What Should We Do with Radioactive Waste?

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Simulation of Flow in Porous Media and
Applications in Waste Management
and CO_2 Sequestration

Radon Special Semester 2011 on
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EPSRC Engineering and Physical Sciences Research Council

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Outline

- Radioactive waste disposal
- Treatment of uncertainty
- Case study - WIPP
- Conclusions

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- Other sources include: military, medical establishments, non-nuclear industries (such the oil industry) and educational and research establishments.
- Whatever your attitude to nuclear power, the existing waste has to be disposed of safely!

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- The HLW and ILW would stand approximately 17m deep on the pitch, with the HLW 10 cms deep.

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- Each country is responsible for dealing with its own waste (no export of waste)
- Minimal assumptions about future development of human society (e.g. can not assume there will be a cure for cancer or other diseases caused by exposure to radioactivity)
- Do not leave a legacy of hazardous and dangerous waste for future generations to manage

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  - Preferred solution of most countries

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- Objective is to make a good decision.
- NB: Not necessarily interested in the worst possible case.

Treatment of uncertainty

As we know, there are known knowns. There are things we know we know. We also know there are known unknowns. That is to say we know there are some things we do not know. But there are also unknown unknowns, the ones we don't know we don't know.

Donald Rumsfeld

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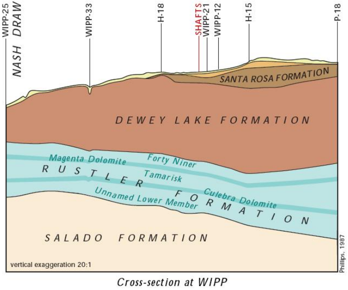
5. Stylised approach
   - eg. biosphere (ICRP and IAEA projects)

Case study - WIPP

- WIPP - Waste Isolation Pilot Plant.



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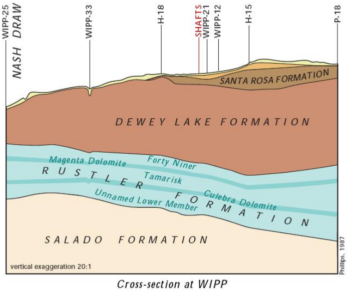

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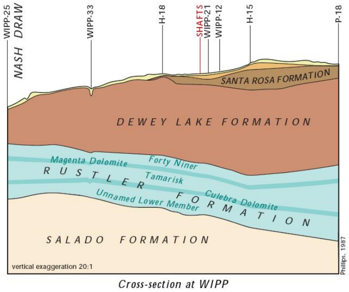

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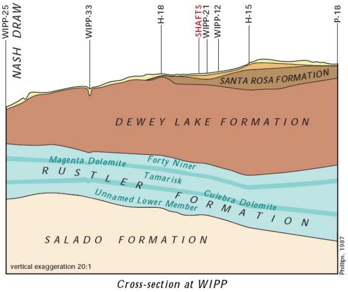

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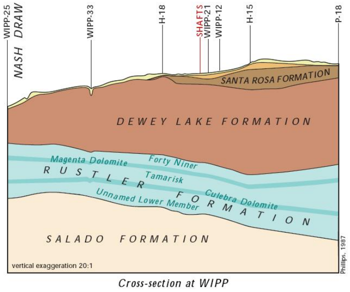
- Culebra would be the principal pathway for transport of radionuclides away from the repository in the event of an accidental breach.



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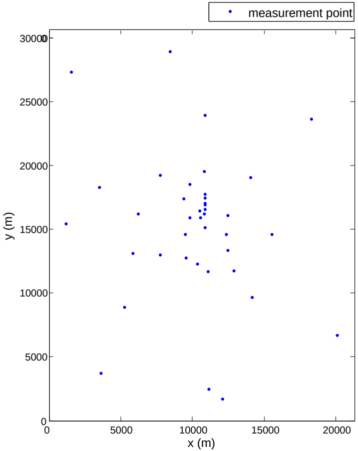

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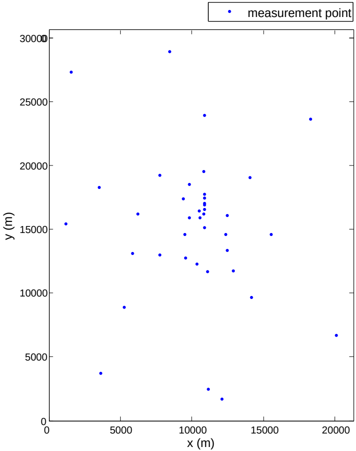

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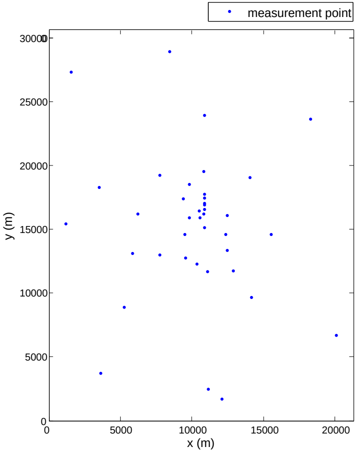

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- The time taken to travel from the repository to the boundary of the WIPP site is an important quantity.

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- Let $s_f = \log(t_f)$.

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- Interpretation of probability is important - a subjective Bayesian approach is used here.

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- Step 3 - post process - calculate the quantities of interest - travel times.
- Step 4 - interpret the results.

Stochastic model for WIPP

- Represent log transmissivity field, $Z = \log(T)$, as a Gaussian random field with mean and covariance:

$$\begin{aligned} \mathrm{E}[Z(\mathbf{x})] &= \beta_0 + \beta_1 x_1 + \beta_2 x_2 \\ \mathrm{Cov}[Z(\mathbf{x}), Z(\mathbf{x}')] &= \omega^2 C(\mathbf{x}, \mathbf{x}') = \omega^2 \exp\left(-\|\mathbf{x} - \mathbf{x}'\|/\lambda\right). \end{aligned}$$

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- The joint probability distribution, $f_\Theta(\theta)$, for the hyperparameters

$$\theta = (\beta_0, \beta_1, \beta_2, \omega^2, \lambda)$$

is estimated using a standard Bayesian analysis and MCMC to generate samples of θ drawn from $f_\Theta(\theta)$.

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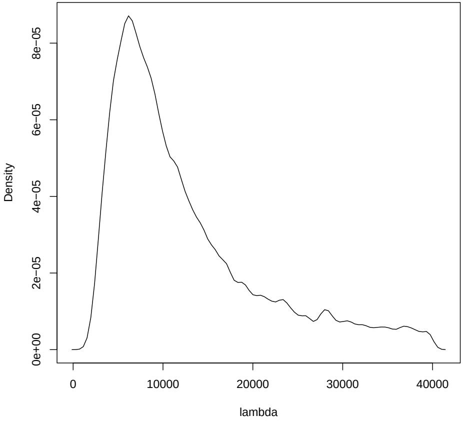

Figure: Probability density function for λ from MCMC calculation.

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- This uncertainty must be taken into account in the analysis.

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- Here $\mathrm{Cov}[Z(x), Z(y)]$ is the covariance function of the conditioned field.

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- Solve

$$\begin{aligned} \nabla \cdot [T(x,\xi)\nabla h(x,\xi)] &= 0 & x \in \mathfrak{D}, a.s. \\ h(x,\xi) &= h_0(x) & x \in \partial\mathfrak{D}, a.s. \end{aligned}$$

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- Compute the log of travel time, s_f, from h and T.
- Thus s_f depends on ξ and θ, and the problem is to compute the stochastic law of s_f from the joint law of ξ and θ.

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- Use Monte-Carlo to compute $F_{S_f}(s)$ as

$$F_{S_f}(s) = \frac{1}{N_\theta} \sum_{i=1}^{N_\theta} F_{S_f|\Theta}(s|\theta_i)$$

where $\{\theta_i\}_{i=1}^{N_\theta}$ is the MCMC sample.

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- Options include:
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  - Generalised polynomial chaos methods
  - Multi-level Monte-Carlo methods
- Many challenges remain in this area, with several promising lines of research.

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- Compute distribution function for s_f.

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Solution of the stochastic PDE

- How big should N be? Possibly too large for some current methods ..

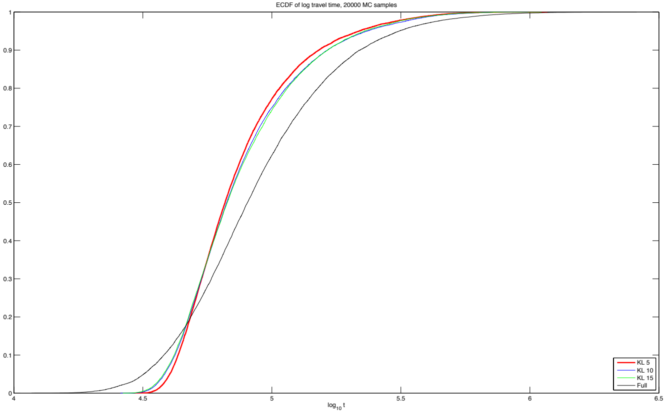

Figure: Distribution function for the log travel time for various N.

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- (Describing Gaussian process emulators would be another talk - think of them as a way of doing interpolation with scattered data in high dimensions - similar to RBF methods.)
- Then use the emulator to evaluate the sum

$$F_{S_f}(s) = \frac{1}{N_\theta} \sum_{i=1}^{N_\theta} \tilde{F}_{S_f|\Theta}(s|\theta_i)$$

to obtain the required solution.

WIPP Results

- For a given θ the stochastic model gives different realisations of T.

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(Loading Movie)

- Different realisations give different pathways and travel times.

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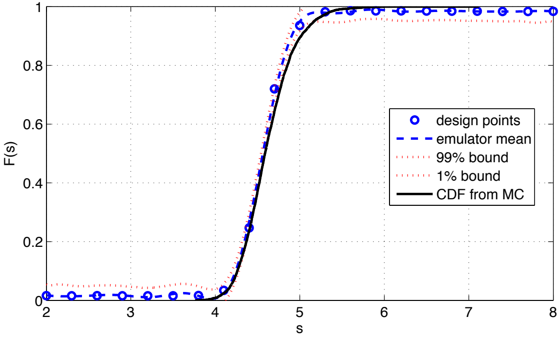

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- Uncertainties in the hyperparameters of the stochastic model for transmissivity can be dealt with in a Bayesian framework.
- The central computational/mathematical problem is solving elliptic PDEs with random coefficients.
- Exciting and important area of research with many promising approaches.