \|_{L^2(\Gamma)} = \left\| G(\cdot,z) + SP_B \varphi \right\|_{L^2(\Gamma)} = \text{field on surface} \quad (7)$$

over some suitable set $U$ of admissible surfaces $\Gamma$. 

---

*Introduction*

Orthogonality Sampling

Time-Domain Probe Method

Data Assimilation

Field & Shape Reconstruction

Time-Domain Probe
Inverse Rough Surface Scattering
Method of Kirsch-Kress or Potential Method I

The idea of the Kirsch-Kress method is to calculate an approximation of the scattered field by a single-layer approach $S\varphi$ defined on a subset of an auxiliary surface $\Gamma_t$.

It is carried out by minimization of the Tikhonov functional

$$J_{\alpha,B} = \left\| S P_B \varphi - v \right\|^2_{L^2(\Gamma_{h,A})} + \alpha \left\| P_B \varphi \right\|^2 = \text{measured data} + \text{regularization}.$$

The unknown scattering surface $\Gamma$ by the minimization of the approximated total field

$$\left\| u \right\|_{L^2(\Gamma)} = \left\| G(\cdot, z) + S P_B \varphi \right\|_{L^2(\Gamma)} = \text{field on surface}$$

over some suitable set $U$ of admissible surfaces $\Gamma$. 

(6) 

(7)
The idea of the Kirsch-Kress method is to calculate an approximation of the scattered field by a single-layer approach \( S\varphi \) defined on a subset of an auxiliary surface \( \Gamma_t \).

It is carried out by minimization of the Tikhonov functional

\[
J_{\alpha,B} = \left\| SP_B\varphi - v \right\|_{L^2(\Gamma_{h,A})}^2 + \alpha \left\| P_B\varphi \right\|_{L^2(\Gamma)}^2.
\]

(6)

The unknown scattering surface \( \Gamma \) by the minimization of the approximated total field

\[
\left\| u \right\|_{L^2(\Gamma)} = \left\| G(\cdot, z) + SP_B\varphi \right\|_{L^2(\Gamma)}
\]

over some suitable set \( U \) of admissible surfaces \( \Gamma \).
Inverse Rough Surface Scattering
Method of Kirsch-Kress or Potential Method II


Inverse Rough Surface Scattering

Numerical Examples I
Inverse Rough Surface Scattering
Numerical Examples I
Inverse Rough Surface Scattering
Numerical Examples II
Inverse Rough Surface Scattering
Numerical Examples II
Inverse Rough Surface Scattering
Numerical Examples III
Inverse Rough Surface Scattering
Numerical Examples IV
Outline

Introduction

Orthogonality Sampling

Time-Domain Probe Method
  Field & Shape Reconstruction
  Time-Domain Probe

Data Assimilation
  3dVar
  4dVar
  Dynamic Inverse Scattering
Time-Domain Probe Method: the Idea

- Incident **time-dependent pulse** coming from some point $z \in D$. 

- When the pulse reaches some point of the scattering surface, a **scattered field** starts to evolve.

- By reconstructing the time-dependent field we can **probe the region** and determine those points where a scattered field evolves right at the moment when the incident pulse first reaches a particular point.

- Use the **potential method** of Kirsch-Kress or the **point-source method** of the author to reconstruct $U^s(x, t)$ for $x \in \Omega, t \in \mathbb{R}$. 
Inverse Rough Surface Scattering

Time-domain probe method, Idea
Inverse Rough Surface Scattering

Time-domain probe method, Idea
Inverse Rough Surface Scattering

Time-domain probe method, Idea
Inverse Rough Surface Scattering

Time-domain probe method, References

Inverse Rough Surface Scattering
Time-domain probe method, Characteristics

We need to study the range of influence of a time-dependent acoustic field...
Inverse Rough Surface Scattering
Time-domain probe method, Characteristics

We need to study the range of influence of a time-dependent acoustic field ...
We need to study the **range of influence** of a time-dependent acoustic field ...
Inverse Rough Surface Scattering
Time-domain probe method, Convergence I

The basic idea behind a convergence proof:
Inverse Rough Surface Scattering

Time-domain probe method, Convergence I

The basic idea behind a convergence proof:
Inverse Rough Surface Scattering

Time-domain probe method, Convergence II

For a point \( x \in \Omega \) we define the first hitting time with respect to the incident field \( U^i \) by

\[
T(x) := \inf_{t \geq 0} |U^i(x, t)| > \rho,
\]

where we usually employ \( \rho = 0 \) or small \( \rho > 0 \) in dependence of the particular choice of the incident field.

**Lemma**

*Let \( U^i \) be an incident spherical pulse. For every point \( x \in \Omega \) we have that*

\[
U^s(x, t) = 0 \text{ for all } t < T(x).
\]
Inverse Rough Surface Scattering
Time-domain probe method, Convergence II

For a point \( x \in \Omega \) we define the first hitting time with respect to the incident field \( U^i \) by

\[
T(x) := \inf_{t \geq 0} |U^i(x, t)| > \rho,
\]

where we usually employ \( \rho = 0 \) or small \( \rho > 0 \) in dependence of the particular choice of the incident field.

**Lemma**

Let \( U^i \) be an incident spherical pulse. For every point \( x \in \Omega \) we have that

\[
U^s(x, t) = 0 \text{ for all } t < T(x).
\]
Inverse Rough Surface Scattering

Time-domain probe method, Convergence II

For a point \( x \in \Omega \) we define the first hitting time with respect to the incident field \( U^i \) by

\[
T(x) := \inf_{t \geq 0} |U^i(x, t)| > \rho,
\]

where we usually employ \( \rho = 0 \) or small \( \rho > 0 \) in dependence of the particular choice of the incident field.

**Lemma**

Let \( U^i \) be an incident spherical pulse. For every point \( x \in \Omega \) we have that

\[
U^s(x, t) = 0 \text{ for all } t < T(x).
\]
For a point \( x \in \Omega \) we define the first hitting time with respect to the incident field \( U^i \) by

\[
T(x) := \inf_{t \geq 0} |U^i(x, t)| > \rho,
\]

where we usually employ \( \rho = 0 \) or small \( \rho > 0 \) in dependence of the particular choice of the incident field.

**Lemma**

*Let \( U^i \) be an incident spherical pulse. For every point \( x \in \Omega \) we have that*

\[
U^s(x, t) = 0 \text{ for all } t < T(x).
\]
Since $U^s(x, t) = -U^i(x, t)$ according to the Dirichlet boundary condition, we know that

$$|U^s(x, t)| > \rho \geq 0, \quad T(x) < t < T(x) + \epsilon, \quad \text{for } x \in \partial \Omega,$$

$$|U^s(x, t)| = 0, \quad T(x) < t < T(x) + \epsilon, \quad \text{for } x \notin \partial \Omega$$

for $\epsilon > 0$ sufficiently small. This can be used to detect the boundary $\partial \Omega$.

**Theorem (Convergence of Time-Domain Probe Method)**

The continuous version of the Time-Domain Probe Method provides a complete reconstruction of the surface $\Gamma$ above the rectangle $Q$. 
Since $U^s(x, t) = -U^i(x, t)$ according to the Dirichlet boundary condition, we know that

$$|U^s(x, t)| > \rho \geq 0, \quad T(x) < t < T(x) + \epsilon, \quad \text{for } x \in \partial\Omega,$$

$$|U^s(x, t)| = 0, \quad T(x) < t < T(x) + \epsilon, \quad \text{for } x \notin \partial\Omega$$

for $\epsilon > 0$ sufficiently small. This can be used to detect the boundary $\partial\Omega$.

**Theorem (Convergence of Time-Domain Probe Method)**

The continuous version of the Time-Domain Probe Method provides a complete reconstruction of the surface $\Gamma$ above the rectangle $Q$. 
Inverse Rough Surface Scattering

Time-domain probe method, Convergence III

Since $U^s(x, t) = -U^i(x, t)$ according to the Dirichlet boundary condition, we know that

$$\left| U^s(x, t) \right| > \rho \geq 0, \quad T(x) < t < T(x) + \epsilon, \quad \text{for } x \in \partial \Omega,$$

$$\left| U^s(x, t) \right| = 0, \quad T(x) < t < T(x) + \epsilon, \quad \text{for } x \notin \partial \Omega$$

for $\epsilon > 0$ sufficiently small. This can be used to detect the boundary $\partial \Omega$.

Theorem (Convergence of Time-Domain Probe Method)

The continuous version of the Time-Domain Probe Method provides a complete reconstruction of the surface $\Gamma$ above the rectangle $Q$. 
Inverse Rough Surface Scattering

Time-domain probe method, Reconstruction 1
Inverse Rough Surface Scattering

Time-domain probe method, Reconstruction 2
Inverse Rough Surface Scattering

Time-domain probe method, Reconstruction 3
Inverse Rough Surface Scattering

Time-domain probe method, Reconstruction 4
Outline

Introduction

Orthogonality Sampling

Time-Domain Probe Method
  Field & Shape Reconstruction
  Time-Domain Probe

Data Assimilation
  3dVar
  4dVar
  Dynamic Inverse Scattering
Data Assimilation Task

- **Dynamical system** $M : \varphi_k \mapsto \varphi_{k+1}$, states at time $t_k$, $k = 1, 2, 3, \ldots$

- **Measurement Operator** $H : \varphi_k \mapsto f_k$ with measurements $f_k$ at time $t_k$

- **Reconstruct** $\varphi_k$ using the knowledge of $M$ and of $f_k$ at $t_k$!

**Basic Notation:**

- We call the reconstruction at time $t_k$ the analysis $\varphi_k^{(a)}$.

- The propagated state $\varphi_k^{(b)} := M(\varphi_k^{(a)})$ is called background.
Data Assimilation Task

- **Dynamical system** $M : \varphi_k \mapsto \varphi_{k+1}$, states at time $t_k$, $k = 1, 2, 3, \ldots$

- **Measurement Operator** $H : \varphi_k \mapsto f_k$ with measurements $f_k$ at time $t_k$

- **Reconstruct** $\varphi_k$ using the knowledge of $M$ and of $f_k$ at $t_k$.

**Basic Notation:**

- We call the reconstruction at time $t_k$ the analysis $\varphi_k^{(a)}$.

- The propagated state $\varphi_{k+1}^{(b)} := M(\varphi_k^{(a)})$ is called background.
Data Assimilation Task

- **Dynamical system** $M : \varphi_k \mapsto \varphi_{k+1}$, states at time $t_k$, $k = 1, 2, 3, \ldots$

- **Measurement Operator** $H : \varphi_k \mapsto f_k$ with measurements $f_k$ at time $t_k$

- **Reconstruct** $\varphi_k$ using the knowledge of $M$ and of $f_k$ at $t_k$.

**Basic Notation:**

- We call the reconstruction at time $t_k$ the analysis $\varphi_k^{(a)}$.

- The propagated state $\varphi_{k+1}^{(b)} := M(\varphi_k^{(a)})$ is called background.
Data Assimilation Task

- **Dynamical system** $M : \varphi_k \mapsto \varphi_{k+1}$, states at time $t_k$, $k = 1, 2, 3, \ldots$

- **Measurement Operator** $H : \varphi_k \mapsto f_k$ with measurements $f_k$ at time $t_k$

- **Reconstruct** $\varphi_k$ using the knowledge of $M$ and of $f_k$ at $t_k$!

Basic Notation:

- We call the reconstruction at time $t_k$ the analysis $\varphi_k^{(a)}$.

- The propagated state $\varphi_{k+1}^{(b)} := M(\varphi_k^{(a)})$ is called background.
Data Assimilation Task

- Dynamical system $M : \varphi_k \mapsto \varphi_{k+1}$, states at time $t_k$, $k = 1, 2, 3, \ldots$

- Measurement Operator $H : \varphi_k \mapsto f_k$ with measurements $f_k$ at time $t_k$

- Reconstruct $\varphi_k$ using the knowledge of $M$ and of $f_k$ at $t_k$!

Basic Notation:

- We call the reconstruction at time $t_k$ the analysis $\varphi_k^{(a)}$.

- The propagated state $\varphi_{k+1}^{(b)} := M(\varphi_k^{(a)})$ is called background.
Data Assimilation Task

- **Dynamical system** \( M : \varphi_k \mapsto \varphi_{k+1} \), states at time \( t_k, k = 1, 2, 3, \ldots \)

- **Measurement Operator** \( H : \varphi_k \mapsto f_k \) with measurements \( f_k \) at time \( t_k \)

- **Reconstruct** \( \varphi_k \) using the knowledge of \( M \) and of \( f_k \) at \( t_k \)!

**Basic Notation:**

- We call the reconstruction at time \( t_k \) the **analysis** \( \varphi^{(a)}_k \).

- The propagated state \( \varphi^{(b)}_{k+1} := M(\varphi^{(a)}_k) \) is called **background**.
Data Assimilation Task

- **Dynamical system** $M : \phi_k \mapsto \phi_{k+1}$, states at time $t_k$, $k = 1, 2, 3, \ldots$

- **Measurement Operator** $H : \phi_k \mapsto f_k$ with measurements $f_k$ at time $t_k$

- **Reconstruct** $\phi_k$ using the knowledge of $M$ and of $f_k$ at $t_k$!

**Basic Notation:**

- We call the reconstruction at time $t_k$ the **analysis** $\phi_k^{(a)}$.

- The propagated state $\phi_{k+1}^{(b)} := M(\phi_k^{(a)})$ is called **background**.
Outline

Introduction

Orthogonality Sampling

Time-Domain Probe Method
  Field & Shape Reconstruction
  Time-Domain Probe

Data Assimilation
  3dVar
  4dVar
  Dynamic Inverse Scattering
Basic Approach

Let $H$ be the data operator mapping the state $\varphi$ onto the measurements $f$. Then we need to find $\varphi$ by solving the equation

$$H\varphi = f$$ (10)

When we have some initial guess $\varphi^{(b)}$, we transform the equation into

$$H(\varphi - \varphi^{(b)}) = f - H\varphi^{(b)}$$ (11)

with the incremental form

$$\varphi = \varphi^{(b)} + H^{-1}(f - H\varphi^{(b)}).$$ (12)
Basic Approach

Let $H$ be the data operator mapping the state $\varphi$ onto the measurements $f$. Then we need to find $\varphi$ by solving the equation

$$H\varphi = f$$  \hfill (10)

When we have some initial guess $\varphi^{(b)}$, we transform the equation into

$$H(\varphi - \varphi^{(b)}) = f - H\varphi^{(b)}$$  \hfill (11)

with the incremental form

$$\varphi = \varphi^{(b)} + H^{-1}(f - H\varphi^{(b)}).$$  \hfill (12)
Basic Approach

Let $H$ be the data operator mapping the state $\varphi$ onto the measurements $f$. Then we need to find $\varphi$ by solving the equation

$$H\varphi = f \quad (10)$$

When we have some initial guess $\varphi^{(b)}$, we transform the equation into

$$H(\varphi - \varphi^{(b)}) = f - H\varphi^{(b)} \quad (11)$$

with the incremental form

$$\varphi = \varphi^{(b)} + H^{-1}(f - H\varphi^{(b)}). \quad (12)$$
Regularization 1

Consider an equation

\[ H\varphi = f \]  

where \( H^{-1} \) is unstable or unbounded.

\[ H\varphi = f \Rightarrow H^*H\varphi = H^*f \]

\[ \Rightarrow (\alpha I + H^*H)\varphi = H^*f. \]  

where \((\alpha I + H^*H)\) has a stable inverse!

**Tikhonov Regularization:** Replace \( H^{-1} \) by the stable version

\[ R_\alpha := (\alpha I + H^*H)^{-1}H^* \]  

with regularization parameter \( \alpha > 0 \).
Consider an equation

$$H\varphi = f$$  \hspace{1cm} (13)\]

where $H^{-1}$ is unstable or unbounded.

$$H\varphi = f \Rightarrow H^* H\varphi = H^* f$$

$$\Rightarrow (\alpha I + H^* H)\varphi = H^* f. \hspace{1cm} (14)$$

where $(\alpha I + H^* H)$ has a stable inverse!

**Tikhonov Regularization:** Replace $H^{-1}$ by the stable version

$$R_{\alpha} := (\alpha I + H^* H)^{-1} H^* \hspace{1cm} (15)$$

with regularization parameter $\alpha > 0$. 

Consider an equation
\[ H\varphi = f \] (13)
where \( H^{-1} \) is unstable or unbounded.

\[
\begin{align*}
H\varphi &= f \\
\Rightarrow H^*H\varphi &= H^*f \\
\Rightarrow (\alpha I + H^*H)\varphi &= H^*f.
\end{align*}
\] (14)

where \( (\alpha I + H^*H) \) has a stable inverse!

**Tikhonov Regularization**: Replace \( H^{-1} \) by the stable version

\[
R_\alpha := (\alpha I + H^*H)^{-1}H^* \] (15)

with regularization parameter \( \alpha > 0 \).
Regularization 2: Least Squares

Tikhonov regularization is equivalent to the minimization of

\[ J(\varphi) := (\alpha \| \varphi \|^2 + \| H\varphi - f \|^2) \]  

The normal equations are obtained from first order optimality conditions

\[ \nabla \varphi J \overset{!}{=} 0. \]  

Differentiation leads to

\[ 0 = 2\alpha \varphi + 2H^*(H\varphi - f) \]
\[ \Rightarrow 0 = (\alpha I + H^*H)\varphi - H^*f, \]  

which is our well-known Tikhonov equation

\[ (\alpha I + H^*H)\varphi = H^*f. \]
Regularization 2: Least Squares

Tikhonov regularization is equivalent to the minimization of

\[ J(\varphi) := \left( \alpha \|\varphi\|^2 + \|H\varphi - f\|^2 \right) \]  

(16)

The normal equations are obtained from first order optimality conditions

\[ \nabla_\varphi J \overset{!}{=} 0. \]  

(17)

Differentiation leads to

\[ 0 = 2\alpha \varphi + 2H^*(H\varphi - f) \]

\[ \Rightarrow 0 = (\alpha I + H^*H)\varphi - H^*f, \]  

(18)

which is our well-known Tikhonov equation

\[ (\alpha I + H^*H)\varphi = H^*f. \]
Regularization 2: Least Squares

Tikhonov regularization is equivalent to the minimization of

\[ J(\varphi) := \left( \alpha \| \varphi \|^2 + \| H\varphi - f \|^2 \right) \]  

The **normal equations** are obtained from *first order optimality conditions*

\[ \nabla_\varphi J \equiv 0. \]  

Differentiation leads to

\[ 0 = 2\alpha \varphi + 2H^*(H\varphi - f) \]

\[ \Rightarrow 0 = (\alpha I + H^*H)\varphi - H^*f, \]  

which is our well-known **Tikhonov equation**

\[ (\alpha I + H^*H)\varphi = H^*f. \]
Covariances and Weighted Norms

Usually, the relation between variables at different points is incorporated by using \textit{covariances / weighted norms}:

\[
J(x) := \left( \| \varphi - \varphi^{(b)} \|_{B^{-1}}^2 + \| H \varphi - f \|_{R^{-1}}^2 \right)
\]

(19)

The \textbf{update formula} is now

\[
\varphi = \varphi^{(b)} + \left( B^{-1} + H^* R^{-1} H \right)^{-1} H^* R^{-1} (f - H \varphi^{(b)})
\]

(20)

or

\[
\varphi = \varphi^{(b)} + BH^* (R + HBH^*)^{-1} (f - H \varphi^{(b)}).
\]

(21)
Outline

Introduction

Orthogonality Sampling

Time-Domain Probe Method
   Field & Shape Reconstruction
   Time-Domain Probe

Data Assimilation
   3dVar
   4dVar
   Dynamic Inverse Scattering
4dVar: Use System Dynamics

So far we have not used the system $M : \varphi_0 \mapsto \varphi(t)$.

Consider some regular grid in time:

\[ t_k = \frac{k}{n} T, \quad (22) \]
\[ \varphi_k := \varphi(t_k) = M(t_k) \varphi_0, \quad k = 0, \ldots \]
\[ (23) \]

The 4dVar functional is given by:

\[ J(\varphi) := \| \varphi - \varphi^{(b)} \|^2 + \sum_{k=1}^{n} \| H \varphi_k - f_k \|^2 \]  

(24)
**4dVar: Use System Dynamics**

So far we have not used the system \( M : \varphi_0 \mapsto \varphi(t) \).

Consider some regular grid in time:

\[
t_k = \frac{k}{n} T, \tag{22}
\]

\[
\varphi_k := \varphi(t_k) = M(t_k) \varphi_0, \quad k = 0, \ldots \tag{23}
\]

The **4dVar functional** is given by:

\[
J(\varphi) := \| \varphi - \varphi^{(b)} \|^2 + \sum_{k=1}^{n} \| H \varphi_k - f_k \|^2 \tag{24}
\]
4dVar: Use System Dynamics

So far we have not used the system \( M : \varphi_0 \mapsto \varphi(t) \).

Consider some regular grid in time:

\[
\begin{align*}
t_k &= \frac{k}{n} T, \\
\varphi_k := \varphi(t_k) &= M(t_k) \varphi_0, \quad k = 0, \ldots
\end{align*}
\]  

The 4dVar functional is given by:

\[
J(\varphi) := \| \varphi - \varphi^{(b)} \|^2 + \sum_{k=1}^{n} \| H\varphi_k - f_k \|^2
\]

(24)
So far we have not used the system

\[ M : \varphi_0 \mapsto \varphi(t). \]

Consider some regular grid in time:

\[ t_k = \frac{k}{n} T, \quad (22) \]

\[ \varphi_k := \varphi(t_k) = M(t_k)\varphi_0, \quad k = 0, \ldots \]

\[ (23) \]

The 4dVar functional is given by:

\[ J(\varphi) := \| \varphi - \varphi^{(b)} \|^2 + \sum_{k=1}^{n} \| H\varphi_k - f_k \|^2 \quad (24) \]
So far we have not used the system $M : \varphi_0 \mapsto \varphi(t)$.

Consider some regular grid in time:

$$t_k = \frac{k}{n} T, \quad (22)$$

$$\varphi_k := \varphi(t_k) = M(t_k) \varphi_0, \quad k = 0, \ldots \quad (23)$$

The 4dVar functional is given by:

$$J(\varphi) := \| \varphi - \varphi^{(b)} \|^2 + \sum_{k=1}^{n} \| H \varphi_k - f_k \|^2 \quad (24)$$
4dVar: Use System Dynamics

So far we have not used the system $M : \varphi_0 \mapsto \varphi(t)$.

Consider some regular grid in time:

$$t_k = \frac{k}{n} T,$$  \hspace{1cm} (22)

$$\varphi_k := \varphi(t_k) = M(t_k) \varphi_0, \quad k = 0, \ldots$$  \hspace{1cm} (23)

The 4dVar functional is given by:

$$J(\varphi) := \| \varphi - \varphi^{(b)} \|^2 + \sum_{k=1}^{n} \| H \varphi_k - f_k \|^2$$  \hspace{1cm} (24)
Outline

Introduction

Orthogonality Sampling

Time-Domain Probe Method
  Field & Shape Reconstruction
  Time-Domain Probe

Data Assimilation
  3dVar
  4dVar
  Dynamic Inverse Scattering
Scattering
Dynamic Inverse Problem: Moving Scatterer

- **Moving Scatterer**

  - Wave scattering at times $t_k$, $k = 1, 2, 3, \ldots$, temporal scales separated!

  - Measurements of the far field patterns $u_k^\infty$ at time $t_k$.

- **Task:** Track Location of the Scatterer

- **Systems $M$:** dynamics is movement to the right with unknown $v_2$-component of the speed $v$, only known approximately!

- For numerical example: form of scatterer known, local inversion using the point source method (P. 1996) or Kirsch-Kress method (1986)
Dynamic Inverse Problem: Moving Scatterer

- Moving Scatterer

- Wave scattering at times $t_k$, $k = 1, 2, 3, \ldots$, temporal scales separated!

- Measurements of the far field patterns $u_k^\infty$ at time $t_k$.

- Task: Track Location of the Scatterer

- Systems $M$: dynamics is movement to the right with unknown $v_2$-component of the speed $v$, only known approximately!

- For numerical example: form of scatterer known, local inversion using the point source method (P. 1996) or Kirsch-Kress method (1986)
Dynamic Inverse Problem: Moving Scatterer

- Moving Scatterer

- Wave scattering at times $t_k$, $k = 1, 2, 3, \ldots$, temporal scales separated!

- Measurements of the far field patterns $u_k^\infty$ at time $t_k$.

- Task: Track Location of the Scatterer

- Systems $M$: dynamics is movement to the right with unknown $v_2$-component of the speed $v$, only known approximately!

- For numerical example: form of scatterer known, local inversion using the point source method (P. 1996) or Kirsch-Kress method (1986)
Dynamic Inverse Problem: Moving Scatterer

▶ Moving Scatterer

▶ Wave scattering at times $t_k$, $k = 1, 2, 3, \ldots$, temporal scales separated!

▶ Measurements of the far field patterns $u_k^\infty$ at time $t_k$.

▶ Task: Track Location of the Scatterer

▶ Systems $M$: dynamics is movement to the right with unknown $v_2$-component of the speed $v$, only known approximately!

▶ For numerical example: form of scatterer known, local inversion using the point source method (P. 1996) or Kirsch-Kress method (1986)
Dynamic Inverse Problem: Moving Scatterer

- Moving Scatterer
- Wave scattering at times $t_k$, $k = 1, 2, 3, \ldots$, temporal scales separated!
- Measurements of the far field patterns $u_k^\infty$ at time $t_k$.
- Task: Track Location of the Scatterer
- Systems $M$: dynamics is movement to the right with unknown $\nu_2$-component of the speed $\nu$, only known approximately!
- For numerical example: form of scatterer known, local inversion using the point source method (P. 1996) or Kirsch-Kress method (1986)
Dynamic Inverse Problem: Moving Scatterer

- Moving Scatterer

- Wave scattering at times $t_k, k = 1, 2, 3, \ldots$, temporal scales separated!

- Measurements of the far field patterns $u_k^\infty$ at time $t_k$.

- Task: Track Location of the Scatterer

- Systems $M$: dynamics is movement to the right with unknown $\nu_2$-component of the speed $\nu$, only known approximately!

- For numerical example: form of scatterer known, local inversion using the point source method (P. 1996) or Kirsch-Kress method (1986)
Original Movement
First Guess and Reconstruction
First Guess and Reconstruction
First Guess and Reconstruction
First Guess and Reconstruction
First Guess and Reconstruction
First Guess and Reconstruction
First Guess and Reconstruction
First Guess and Reconstruction
First Guess and Reconstruction
First Guess and Reconstruction
Reconstructed Movement
Reconstructed Movement with random speed
Comments/Further Questions

- **Stability Analysis** (Marx/P., Moodey/P./Lawless/van Leeuwen, Preprint 2011) many open questions on the interaction of the ill-posedness of the inverse problem with the deterministic and stochastic properties of the evolution of the reconstructions

- Observability is increased by using the systems dynamics, Control Theory, generic insight and many interesting questions for a particular application area

- Active use/Design of dynamical setup to increase reconstructability!

- Large toolbox of data assimilation methods: variational, ensemble, hybrid ...
Comments/Further Questions

- **Stability Analysis** (Marx/P., Moodey/P./Lawless/van Leeuwen, Preprint 2011) many open questions on the interaction of the ill-posedness of the inverse problem with the deterministic and stochastic properties of the evolution of the reconstructions.

- **Observability** is increased by using the systems dynamics, Control Theory, generic insight and many interesting questions for a particular application area.

- Active use/Design of dynamical setup to increase reconstructability!

- Large toolbox of data assimilation methods: variational, ensemble, hybrid...
Comments/Further Questions

▶ **Stability Anaysis** (Marx/P., Moodey/P./Lawless/van Leeuwen, Preprint 2011) many open questions on the interaction of the ill-posedness of the inverse problem with the deterministic and stochastic properties of the evolution of the reconstructions

▶ **Observability** is increased by using the systems dynamics, Control Theory, generic insight and many interesting questions for a particular application area

▶ **Active use/Design** of dynamical setup to increase reconstructability!

▶ Large toolbox of data assimilation methods: variational, ensemble, hybrid...
Comments/Further Questions

- **Stability Analysis** (Marx/P., Moodey/P./Lawless/van Leeuwen, Preprint 2011) many open questions on the interaction of the ill-posedness of the inverse problem with the deterministic and stochastic properties of the evolution of the reconstructions

- **Observability** is increased by using the systems dynamics, **Control Theory**, generic insight and many interesting questions for a particular application area

- **Active use/Design** of dynamical setup to increase reconstructability!

- Large **toolbox of data assimilation methods**: variational, ensemble, hybrid ...
Thank You