Research Kitchen on Coarse Spaces for Multiscale Problems

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RICAM Special Semester on
Multiscale Simulation & Analysis
in Energy and the Environment

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Model Problem

PDE with strongly heterogeneous coefficients (high contrast)

Examples:

- Darcy's equation: $p \in H^1_D(\Omega)$:

$$\int_\Omega \boldsymbol{\kappa} \nabla p \cdot \nabla v \, dx = f(v) \qquad \forall v \in H^1_D(\Omega)$$

- Linearized Elasticity: $\boldsymbol{u} \in H^1_D(\Omega)^3$:

$$\int_\Omega \boldsymbol{C} \, \varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}) \, dx \ = \ f(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in H^1_D(\Omega)^3$$

- Your favourite problem here

Fine Discretization: $\rightsquigarrow$ LARGE PROBLEM

- Continuous P^1 finite elements
- Non-conforming FE
- Finite Volume
- ...

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Two Different Goals

1. **Solving** fine scale equation

$$u_h \in V_h : \qquad A_h u_h = f_h$$

efficiently.

E.g. DD / Schwarz method / subspace correction / etc.

2. **Upscaling**: find

$$u_H \in V_H : \qquad A_H u_H = f_H$$

to approximate u_h by u_H

Central aims:

- $\|u_h - u_H\|_{??} \leq C(H) \|u_h\|_{??}$
- $\inf_{v_H \in V_H} \|u_h - v_H\|_{??} \leq C(H) \|u_h\|_{??}$

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For elliptic PDEs:

Carefully defined **coarse space** needed to be robust

- w.r.t. h
- w.r.t. heterogeneities

Typical requirement:

Coarse interpolant / projection $\Pi_H : V_h \to V_H$

- stability: $\|\Pi_H v\|_{??} \leq C\,\|v\|_{??}$
- approximation: $\|v - \Pi_H v\|_{??} \leq C\,\|v\|_{??}$

Tool for Darcy: **Weighted Poincaré Inequality (WPI)**:

$$\inf_{c \in \mathbb{R}} \int_D \alpha\,|v - c|^2\,dx \leq C_{P,\alpha}\,\mathrm{diam}\,(D)^2 \int_D \alpha\,|\nabla v|^2\,dx$$

$C_{P,\alpha}$ robust w.r.t. high contrast if α quasi-monotone on D

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