Coupling free and porous-media flows: modeling, analysis and numerical approximation

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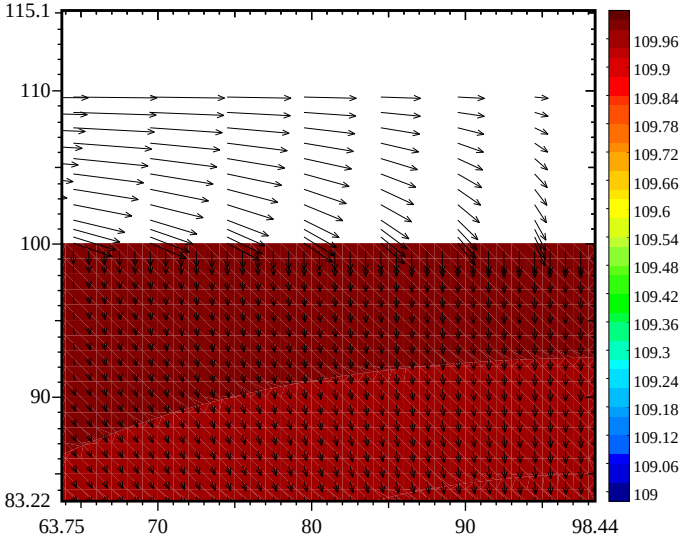
MOTIVATION

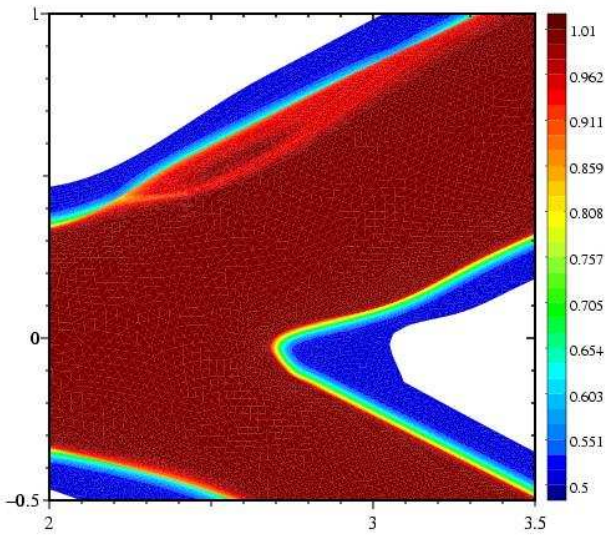
Modeling **free and porous media flows** requires to consider coupled differential models featuring *Navier-Stokes equations* in the fluid domain and a filtration model in the porous domain, like the *Darcy equation*.

$\Rightarrow$ Global coupled *heterogeneous* differential problem.

Environmental application



Blood flow simulations



Industrial applications: filters, porous foams, fuel cells...

PROBLEM SETTING

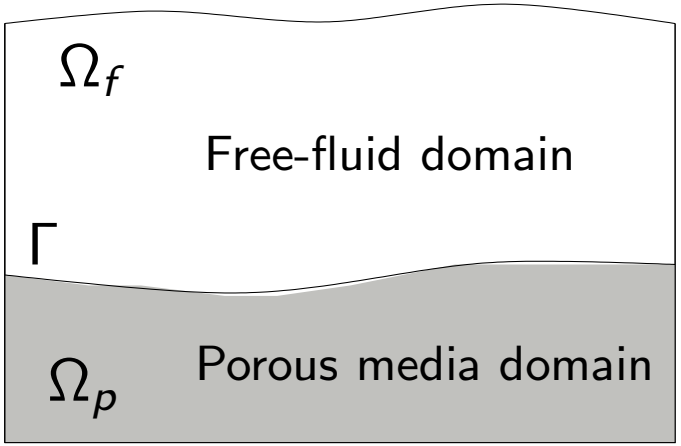
- **Fluid flow:** Navier-Stokes equations

$$\begin{aligned} -\mathbf{div}\,\mathsf{T}(\mathbf{u}_f, p_f) + (\mathbf{u}_f \cdot \nabla)\mathbf{u}_f &= \mathbf{f} \\ \operatorname{div}\mathbf{u}_f &= 0 \end{aligned} \quad \text{in } \Omega_f$$

where $\mathsf{T}(\mathbf{u}_f, p_f) = \nu(\nabla\mathbf{u}_f + \nabla^T\mathbf{u}_f) - p_f\mathsf{I}$ is the Cauchy stress tensor.

- **Fluid through porous media:** Darcy's equations

$$\begin{aligned} \mathsf{K}^{-1}\mathbf{u}_p + \nabla p_p &= \mathbf{0} \\ \operatorname{div}\mathbf{u}_p &= 0 \end{aligned} \quad \text{in } \Omega_p \quad \Leftrightarrow \quad -\operatorname{div}(\mathsf{K}\nabla p_p) = 0 \quad \text{in } \Omega_p$$



COUPLING (INTERFACE) CONDITIONS

The solution must satisfy three regularity conditions across Γ:

- the continuity of the normal velocities

$$\boxed{\mathbf{u}_f \cdot \mathbf{n} = \mathbf{u}_p \cdot \mathbf{n} \;\Leftrightarrow \mathbf{u}_f \cdot \mathbf{n} = -\mathsf{K}\nabla p_p \cdot \mathbf{n}}$$

  a consequence of the incompressibility;

- the continuity of the normal stresses

$$\boxed{-\mathbf{n} \cdot \mathsf{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n} = p_p}$$

  (pressures can be discontinuous across Γ);

- a condition on the tangential component of the normal stress: Beavers–Joseph–Saffman equation

$$\boxed{-\boldsymbol{\tau} \cdot \mathsf{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n} = \alpha \mathbf{u}_f \cdot \boldsymbol{\tau}(-\alpha \mathbf{u}_p \cdot \boldsymbol{\tau})}$$

[Miglio, Discacciati, Quarteroni (2002); Layton, Schieweck, Yotov (2003)]

COUPLING (INTERFACE) CONDITIONS

- Experimental approach:
  - 1967: Beavers and Joseph;
  - 1971: corrected by Saffman (removed $\mathbf{u}_p$ in the tangential part).

- Mathematical approach:
  - 1996, 2000, 2001: justification by Jäger and Mikelić via homogenization theory.

LITERATURE

A far-from-complete list of names:

Arbogast et al.;
Becker et al;
Burman, Hansbo;
D'Angelo, Zunino;
Iliev, Laptev;
Layton;
Galvis, Sarkis;
Girault;
Jäger, Mikelić, Neuss;
Nassehi et al;
Urquiza et al.;
Yotov;

Badia, Codina;
Bernardi et al.;
Correa, Loula;
Discacciati, Quarteroni;
Kanschat;
Fuhrmann et al.;
Gatica, Oyarzua;
Gunzburger, Hua;
Mu, Xu, Zhu;
Rivière;
Wolmuth, Helmig;
...

MATHEMATICAL ANALYSIS OF THE COUPLED PROBLEM

WEAK FORM OF THE COUPLED DARCY – NAVIER-STOKES PROBLEM (I)

Find $\mathbf{u}_f \in H^1(\Omega_f)$, $p_f \in L^2(\Omega_f)$, $p_p \in H^1(\Omega_p)$:

$$\int_{\Omega_f} \nu \nabla \mathbf{u}_f \cdot \nabla \mathbf{v} + \int_{\Gamma} \alpha (\mathbf{u}_f \cdot \boldsymbol{\tau})(\mathbf{v} \cdot \boldsymbol{\tau}) + \int_{\Omega_f} [(\mathbf{u}_f \cdot \nabla)\mathbf{u}_f]\mathbf{v}$$

$$- \int_{\Omega_f} p_f \mathrm{div} \mathbf{v} + \int_{\Gamma} p_p (\mathbf{v} \cdot \mathbf{n}) = \int_{\Omega_f} \mathbf{f} \cdot \mathbf{v}$$

$$\int_{\Omega_f} q \mathrm{div} \mathbf{u}_f = 0$$

$$\int_{\Omega_p} \mathrm{K} \nabla p_p \cdot \nabla \psi - \int_{\Gamma} \psi (\mathbf{u}_f \cdot \mathbf{n}) = 0$$

WEAK FORM OF THE COUPLED DARCY–NS PROBLEM (II)

Find $\mathbf{u}_f \in H^1(\Omega_f)$, $p_f \in L^2(\Omega_f)$, $\mathbf{u}_p \in L^2(\Omega_p)$, $p_p \in H^1(\Omega_p)$:

$$\int_{\Omega_f} \nu \nabla \mathbf{u}_f \cdot \nabla \mathbf{v} + \int_{\Gamma} \alpha (\mathbf{u}_f \cdot \boldsymbol{\tau})(\mathbf{v} \cdot \boldsymbol{\tau}) + \int_{\Omega_f} [(\mathbf{u}_f \cdot \nabla)\mathbf{u}_f]\mathbf{v}$$

$$- \int_{\Omega_f} p_f \mathrm{div} \mathbf{v} + \int_{\Gamma} p_p (\mathbf{v} \cdot \mathbf{n}) = \int_{\Omega_f} \mathbf{f} \cdot \mathbf{v}$$

$$\int_{\Omega_f} q \mathrm{div} \mathbf{u}_f = 0$$

$$\int_{\Omega_p} \mathrm{K}^{-1} \mathbf{u}_p \cdot \mathbf{w} + \int_{\Omega_p} \mathbf{w} \cdot \nabla p_p = 0$$

$$\int_{\Omega_p} \mathbf{u}_p \cdot \nabla \psi + \int_{\Gamma} (\mathbf{u}_f \cdot \mathbf{n}) \psi = 0$$

[Urquiza et al. (2008); Masud, Hughes (2005)]

ON THE WELL-POSEDNESS OF THE COUPLED PROBLEMS

- The Darcy-Stokes case: well-posedness can be easily proved using the theory by Brezzi for saddle point problems.
  [Miglio et al. (2002); Layton et al. (2003); Urquiza et al. (2008)]

- The Darcy-Navier–Stokes case: if there holds

$$\boxed{\|\mathbf{f}\|_{L^2(\Omega_f)} \leq C\nu^2}$$

the Darcy-Navier–Stokes problem has a solution which is unique if the normal velocity across the interface is *'small enough'*:

$$\boxed{\mathbf{u}_f \cdot \mathbf{n} \in S_{r_m} = \{\eta \in H^{1/2}_{00}(\Gamma) \,:\, \|\eta\|_{H^{1/2}_{00}(\Gamma)} \leq r_m\} \subset H^{1/2}_{00}(\Gamma)}$$

for a suitably defined radius r_m.
[Girault, Rivière (2009); Badea, Discacciati, Quarteroni (2010)]

FINITE ELEMENT APPROXIMATION

A conforming FE approximation of this problem would lead to solve a global nonlinear system, generally large, sparse and ill-conditioned.

- "Laplace" – Navier-Stokes problem:

$$\left(\begin{array}{ccc|cc} \mathrm{A}_{ff}(\mathbf{u}_f) & \mathrm{D}_f^T & \mathrm{A}_{f\Gamma}(\mathbf{u}_f) & 0 & 0 \\ \mathrm{D}_f & 0 & \mathrm{D}_{f\Gamma} & 0 & 0 \\ \mathrm{A}_{\Gamma f}(\mathbf{u}_f) & \mathrm{D}_{f\Gamma}^T & \mathrm{A}_{\Gamma\Gamma}^f(\mathbf{u}_f) & 0 & \mathrm{M}_{\Gamma\Gamma} \\ \hline 0 & 0 & 0 & \mathrm{A}_{pp} & \mathrm{A}_{\Gamma p}^T \\ 0 & 0 & -\mathrm{M}_{\Gamma\Gamma}^T & \mathrm{A}_{\Gamma p} & \mathrm{A}_{\Gamma\Gamma}^p \end{array}\right) \left(\begin{array}{c} \mathbf{u}_f \\ \mathbf{p}_f \\ \mathbf{u}_f^\Gamma \\ \mathbf{p}_p \\ \mathbf{p}_p^\Gamma \end{array}\right) = \mathbf{F}$$

with $\mathbf{u}_f^\Gamma \to$ nodal values of $\mathbf{u}_f^h \cdot \mathbf{n}$ on Γ
$\mathbf{p}_p^\Gamma \to$ nodal values of p_p^h on Γ

- Darcy – Navier-Stokes problem:

$$\left(\begin{array}{ccc|ccc} \mathrm{A}_{ff}(\mathbf{u}_f) & \mathrm{D}_f^T & \mathrm{A}_{f\Gamma}(\mathbf{u}_f) & 0 & 0 & 0 \\ \mathrm{D}_f & 0 & \mathrm{D}_{f\Gamma} & 0 & 0 & 0 \\ \mathrm{A}_{\Gamma f}(\mathbf{u}_f) & \mathrm{D}_{f\Gamma}^T & \mathrm{A}_{\Gamma\Gamma}^f(\mathbf{u}_f) & 0 & 0 & \mathrm{M}_{\Gamma\Gamma} \\ \hline 0 & 0 & 0 & \mathrm{A}_p & \mathrm{G}_p^T & \mathrm{G}_{\Gamma p}^T \\ 0 & 0 & 0 & \mathrm{G}_p & \mathrm{S}_{pp} & \mathrm{S}_{p\Gamma} \\ 0 & 0 & \mathrm{M}_{\Gamma\Gamma}^T & \mathrm{G}_{\Gamma p} & \mathrm{S}_{\Gamma p} & \mathrm{S}_{\Gamma\Gamma}^p \end{array}\right) \begin{pmatrix} \mathbf{u}_f \\ \mathbf{p}_f \\ \mathbf{u}_f^\Gamma \\ \mathbf{u}_p \\ \mathbf{p}_p \\ \mathbf{p}_p^\Gamma \end{pmatrix} = \mathbf{F}$$

NUMERICAL ALGORITHMS

THE DARCY – NAVIER-STOKES CASE

- Fixed-point (Picard) method

- Newton method
  Convergence result: *if*
  - $\|\mathbf{f}\|_{L^2(\Omega_f)} \leq \tilde{C}\nu^2$

  then
  - *the Navier-Stokes/Darcy problem has a unique solution*
  - *the Newton method converges to this solution provided the initial normal velocity* $\mathbf{u}_f^0 \cdot \mathbf{n}$ *on* Γ *is chosen 'close enough' to the solution.*

We have to solve a linearized coupled problem at each iteration.

[Badea, Discacciati, Quarteroni (2010)]

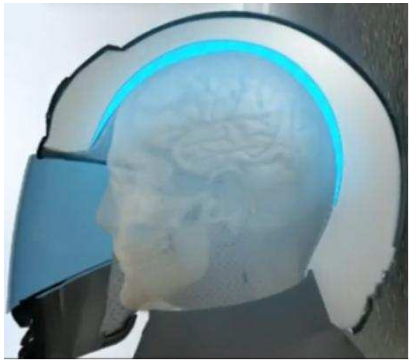
NUMERICAL RESULTS

We take $\Omega_f = (0,1) \times (1,2)$ and $\Omega_p = (0,1) \times (0,1)$.
We use Taylor-Hood elements for the Navier-Stokes equations and quadratic Lagrangian elements for the Darcy equation.
The exact solution is $\mathbf{u}_f = ((y-1)^2 + (y-1) + \sqrt{\mathrm{K}}x(x-1))$, $p_f = 2\nu(x+y-1)$, $\varphi = \mathrm{K}^{-1}(x(1-x)(y-1) + (y-1)^3/3) + 2\nu x$.

Number of iterations with respect to the parameters ν and K:

ν	K	$h = 0.1429$		$h = 0.0714$		$h = 0.0357$	
		FP	N	FP	N	FP	N
1	1	7	4	7	4	7	4
1	10^{-4}	5	4	5	4	5	4
10^{-1}	10^{-1}	10	5	10	5	10	5
10^{-2}	10^{-1}	15	6	15	6	15	6
10^{-2}	10^{-3}	13	6	13	6	13	6

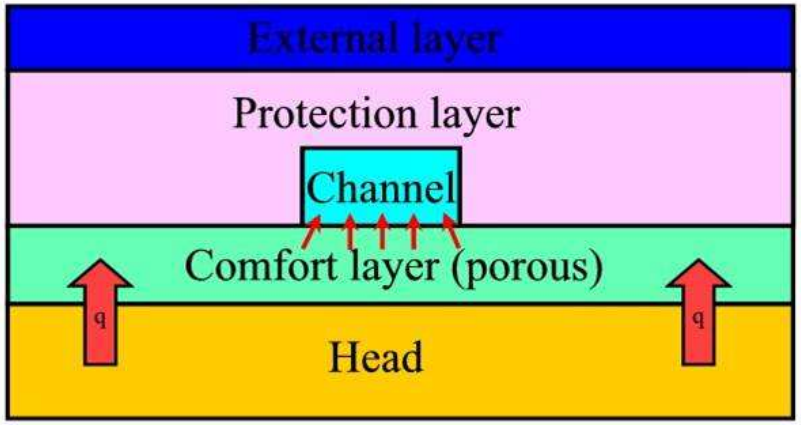
AN APPLICATION: internal ventilation of motorcycle helmets *(collaboration with F. Cimolin, Politecnico di Torino)*



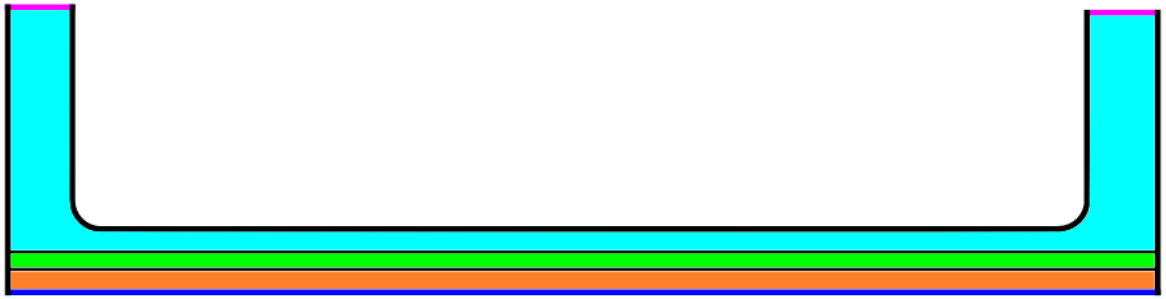
The ventilation system is realized by means of a series of channels crossing the helmet. The air enters the channels from the air intakes, and the objective is to extract as much heat as possible.

[Cimolin, Discacciati (2010)]

This simplified scheme shows how the heat is extracted by the fresh air, which flows above and through the porous comfort layer surrounding the head:



(Schematic frontal cross section of the helmet)

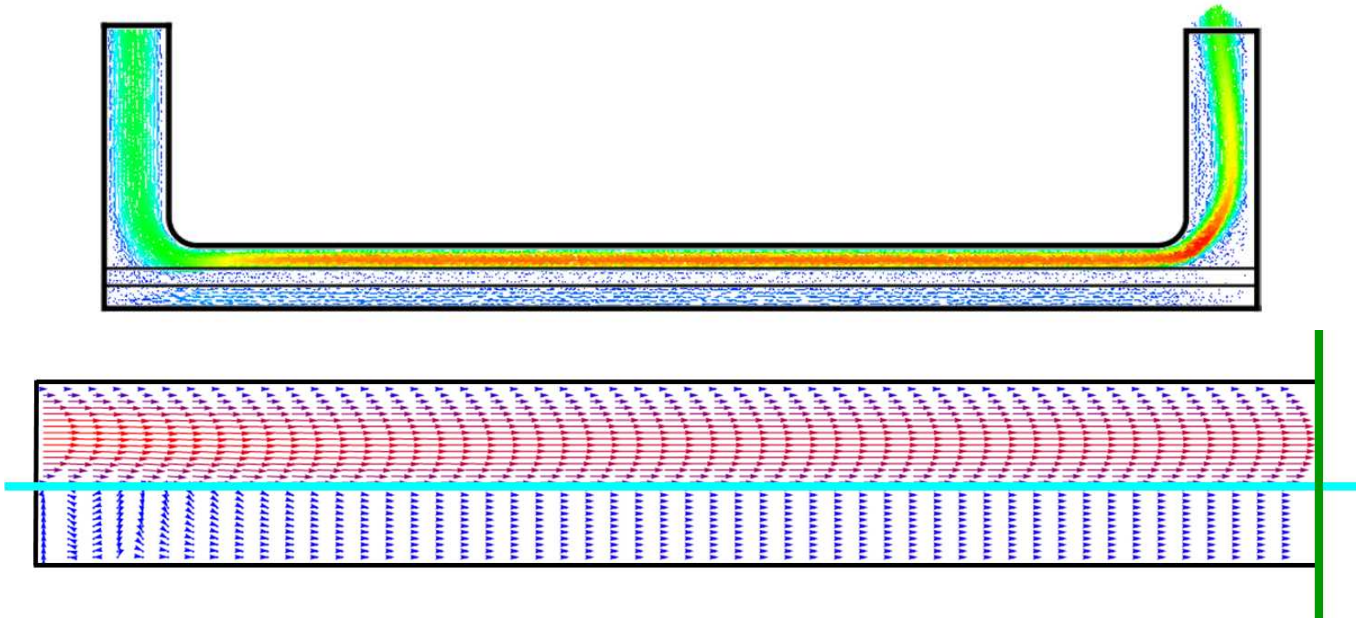


(Schematic longitudinal cross section of an air channel)

NUMERICAL RESULTS (I)

2D problem discretized using about 200,000 elements.
Navier-Stokes/Darcy problem solved by the Newton method.

Velocity field



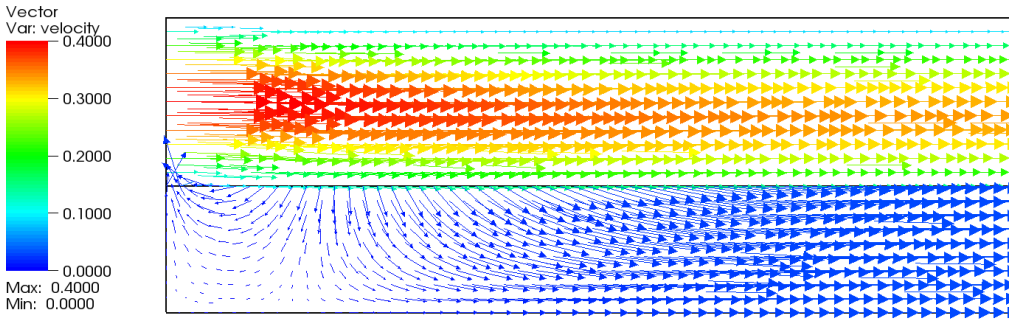
A "DECOUPLED" STRATEGY

- We would like (in this case for software availability reasons) to solve the coupled problem exploiting the "intrinsic" decoupled structure of our physical problem $\Rightarrow$ alternate the solution of the Navier-Stokes problem in Ω_f and of Darcy equations in Ω_p:
  1. Given a normal stress on the interface, solve the Navier-Stokes equations in Ω_f and recover the corresponding normal velocity across the interface;
  2. Use the computed normal velocity across the interface as boundary condition for the Darcy equations in Ω_p, solve them and recover the corresponding normal stress on the interface;
  3. Iterate using a suitable convergence criterion and a relaxation procedure to enhance convergence (if necessary).

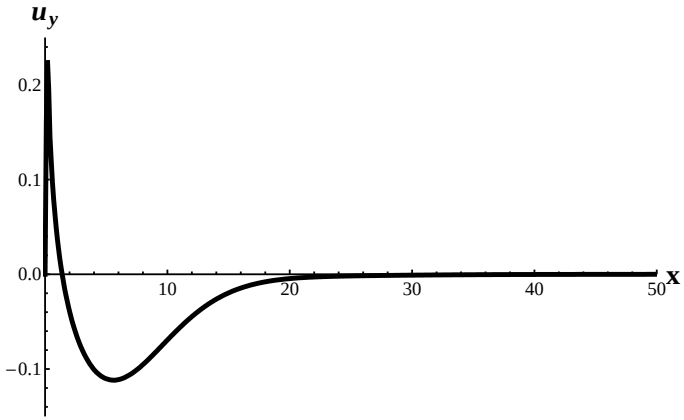
- Use a *domain decomposition* approach
  $\Rightarrow$ write the global problem as an interface problem, choosing suitable interface variables.

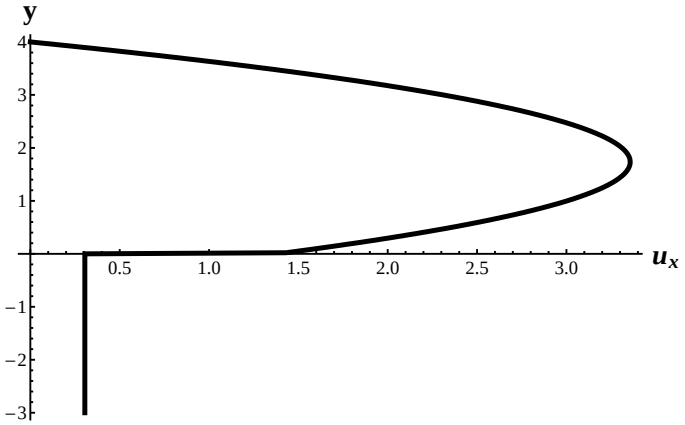
NUMERICAL RESULTS (II)

Steady-state flow field computed using a Navier-Stokes/ Forchheimer model:



The normal component of the velocity through the interface and velocity profile at the outlet:





THE DOMAIN DECOMPOSITION FRAMEWORK

CHOICE OF THE INTERFACE VARIABLE

There are two possible strategies to choose the interface variable:

- $\lambda = \mathbf{u}_f \cdot \mathbf{n}$ on Γ; in that case we aim at satisfying

$$\boxed{-\mathrm{K}\nabla p_p \cdot \mathbf{n} = \lambda \text{ on } \Gamma}$$

- $\sigma = p_p$ on Γ; here, we aim at satisfying

$$\boxed{-\mathbf{n} \cdot \mathrm{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n} = \sigma \text{ on } \Gamma}$$

Both choices are suitable from a mathematical standpoint since they yield well-posed subproblems in the fluid and the porous part.

INTERFACE EQUATION FOR DARCY–STOKES

We can equivalently express the Darcy-Stokes problem in terms of the solution λ (normal velocity across Γ) of the interface problem

$$\boxed{S_s\lambda + S_d\lambda = \chi \quad \text{on } \Gamma} \tag{1}$$

- S_s continuous and coercive *fluid* operator:

$$S_s : \lambda \text{ (normal velocities on } \Gamma) \xrightarrow[Stokes]{solve} \xi = -\mathbf{n}\cdot\mathrm{T}(\mathbf{u}_f, p_f)\cdot\mathbf{n} \text{ (normal stresses on } \Gamma).$$

- S_d continuous, positive *porous media* operator:

$$S_d : \lambda \text{ (fluxes of } p_p \text{ on } \Gamma) \xrightarrow[Darcy]{solve} \xi = p_{p|\Gamma}.$$

- S_s is *spectrally equivalent* to $S_s + S_d$: there exist two positive constants k_1 and k_2 (independent of η) such that

$$k_1\langle S_s\eta, \eta\rangle \;\leq\; \langle (S_s + S_d)\eta, \eta\rangle \;\leq\; k_2\langle S_s\eta, \eta\rangle \qquad \forall \eta \in \Lambda_0 \subset H^{1/2}_{00}(\Gamma).$$

- There exists a unique solution $\lambda \in H^{1/2}_{00}(\Gamma)$ for (1).

INTERFACE EQUATIONS AT THE DISCRETE LEVEL

The discrete counterpart of the interface equations are the *symmetric positive definite* Schur complement systems:

- Discrete interface equation for the normal velocity:

$$\boxed{\Sigma_s \mathbf{u}_f^\Gamma + \Sigma_d \mathbf{u}_f^\Gamma = \chi_s + \chi_d}$$

- Discrete interface equation for the piezometric head:

$$\boxed{\Sigma_f \mathbf{p}_p^\Gamma + \Sigma_p \mathbf{p}_p^\Gamma = \chi_f + \chi_p}$$

PRECONDITIONING TECHNIQUES (I)

Then, we can characterize the preconditioners:

- for the interface equation involving $\mathbf{u}_f^\Gamma$:

$$\boxed{P_1 = (2\alpha_1)^{-1}(\Sigma_s + \alpha_1 I)(\Sigma_d + \alpha_1 I)} \qquad \alpha_1 \simeq \sqrt{\nu}$$

- for the interface equation involving $\mathbf{p}_p^\Gamma$:

$$\boxed{P_2 = (2\alpha_2)^{-1}(\Sigma_p + \alpha_2 I)(\Sigma_f + \alpha_2 I)} \qquad \alpha_2 \simeq \sqrt{\mathrm{K}}$$

These preconditioners

- have a multiplicative structure;
- can be used within GMRES iterations;
- generalize from the algebraic viewpoint the Robin-Robin method.

[see also Benzi (2009), Bai et al. (2003)]

NUMERICAL RESULTS (I)

Comparison between CG iterations without preconditioner and GMRES iterations...

- with preconditioner P_1 for the system involving $\mathbf{u}_f^\Gamma$

	$\nu = 10^{-4}$, K $= 10^{-3}$			$\nu = 10^{-6}$, K $= 10^{-5}$			$\nu = 10^{-6}$, K $= 10^{-8}$		
	CG	**GMRES + P_1**		CG	**GMRES + P_1**		CG	**GMRES + P_1**	
h_1	9	5	$(\alpha_1 = 10^{-2})$	9	4	$(\alpha_1 = 10^{-3})$	9	4	$(\alpha_1 = 10^{-3})$
h_2	20	7	$(\alpha_1 = 10^{-2})$	20	4	$(\alpha_1 = 10^{-3})$	20	4	$(\alpha_1 = 10^{-3})$
h_3	42	9	$(\alpha_1 = 10^{-3})$	42	4	$(\alpha_1 = 10^{-3})$	42	4	$(\alpha_1 = 10^{-3})$
h_4	64	9	$(\alpha_1 = 10^{-3})$	66	4	$(\alpha_1 = 10^{-3})$	66	4	$(\alpha_1 = 10^{-3})$

- with preconditioner P_2 for the system involving $\mathbf{p}_p^\Gamma$

	$\nu = 10^{-4}$, K $= 10^{-3}$			$\nu = 10^{-6}$, K $= 10^{-5}$			$\nu = 10^{-6}$, K $= 10^{-8}$		
	CG	**GMRES + P_2**		CG	**GMRES + P_2**		CG	**GMRES + P_2**	
h_1	11	8	$(\alpha_2 = 10^{-2})$	13	5	$(\alpha_2 = 10^{-3})$	-	3	$(\alpha_2 = 10^{-3})$
h_2	22	9	$(\alpha_2 = 10^{-2})$	24	5	$(\alpha_2 = 10^{-3})$	-	4	$(\alpha_2 = 10^{-3})$
h_3	47	10	$(\alpha_2 = 10^{-2})$	52	6	$(\alpha_2 = 10^{-3})$	57	4	$(\alpha_2 = 10^{-3})$
h_4	84	10	$(\alpha_2 = 10^{-2})$	108	6	$(\alpha_2 = 10^{-3})$	124	4	$(\alpha_2 = 10^{-3})$

PRECONDITIONING TECHNIQUES (II)

Effective preconditioners with additive structure can be characterized as well, considering the following *augmented interface systems*:

- Discrete augmented Dirichlet-Dirichlet (aDD) problem:

$$\begin{pmatrix} \Sigma_s & -M_\Gamma \\ M_\Gamma^T & \Sigma_p \end{pmatrix} \begin{pmatrix} \mathbf{u}_f^\Gamma \\ \mathbf{p}_p^\Gamma \end{pmatrix} = \begin{pmatrix} \chi_s \\ \chi_p \end{pmatrix}$$

- Discrete augmented Neumann-Neumann (aNN) problem:

$$\begin{pmatrix} \Sigma_d & M_\Gamma \\ -M_\Gamma^T & \Sigma_f \end{pmatrix} \begin{pmatrix} \mathbf{u}_f^\Gamma \\ \mathbf{p}_p^\Gamma \end{pmatrix} = \begin{pmatrix} \chi_d \\ \chi_f \end{pmatrix}$$

In this case, we can characterize the preconditioners

- for the aDD problem

$$\boxed{P_3 = (2\alpha_3)^{-1} \begin{pmatrix} \Sigma_s + \alpha_3 I & 0 \\ 0 & \Sigma_p + \alpha_3 I \end{pmatrix} \begin{pmatrix} \alpha_3 I & -M_\Gamma \\ M_\Gamma^T & \alpha_3 I \end{pmatrix}}$$

- for the aNN problem

$$\boxed{P_4 = (2\alpha_4)^{-1} \begin{pmatrix} \Sigma_d + \alpha_4 I & 0 \\ 0 & \Sigma_f + \alpha_4 I \end{pmatrix} \begin{pmatrix} \alpha_4 I & M_\Gamma \\ -M_\Gamma^T & \alpha_4 I \end{pmatrix}}$$

These preconditioners

- allow solving the fluid and the porous-media subproblems independently in a parallel fashion;
- can be used within GMRES iterations.

NUMERICAL RESULTS (II)

Comparison between GMRES iterations without preconditioner for the augmented systems...

- with preconditioner P_3 for the aDD problem

	$\nu = 10^{-4}$, K $= 10^{-3}$			$\nu = 10^{-6}$, K $= 10^{-5}$			$\nu = 10^{-6}$, K $= 10^{-8}$		
	GMRES	**GMRES + P_3**		GMRES	**GMRES + P_3**		GMRES	**GMRES + P_3**	
h_1	17	14	$(\alpha_3 = 10^{-3})$	17	7	$(\alpha_3 = 10^{-3})$	17	8	$(\alpha_3 = 10^{-3})$
h_2	33	17	$(\alpha_3 = 10^{-3})$	33	8	$(\alpha_3 = 10^{-3})$	33	10	$(\alpha_3 = 10^{-3})$
h_3	63	22	$(\alpha_3 = 5 \cdot 10^{-4})$	65	8	$(\alpha_3 = 5 \cdot 10^{-4})$	65	10	$(\alpha_3 = 5 \cdot 10^{-4})$
h_4	67	23	$(\alpha_3 = 5 \cdot 10^{-4})$	79	9	$(\alpha_3 = 5 \cdot 10^{-4})$	101	11	$(\alpha_3 = 5 \cdot 10^{-4})$

- with preconditioner P_4 for the aNN problem

	$\nu = 10^{-4}$, K $= 10^{-3}$			$\nu = 10^{-6}$, K $= 10^{-5}$			$\nu = 10^{-6}$, K $= 10^{-8}$		
	GMRES	**GMRES + P_4**		GMRES	**GMRES + P_4**		GMRES	**GMRES + P_4**	
h_1	17	16	$(\alpha_4 = 0.1)$	16	9	$(\alpha_4 = 0.5)$	9	8	$(\alpha_4 = 1)$
h_2	32	18	$(\alpha_4 = 0.1)$	32	8	$(\alpha_4 = 0.5)$	16	7	$(\alpha_4 = 0.5)$
h_3	59	20	$(\alpha_4 = 5 \cdot 10^{-2})$	58	10	$(\alpha_4 = 0.1)$	30	5	$(\alpha_4 = 0.8)$
h_4	82	27	$(\alpha_4 = 5 \cdot 10^{-2})$	81	8	$(\alpha_4 = 0.1)$	44	5	$(\alpha_4 = 0.8)$

[Discacciati (2011)]

THE NONLINEAR DARCY – NAVIER-STOKES CASE

- An interface equation depending solely on the normal velocity λ across the interface can be characterized also for the Darcy – Navier-Stokes problem.

- We can define a "fluid" *non-linear* pseudo-differential operator $\mathcal{S}_{ns}$ analogous to S_s that associates to the normal velocity λ on Γ the normal component of the corresponding Cauchy stress tensor on Γ.

- We can write the interface equation:

$$\boxed{\text{find } \lambda \in \Lambda_0 \subset H^{1/2}_{00}(\Gamma): \quad \langle \mathcal{S}_{ns}(\lambda) + S_d\lambda, \mu \rangle = 0 \qquad \forall \mu \in \Lambda_0}$$

- Preconditioning?

[Badea, Discacciati, Quarteroni (2010)]

SUMMARIZING...

Using domain decomposition techniques (without overlap)

- we can reduce the global coupled problem to an equation defined only on the interface, so that we get a linear system whose size coincides with the number of dofs on Γ;
- we can characterize (optimal) preconditioners that we can use within iterative methods (CG, GMRES, ...);
- we can solve the coupled problem considering separately each (simpler) subproblem.

This methodology relies strongly on the coupling conditions.
We would like to study alternative approaches where the role of coupling conditions is not so strong.

A PENALIZATION APPROACH

- Due to the difficulty of dealing with different type of equations in the subdomains, a penalization approach (e.g., Iliev et al. 2004, 2007) is often adopted to model the flow over porous media.

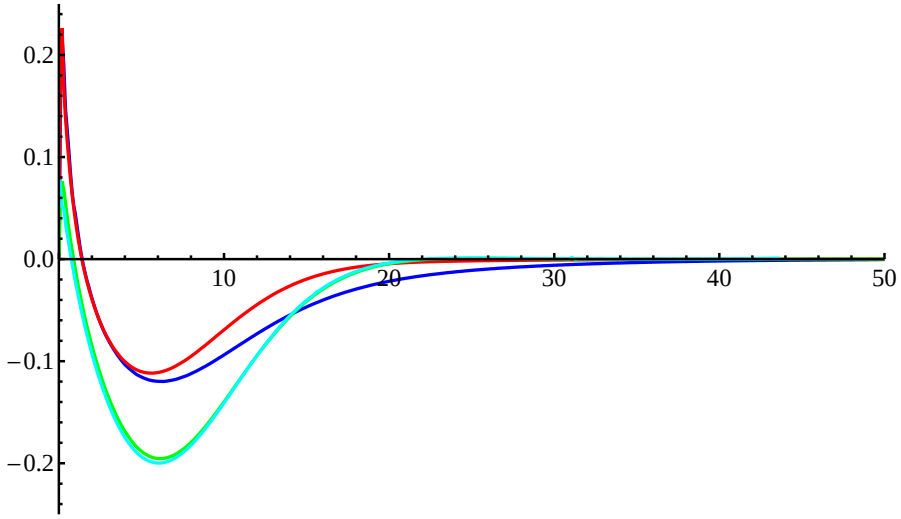
- This approach, similar to the *fictitious domain approach* of Angot (1999), models the resistance induced by the porous medium via penalization terms in the Navier-Stokes equations:

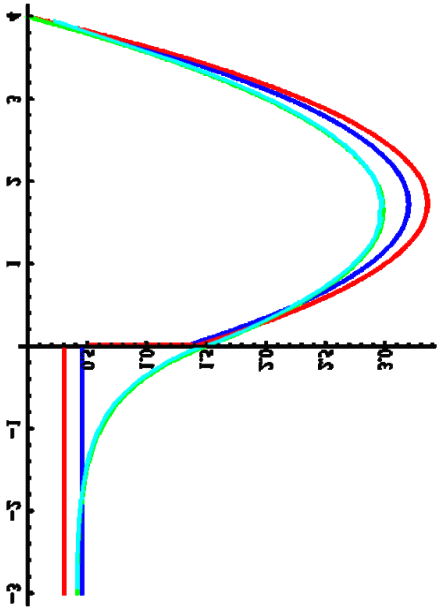
$$\begin{aligned} -\mathbf{div}\,\mathrm{T}(\mathbf{u},p) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{\nu}{\mathrm{K}}\mathbf{u} + \frac{C_F}{\sqrt{\mathrm{K}}}|\mathbf{u}|\mathbf{u} &= \mathbf{f} \\ \mathrm{div}\,\mathbf{u} &= 0 \end{aligned} \quad \text{in } \Omega_f \cup \Omega_p$$

C_F is the inertial resistance coefficient.
The penalization terms are set to zero in Ω_f using discontinous coefficients.

- Solvers comparison

  The normal component of the velocity through the interface and velocity profile at the outlet:





NS-Darcy – NS-Forchheimer – Penalization

THE VIRTUAL CONTROL APPROACH

(Joint work with P. Gervasio and A. Quarteroni)

A TOY PROBLEM

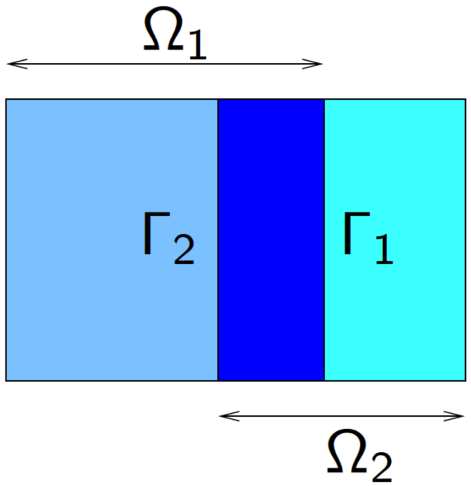
- Consider the boundary value problem

$$\begin{aligned} Lu &= f && \text{in } \Omega \\ u &= \phi_D && \text{on } \Gamma_D \\ \partial_{n_L} u &= \phi_N && \text{on } \Gamma_N \end{aligned}$$

where L is the linear elliptic second-order operator

$$\boxed{Lu = -\mathrm{div}(K\nabla u) + b_0 u}$$

- We split Ω into two overlapping subdomains:



The problem can be equivalently rewritten as

$$\begin{array}{lll} Lu_1 = f & \text{in } \Omega_1 \\ Lu_2 = f & \text{in } \Omega_2 \\ u_1 = u_2 & \text{in } \Omega_{12} \\ b.c. & \text{on } \partial\Omega_i \setminus \Gamma_i \end{array} \quad \text{or} \quad \begin{array}{lll} Lu_1 = f & \text{in } \Omega_1 \\ Lu_2 = f & \text{in } \Omega_2 \\ \Psi(u_1) = \Psi(u_2) & \text{on } \Gamma_1 \cup \Gamma_2 \\ b.c. & \text{on } \partial\Omega_i \setminus \Gamma_i \end{array}$$

where

$$\boxed{\Psi(u_i) = \begin{cases} u_i & \\ \text{or} \quad \beta u_i + \partial_{n_L} u_i & \text{on } \Gamma_1 \cup \Gamma_2 \end{cases}}$$

so that in the second problem we impose either

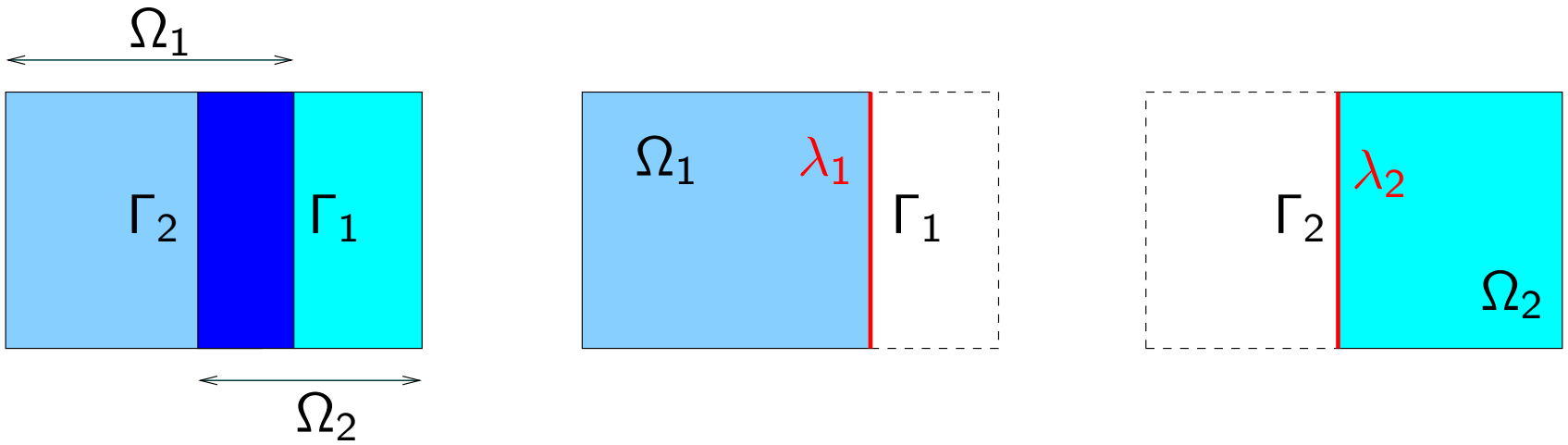
$$u_1 = u_2 \quad \text{on } \Gamma_1 \cup \Gamma_2$$

or

$$\beta u_1 + \partial_{n_L} u_1 = \beta u_2 + \partial_{n_L} u_2 \quad \text{on } \Gamma_1 \cup \Gamma_2$$

VIRTUAL CONTROL APPROACH (with overlap)

$\Omega_1,\ \Omega_2 \subset \Omega,\ \ \Omega_{12} = \Omega_1 \cap \Omega_2 \neq \emptyset,\ \ \Gamma_k = \partial\Omega_k \setminus \partial\Omega,\ k = 1, 2.$



$$\begin{cases} Lu_1 = f & \text{in } \Omega_1 \\ \Psi(u_1) = \lambda_1 & \text{on } \Gamma_1 \\ b.c. & \text{on } \partial\Omega_1 \setminus \Gamma_1 \end{cases} \qquad \begin{cases} Lu_2 = f & \text{in } \Omega_2 \\ \Psi(u_2) = \lambda_2 & \text{on } \Gamma_2 \\ b.c. & \text{on } \partial\Omega_2 \setminus \Gamma_2 \end{cases}$$

λ_1, λ_2 are solutions of a suitable minimization problem

$$\boxed{\inf_{\lambda_1,\lambda_2} J(\lambda_1, \lambda_2)}$$

VIRTUAL CONTROL APPROACH (with overlap)

- It represents the formal mathematical justification of engineering practice.
- Origin and mathematical foundations: theory of optimal control [J.-L. Lions (1971); Glowinski, Dinh, Periaux (1983); Glowinski, Periaux, Terrasson (1990); Lions, Pironneau (1998, 1999)]
- It is more "indifferent" w.r.t. interface conditions (no a-priori info required), contrary to the domain decomposition approach without overlap.

FUNCTIONAL SETTING

λ_i are the virtual controls that, depending on the choice of Ψ, may be either

- admissible *Dirichlet* controls

$$\boxed{\lambda_i \in \Lambda_i^D \subset H^{1/2}(\Gamma_i)}$$

- or admissible *Robin* controls

$$\boxed{\lambda_i \in \Lambda_i^R \subset H^{-1/2}(\Gamma_i)}$$

We have to solve a control problem with boundary controls and...

... either distributed or boundary (interface) observation depending on the choice of the cost functional J:

- Minimization in the norm $L^2(\Omega_{12})$

$$\boxed{J_0(\lambda_1, \lambda_2) = \frac{1}{2}\|u_1(\lambda_1) - u_2(\lambda_2)\|^2_{L^2(\Omega_{12})}}$$

- Minimization in the norm $H^1(\Omega_{12})$

$$\boxed{J_1(\lambda_1, \lambda_2) = \frac{1}{2}\|u_1(\lambda_1) - u_2(\lambda_2)\|^2_{H^1(\Omega_{12})}}$$

- Minimization in the norm $H^{1/2}(\Gamma_1 \cup \Gamma_2)$

$$\boxed{J_{1/2}(\lambda_1, \lambda_2) = \frac{1}{2}\|u_1(\lambda_1) - u_2(\lambda_2)\|^2_{H^{1/2}(\Gamma_1 \cup \Gamma_2)}}$$

- Minimization in the norm $H^{-1/2}(\Gamma_1 \cup \Gamma_2)$

$$\boxed{J_{-1/2}(\lambda_1, \lambda_2) = \frac{1}{2}\|\partial_{n_L} u_1(\lambda_1) - \partial_{n_L} u_2(\lambda_2)\|^2_{H^{-1/2}(\Gamma_1 \cup \Gamma_2)}}$$

Theorem. For all the choices of functionals and for either Dirichlet or Robin controls, the minimization problem $\inf_{\lambda_1,\lambda_2} J(\lambda_1, \lambda_2)$ has a unique solution.
The solution of the minimization problem coincides with the solution of the original problem.

Moreover, we can prove that the functionals $J_{1/2}$ and $J_{-1/2}$ are equivalent to J_1.

[Discacciati, Gervasio, Quarteroni (2011, in preparation)]

The solution of the virtual control approach satisfies (in weak sense) the optimality system.
Considering, e.g., Dirichlet controls and minimization in the norm $L^2(\Omega_{12})$, we obtain:

- State equations:

$$\begin{array}{ll} Lu_i = f & \text{in } \Omega_i \\ u_i = \lambda_i & \text{on } \Gamma_i \\ b.c. & \text{on } \partial\Omega_i \setminus \Gamma_i \end{array}$$

- Adjoint equations:

$$\begin{array}{ll} L^* p_i = (-1)^{i+1}(u_1 - u_2)\chi_{12} & \text{in } \Omega_i \\ p_i = 0 & \text{on } \Gamma_i \\ b.c. & \text{on } \partial\Omega_i \setminus \Gamma_i \end{array}$$

- Euler equations:

$$\partial_{n_L} p_i = 0 \quad \text{on } \Gamma_i$$

ASSOCIATED ALGEBRAIC SYSTEM

Considering a conforming finite element approximation with matching grids on Ω_{12}, we obtain the linear system

$$\left(\begin{array}{cc|cc|cc} A_1 & 0 & 0 & 0 & D_{\Gamma_1}^T & 0 \\ 0 & A_2 & 0 & 0 & 0 & D_{\Gamma_2}^T \\ \hline M_1^{12} & -M_2^{12} & A_1 & 0 & 0 & 0 \\ -M_1^{12} & M_2^{12} & 0 & A_2 & 0 & 0 \\ \hline 0 & 0 & D_{\Gamma_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{\Gamma_2} & 0 & 0 \end{array}\right) \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \boldsymbol{\lambda}_1 \\ \boldsymbol{\lambda}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

We can use two possible strategies to solve this system:

- compute the Schur complement wrt the controls $\boldsymbol{\lambda}_i$
- solve the system at once for all variables

In both cases suitable preconditioners should be designed.

NUMERICAL RESULTS

We consider a test case with $b_0 = 1$, $\mathbb{Q}_1$ elements.

- Virtual control method: cost functional J_0

h	H_δ	$K = 10^{-4}$	$K = 1$	$K = 10^4$
10^{-1}	10^{-1}	28	25	25
$5 \cdot 10^{-2}$	10^{-1}	35	37	36
$2.5 \cdot 10^{-2}$	10^{-1}	32	30	31
$1.25 \cdot 10^{-2}$	10^{-1}	32	33	31
$6.25 \cdot 10^{-3}$	10^{-1}	36	35	37

h	H_δ	$K = 10^{-4}$	$K = 1$	$K = 10^4$
$2 \cdot 10^{-2}$	10^{-1}	30	30	30
$2 \cdot 10^{-2}$	$8 \cdot 10^{-2}$	50	43	45
$2 \cdot 10^{-2}$	$6 \cdot 10^{-2}$	70	68	81
$2 \cdot 10^{-2}$	$4 \cdot 10^{-2}$	127	115	123
$2 \cdot 10^{-2}$	$2 \cdot 10^{-2}$	236	213	266

- Virtual control method: cost functional J_1

h	H_δ	$K = 10^{-4}$	$K = 1$	$K = 10^4$
10^{-1}	10^{-1}	13	28	25
$5 \cdot 10^{-2}$	10^{-1}	15	15	15
$2.5 \cdot 10^{-2}$	10^{-1}	15	15	15
$1.25 \cdot 10^{-2}$	10^{-1}	14	14	14
$6.25 \cdot 10^{-3}$	10^{-1}	14	14	14

h	H_δ	$K = 10^{-4}$	$K = 1$	$K = 10^4$
$2 \cdot 10^{-2}$	10^{-1}	14	14	14
$2 \cdot 10^{-2}$	$8 \cdot 10^{-2}$	15	15	15
$2 \cdot 10^{-2}$	$6 \cdot 10^{-2}$	18	18	18
$2 \cdot 10^{-2}$	$4 \cdot 10^{-2}$	22	22	22
$2 \cdot 10^{-2}$	$2 \cdot 10^{-2}$	30	156	145

BiCGStab iterations on the Schur complement system;
$tol = 10^{-14}$.

- Additive Schwarz method

h	H_δ	$K = 10^{-4}$	$K = 1$	$K = 10^4$
10^{-1}	10^{-1}	95	95	94
$5 \cdot 10^{-2}$	10^{-1}	94	94	94
$2.5 \cdot 10^{-2}$	10^{-1}	94	94	95
$1.25 \cdot 10^{-2}$	10^{-1}	95	94	94
$6.25 \cdot 10^{-3}$	10^{-1}	94	95	94

h	H_δ	$K = 10^{-4}$	$K = 1$	$K = 10^4$
$2 \cdot 10^{-2}$	10^{-1}	95	94	94
$2 \cdot 10^{-2}$	$8 \cdot 10^{-2}$	117	117	117
$2 \cdot 10^{-2}$	$6 \cdot 10^{-2}$	153	153	153
$2 \cdot 10^{-2}$	$4 \cdot 10^{-2}$	226	226	226
$2 \cdot 10^{-2}$	$2 \cdot 10^{-2}$	441	441	441

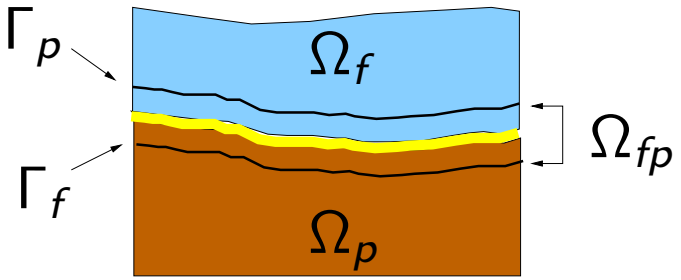
- Multiplicative Schwarz method

h	H_δ	$K = 10^{-4}$	$K = 1$	$K = 10^4$
10^{-1}	10^{-1}	49	49	49
$5 \cdot 10^{-2}$	10^{-1}	48	48	48
$2.5 \cdot 10^{-2}$	10^{-1}	48	48	49
$1.25 \cdot 10^{-2}$	10^{-1}	49	49	48
$6.25 \cdot 10^{-3}$	10^{-1}	49	48	49

h	H_δ	$K = 10^{-4}$	$K = 1$	$K = 10^4$
$2 \cdot 10^{-2}$	10^{-1}	48	48	48
$2 \cdot 10^{-2}$	$8 \cdot 10^{-2}$	60	60	60
$2 \cdot 10^{-2}$	$6 \cdot 10^{-2}$	78	78	78
$2 \cdot 10^{-2}$	$4 \cdot 10^{-2}$	116	116	116
$2 \cdot 10^{-2}$	$2 \cdot 10^{-2}$	224	224	224

$Tol = 10^{-14}$.

ONGOING WORK: VIRTUAL CONTROLS FOR DARCY–STOKES



Stokes equations
$$\begin{cases} -\mathbf{div}\,\mathrm{T}(\mathbf{u}_f, p_f) = \mathbf{f}, \quad \mathrm{div}\ \mathbf{u}_f = 0 \text{ in } \Omega_f \\ \mathbf{u}_f = \boldsymbol{\lambda}_1 \text{ on } \Gamma_f \end{cases}$$

Darcy equations
$$\begin{cases} -\mathrm{div}\ (\mathrm{K}\nabla p_p) = 0 \text{ in } \Omega_p \\ p_p = \lambda_2 \text{ on } \Gamma_p \end{cases}$$

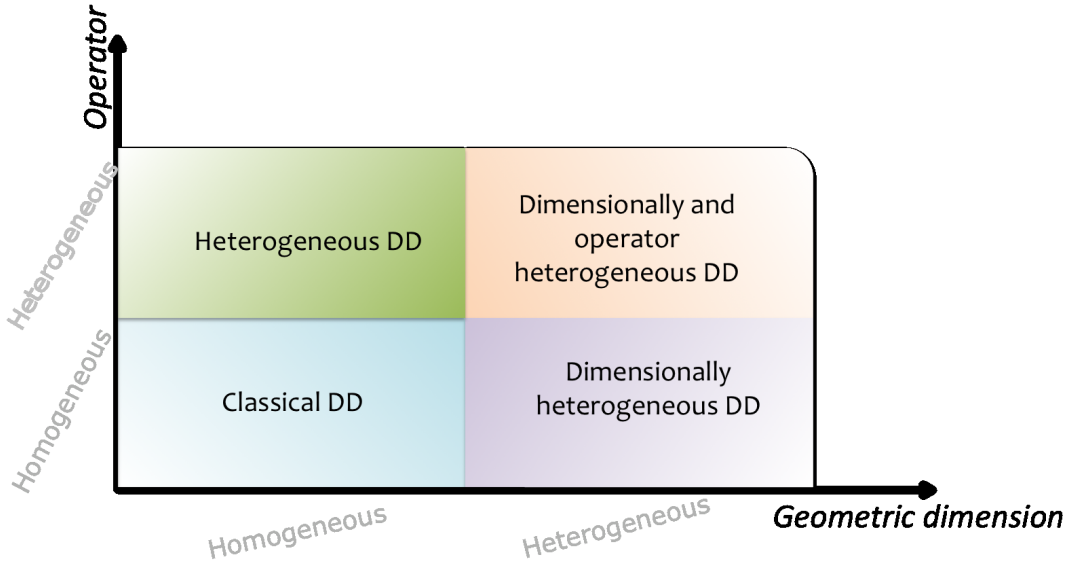
$\boldsymbol{\lambda}_1,\ \lambda_2$ virtual controls, solutions of the MINIMUM PROBLEM

$$\boxed{\inf_{\boldsymbol{\lambda}_1,\lambda_2} J(\boldsymbol{\lambda}_1, \lambda_2)}$$

A possible choice for J may be: $\boxed{J(\boldsymbol{\lambda}_1, \lambda_2) = \int_{\Omega_{fp}} (\mathrm{K}\nabla p_p - \mathbf{u}_f)^2}$

[Discacciati, Gervasio, Quarteroni (2010)]

OVERALL FRAMEWORK

Domain decomposition

With overlap	Without overlap
Schwarz methods	Steklov-Poincaré equation - *Schur complement system* - *DN/RR preconditioners*
Virtual controls - *boundary observation* - *distributed observation*	Virtual controls - *boundary observation*