Market environments, stability and equlibria

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A Toy Model

The Information Flow

- two states of the world: $\Omega = \{\omega_1, \omega_2\}$
- one period $t \in \{0, 1\}$
- nothing is known at $t = 0$, everything is known at $t = 1$: $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = 2^{\Omega}$.

Agents

two economic agents characterized by

- random endowments (stochastic income)

$$\mathcal{E}^1 = \begin{Bmatrix} 3 \\ 1 \end{Bmatrix}, \mathcal{E}^2 = \begin{Bmatrix} 1 \\ 4 \end{Bmatrix}$$

- utility functions

$$\mathbb{U}^1(\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}) = \tfrac{1}{2}\log(x_1) + \tfrac{1}{2}\log(x_2)$$

$$\mathbb{U}^2(\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}) = \tfrac{1}{7}x_1^{1/3} + \tfrac{6}{7}x_2^{1/3}$$

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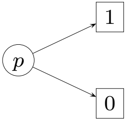
A TOY EXAMPLE

THE FINANCIAL INSTRUMENT

$$S_0 = p,\ S_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix},\ B_0 = B_1 = 1:$$



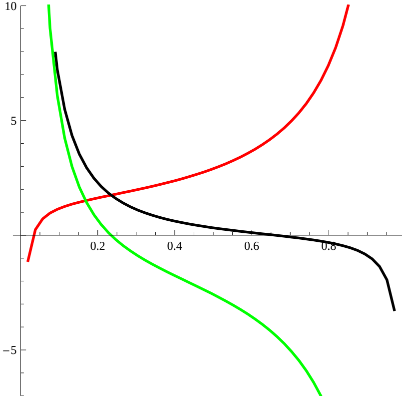
MARKET CLEARING

- The demand functions:

$$\Delta^i(p) = \underset{q}{\operatorname{argmax}}\, \mathbb{U}^i(\mathcal{E}^i + q(S_1 - p))$$

- Equilibrium conditions:

$$\Delta^1(p) + \Delta^2(p) = 0$$



A toy example

The financial instrument

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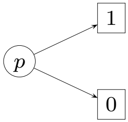

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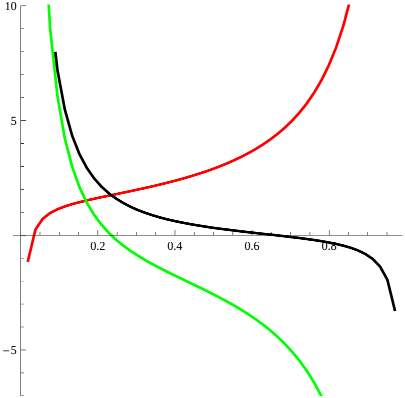
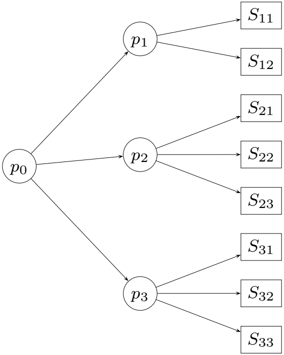
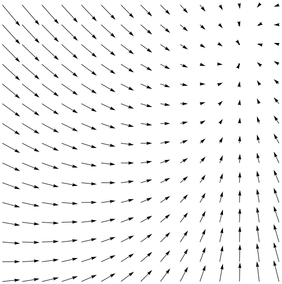


What happens when markets are incomplete and trading is dynamic?



- In the IC&mp case, the equilibrium conditions determine both prices **and** the geometry (degree of incompleteness) of the market.
- Another complication : no representative-agent analysis. The *First Welfare Theorem* does not hold anymore.



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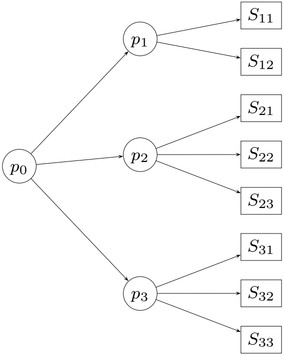

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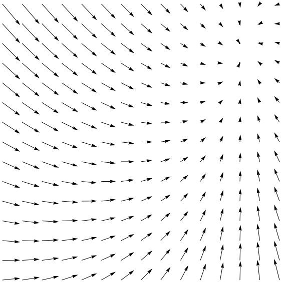

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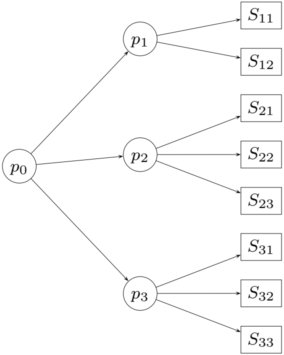

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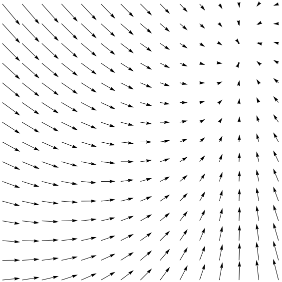

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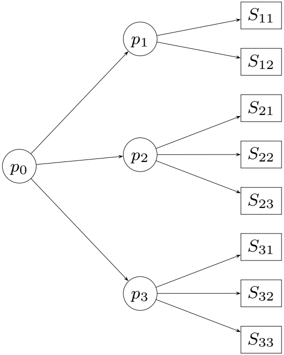

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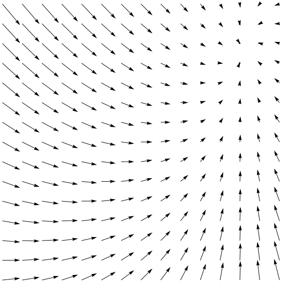

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Information

A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, where $\mathbb{P}$ is used only to determine the null-sets.

Agents

A number I (finite or infinite) of economic agents, each of which is characterized by

- a random endowment $\mathcal{E}^i \in \mathcal{F}_T$,
- a utility function $U : \mathrm{Dom}(U) \to \mathbb{R}$,
- a subjective probability measure $\mathbb{P}^i \sim \mathbb{P}$.

Completeness constraints

A set $\mathcal{S}$ of $(\mathcal{F}_t)_{t\in[0,T]}$-semimartingales (possibly several-dimensional) - the allowed asset-price dynamics.

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Does there exist $S \in \mathcal{S}$ such that

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where

$$\hat{\pi}^i(S) = \operatorname*{argmax}_{\pi} \mathbb{E}^{\mathbb{P}^i}[U^i(\mathcal{E}^i + \int_0^T \pi_u \, dS_u)]$$

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Examples of Completeness Constraints

- **Complete markets.** $\mathcal{S}$ contains all $(\mathcal{F}_t)_{t\in[0,T]}$-semimartingales. (If an equilibrium exists, a complete one will exist).
- **Constraints on the number of assets.** $\mathcal{S}$ is the set of all d-dimensional $(\mathcal{F}_t)_{t\in[0,T]}$-semimartingales. If $d < n$, where n is the spanning number of the filtration, no complete markets are allowed.
- **Information-constrained markets.** Let $(\mathcal{G}_t)_{t\in[0,T]}$ be a sub-filtration of $(\mathcal{F}_t)_{t\in[0,T]}$, and let $\mathcal{S}$ be the class of all $(\mathcal{G}_t)_{t\in[0,T]}$-semimartingales.
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Examples of Completenes Constraints

- **Markets with "fast-and-slow" information.** Let $(\mathcal{F}_t)_{t\in[0,T]}$ be generated by two orthogonal martingales M^1 and M^2, and let $\mathcal{S}$ be the collection of all processes of the form

  $$S_t = D_t + M^1_t,$$

  where D is any predictable process of finite variation. For example, $M^1 = B$ (Brownian motion), $M^2 = N_t - t$ (compensated Poisson process) so that a "typical" element of $\mathcal{S}$ is given by

  $$S_t = \int \lambda(u, B_u, N_u)\, du + dB_u.$$

  The information in B is "fast", and that in N is "slow".

  Another interesting situation: $M^1 = B$, $M^2 = W$, where B and W independent Brownian motions.

Two paths to existence

- **Representative agents.** Uses the fact that equilibrium allocations are Pareto optimal; works (essentially) only for complete markets.

  Literature in continuous time:

  - **Complete markets:** Bank, Dana, Duffie, Huang, Karatzas, Lakner, Lehoczky, Riedel, Shreve, Ž., etc.
  - **Incomplete markets:** Basak and Cuoco *'98* (incompleteness from restrictions in stock-market participation, logarithmic utility)

- **Excess-demand approach.** Introduced by Walras (1874):

  1. Establish good topological/convexity properties of the excess demand $\hat{\pi}(S)$, and then
  2. use a suitable fixed-point-type theorem to show existence (Brouwer, KKM, degree-based, etc.)

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A convex-analytic (sub)approach

A first step towards a solution

Work with random variables instead of processes; for example in the fast-and-slow model with

$$dS_u^\lambda = \lambda_u \, du + dB_u,$$

we perform the following transformations

$$\pi \mapsto X_T^{\lambda,\pi} = \int_0^T \pi_u \, dS_u^\lambda, \ \lambda \mapsto Z_T^\lambda = \mathcal{E}(-\lambda \cdot M),$$

and consider a more tractable version Δ^i of the demand function

$$\Delta^i(Z_T^\lambda) = X_T^{\lambda,\hat{\pi}^i(S^\lambda)},$$

so that

$$\Delta^i : E_M \subseteq \mathbb{L}_+^0 \to \mathbb{L}_+^0 - \mathbb{L}_+^\infty.$$

The problem now becomes simple to state:

Can we solve the equation $\Delta(Z) = 0$, a.s. on E_M?

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Stability of utility maximization in incomplete markets

(**Note:** fix an agent and drop the index i.)

Theorem (Larsen and Ž. (2006), to appear in SPA)
Suppose that $\mathcal{E} \equiv 0$. Let $\{\lambda_n\}_{n \in \mathbb{N}} \subseteq \Lambda$ be a sequence such that

- Z^{λ_n} is a martingale for each n,
- the collection $\{V^+(Z_T^{\lambda_n}) : n \in \mathbb{N}\}$ is uniformly integrable, and
- $Z_T^{\lambda_n} \to Z_T^{\lambda}$ in probability.

Then, for $x_n \to x > 0$ we have

$$u^{\lambda_n}(x_n) \to u^{\lambda}(x), \text{ and } \hat{X}_T^{\lambda_n, x_n} \to \hat{X}_T^{\lambda, x} \text{ in probability.}$$

Here, V is the convex conjugate of the utility function U, i.e., $V(y) = \sup_{x>0}(U(x) - xy)$, and $\hat{X}_T^{x,\lambda}$ is the optimal terminal wealth in the market S^{λ} with initial wealth x.

Remarks:

- The uniform-integrability condition is practically necessary.
- Completes the Hadamard-style analysis of the utility maximization problem - repercussions for estimation.
- Further generalized to the general semimartingale case - under a different perturbation family - including general $\mathcal{E} \in \mathbb{L}^{\infty}$ (Kardaras and Ž. (2007)).
- Therefore, (under suitable conditions) Δ is $(\mathbb{L}^0, \mathbb{L}^0)$-continuous.

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Theorem (Larsen and Ž. (2006), to appear in SPA)
Suppose that $\mathcal{E} \equiv 0$. Let $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq \Lambda$ be a sequence such that

- Z^{λ_n} is a martingale for each n,
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- $Z_T^{\lambda_n} \to Z_T^{\lambda}$ in probability.

Then, for $x_n \to x > 0$ we have

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Here, V is the convex conjugate of the utility function U, i.e., $V(y) = \sup_{x>0}(U(x) - xy)$, and $\hat{X}_T^{x,\lambda}$ is the optimal terminal wealth in the market S^{λ} with initial wealth x.

Remarks:

- The uniform-integrability condition is practically necessary.
- Completes the Hadamard-style analysis of the utility maximization problem - repercussions for estimation.
- Further generalized to the general semimartingale case - under a different perturbation family - including general $\mathcal{E} \in \mathbb{L}^{\infty}$ (Kardaras and Ž. (2007)).
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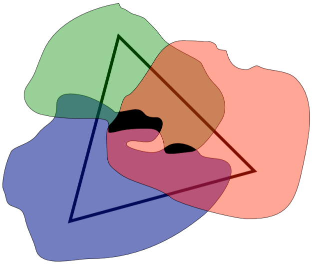
The KKM-theorem

Theorem (Knaster, Kuratowski and Mazurkiewicz, 1929) Let S be the unit simplex in $\mathbb{R}^m$, and let $V = \{e_1, \dots, e_m\}$ be the set of its vertices. A mapping $F : V \to 2^{\mathbb{R}^m}$ is said to be a *KKM-map* if

$$\mathrm{conv}(e_i, i \in J) \subseteq \cup_{i\in J} F(e_i), \ \forall J \subseteq \{1, \dots, m\}.$$

If $F(e_i)$ is a closed subset of $\mathbb{R}^m$ for all $i \in \{1, \dots, m\}$, then

$$\cap_{i\in\{1,\dots,n\}} F(e_i) \neq \emptyset.$$



Convex compactness

The KKM-Theorem can easily be extended to infinite-dimensional vector-spaces as long as mild topological properties are imposed and **local convexity** is required (Kakutani, Fan, Browder, etc.)

How about $\mathbb{L}^0$ - the prime example of a non-locally-convex space? Yes, if one can fake compactness there:

Convex-compactness
(Nikišin, Buhvalov, Lozanovskii, Delbaen, Schahermayer, etc.)

A subset B of a topological vector space is said to be **convex-compact** if any family $(F_\alpha)_{\alpha\in A}$ of closed and convex subsets of B has the finite-intersection property, i.e.

$$\left(\forall D \subseteq_{fin} A \quad \bigcap_{\alpha\in D} F_\alpha \neq 0\right) \Rightarrow \bigcap_{\alpha\in A} F_\alpha \neq \emptyset.$$

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A characterization

Proposition. A closed and convex subset C of a topological vector space X is convex-compact if and only if for any net $(x_\alpha)_{\alpha\in A}$ in C there exists a subnet $(y_\beta)_{\beta\in B}$ of convex combinations of $(x_\alpha)_{\alpha\in A}$ such that $y_\beta \to y$ for some $y \in C$.

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Examples.

- Any convex and compact subset of a TVS is convex-compact.
- A closed and convex subset of a unit ball in a dual X^* of a normed vector space X is convex-compact under any compatible topology (essentially *Mazur*),
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Attainment of minima

Theorem. Let A be a convex-compact subset of X, and let $f : A \to \mathbb{R}$ be a convex lower-semicontinuous function. Then f attains its minimum on A.

A minimax-type theorem

Theorem. Let A, B be a convex-compact subsets of TVS X and Y, respectively. Let $f : A \times B \to \mathbb{R}$ be a function with the following properties:

- $x \mapsto f(x, y)$ is usc and (quasi)-concave for each $y \in B$,
- $y \mapsto f(x, y)$ is lsc and (quasi)-convex for each $x \in A$.

Then

$$\max_x \min_y f(x, y) = \min_y \max_x f(x, y).$$

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Generalized KKM theorem

Theorem. Let A be convex-compact subset of a TVS X. Let $\{F(x)\}_{x \in A}$ be a family of closed and convex subsets of A such that

$$\mathrm{conv}(x_1, \dots, x_n) \subseteq \cup_{i=1}^{n} F(x_i), \ \forall\, n \in \mathbb{N}, \ \forall\, x_1, \dots, x_n \in A.$$

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$$\cap_{x \in A} F(x) \neq \emptyset.$$

The state of affairs

Using the generalized KKM theorem, we can show existence of equilibria in many cases of some interest (it works for an infinity of agents, too).

The requirement of (quasi)-convexity it places on the excess-demand function is a serious one. We are trying to sort the situation out (work in progress with Malamud, Anthropelos) ...

Kardaras ('08) uses convex-compactness to give a general abstract framework for existence of numéraire portfolios.

The direct (sub)approach

Let us consider the fast-and-slow model with Let $(\mathcal{F}_t)_{t\in[0,T]}$ be generated by a Brownian motion B and a one-jump-Poisson process N with intensity $\mu > 0$. We let $\mathcal{S}$ be the collection of all processes of the form

$$S_t = \int_0^t \lambda(u, B_u, N_u)\, du + dB_t,$$

where $\lambda : [0,T] \times \mathbb{R} \times \{0,1\} \to \mathbb{R}$ ranges through bounded measurable functions.

- There is a finite number I of agents,
- each agent has the exponential utility $U^i(x) = -\exp(-\gamma_i x)$,
- the random endowments are of the form $\mathcal{E}^i = g^i(B_T, N_T)$.

Theorem. Under the assumption that $g^i \in C_{2+\delta}(\mathbb{R})$, $i \in I$, $\delta \in (0,1)$, there exists $T_0 > 0$ such that an equilibrium market, unique in the class $C_{2+\delta,1+\delta/2}([0,T]\times\mathbb{R})$, exists whenever $T \le T_0$.

Theorem* The restriction $T < T_0$ is superfluous.

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Sketch of the proof

- Express the optimal portfolio in the form

$$\pi_t^i = \frac{1}{\gamma_i}\lambda(t, B_T, N_t) - u_b^i(t, B_t, N_t),$$

where solves the semi-linear system of two interacting PDEs

$$\begin{cases} 0 = u_t^i + \frac{1}{2}u_{bb}^i - \lambda u_b^i + \frac{1}{2\gamma_i}\lambda^2 - \frac{\mu}{\gamma}(\exp(-\gamma u_n^i) - 1) \\ u^i(T, \cdot, \cdot) = g^i. \end{cases}$$

where $u_n^i(t, b, 0) = u^i(t, b, 1) - u^i(t, b, 0)$, $u_n^i(t, b, 1) = 0$.

- Write the market-clearing condition

$$0 = \sum_{i=1}^{I} \hat{\pi}_t^i(\lambda) = \frac{1}{\bar{\gamma}}\lambda - \sum_{i=1}^{I} u_b^i(\lambda),$$

in the form

$$F(\lambda) = \sum_{i=1}^{I} \bar{\gamma} u_b^i(\lambda) = \lambda.$$

where $\frac{1}{\bar{\gamma}} = \sum_{i=1}^{I} \frac{1}{\gamma_i}$.

Sketch of the proof

- Express the optimal portfolio in the form

$$\pi_t^i = \frac{1}{\gamma_i}\lambda(t, B_T, N_t) - u_b^i(t, B_t, N_t),$$

where solves the semi-linear system of two interacting PDEs

$$\begin{cases} 0 = u_t^i + \frac{1}{2}u_{bb}^i - \lambda u_b^i + \frac{1}{2\gamma_i}\lambda^2 - \frac{\mu}{\gamma}(\exp(-\gamma u_n^i) - 1) \\ u^i(T, \cdot, \cdot) = g^i. \end{cases}$$

where $u_n^i(t, b, 0) = u^i(t, b, 1) - u^i(t, b, 0)$, $u_n^i(t, b, 1) = 0$.

- Write the market-clearing condition

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Sketch of the proof

- Show that the mapping
$$\lambda \mapsto u_b^i(\lambda)$$
is Lipschitz with a small Lipschitz coefficient in a well-chosen function space. The right one turns out to be the weighted Hölder space $C_{(\beta);1+\delta}([0,T] \times \mathbb{R})$.
- Apply the Banach fixed-point theorem to the function F.

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What next

Some research directions:

- **(Aumann models)** let the number of agents $\to \infty$, and study the limiting behavior (mean-field-type ideas)
- **(Alternative sources of incompleteness)** jumps, transactions costs, default, etc.
- **(Numerical methods)** forward-backward SDEs, iterative approaches
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