Dual Formulation of Second order Target Problems

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Outline

1. Introduction: the Cetin-Jarrow-Protter liquidity model
2. A Reference Dominating Measure
3. Second Order Stochastic Target Problems
Price of an order depends on volumes $S_t(\omega, \nu)$. $S$ may be estimated from orders book:

<table>
<thead>
<tr>
<th>Quantity</th>
<th>10</th>
<th>35</th>
<th>20</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>110</td>
<td>112</td>
<td>117</td>
<td>125</td>
</tr>
</tbody>
</table>

Note that the price by share is non-decreasing. But there is no influence of a large trade on the next moment orders book... (Çetin-Jarrow-Protter ’06, Rogers-Singh ’05)

This includes Proportional Transaction Costs models

$$S_t(\nu) = (1 + \lambda)S_t(0)\mathbb{1}_{\mathbb{R}^+}(\nu) + (1 - \mu)S_t(0)\mathbb{1}_{\mathbb{R}^-}(\nu)$$
The discrete-time model (Çetin, Jarrow and Protter 2004, 2006)

Risky asset price is defined by

- the marginal price \( S_t, t \geq 0 \)
- the supply curve \( \nu \mapsto S(., \nu) \):
  \[
  S(S_t, \nu) \quad \text{price per share of } \nu \text{ risky assets}
  \]
  with \( S(s, 0) = s \)

\( Z^0_t \) : holdings in cash, \( Z_t \) : holdings in risky asset

\[
Z^0_{t+dt} - Z^0_t + (Z_{t+dt} - Z_t) S(S_t, Z_{t+dt} - Z_t) = 0
\]

\[
\implies Z^0_T = Z^0_0 - \sum (Z_{t+dt} - Z_t) S(S_t, Z_{t+dt} - Z_t)
\]

\[
= Z^0_0 + \sum Z_t (S_{t+dt} - S_t) + \ldots
\]
Continuous-time formulation of Model

Set \( Y_t := Z_t^0 + Z_t S_t \), then:

\[
Y_T = Y_0 + \sum Z_t \left( S_{t+dt} - S_t \right) - \sum \left( Z_{t+dt} - Z_t \right) \left[ \mathbb{S} \left( S_t, Z_{t+dt} - Z_t \right) - \mathbb{S} \left( S_t, Z_{t+dt} - Z_t \right) \right]
\]

Assume \( \nu \mapsto \mathbb{S}(., \nu) \) is smooth (unlike proportional transaction costs models), then:

\[
Y_T = Y_0 + \int_0^T Z_t dS_t - \int_0^T \frac{\partial \mathbb{S}}{\partial \nu} \left( S_t, 0 \right) d\langle Z^c \rangle_t - \sum_{t \leq T} \Delta Z_t \left[ \mathbb{S} \left( S_t, \Delta Z_t \right) - S_t \right]
\]

- \( d\langle Z^c \rangle_t = \Gamma_t^2 d\langle Z^c \rangle_t \) : the so-called Gamma...
The Hedging Problem

Option / contingent claim: \( g(S_T) \), where

\[ g : \mathbb{R}_+ \rightarrow \mathbb{R} \] has linear growth

Super-hedging problem

\[ V := \inf \left\{ y : Y^{y,Z}_T \geq g(S_T) \quad \mathbb{P} - \text{a.s. for some "admissible" } Z \right\} \]

- For this formulation to be consistent with the financial problem, we assume there is no liquidity cost at maturity \( T \)
- Here, admissibility is the crucial issue
- Non-Markov case: with new results, should be possible...
Without further restrictions on trading strategies, the problem reduces to Black-Scholes! Reason for this result is the following result of Bank-Baum 04

**Lemma** For predictable $W-$integ. càdlàg process $\phi$, and $\varepsilon > 0$

$$\sup_{0 \leq t \leq 1} \left| \int_0^t \phi_r dW_r - \int_0^t \phi_r^\varepsilon dW_r \right| \leq \varepsilon$$

for some a.c. predictable process $\phi_t^\varepsilon = \phi_0^\varepsilon + \int_0^t \alpha_r dr$

$\implies$ If the "admissibility" set allows for arbitrary a.c. portfolio $Z_t = Z_0 + \int_0^t \alpha_u du$, then $V = V^{\text{BS}}$ (with $\Gamma = 0$!)

**The Çetin-Jarrow-Protter Negative Result**

Introduction : the Cetin-Jarrow-Protter liquidity model
A Reference Dominating Measure
Second Order Stochastic Target Problems
A Convenient Set of Admissible Strategies

We show that liquidity cost does affect $V$, perfect replication is possible, and hedging strategy can be described (formally)

Definition: $Z \in \mathcal{A}$ if it is of the form

$$Z_t = \sum_{n=0}^{N-1} z_n \mathbb{I}_{\{t<\tau_{n+1}\}} + \int_0^t \alpha_u du + \int_0^t \Gamma_u dS_u$$

- $(\tau_n)$ is an ↗ seq. of stop. times, $z_n$ are $\mathcal{F}_{\tau_n}$-measurable, $\|N\|_\infty < \infty$
- $Z$ and $\Gamma$ are $\mathbb{L}_\infty$-bounded up to some polynomial of $S$
- $\Gamma_t = \Gamma_0 + \int_0^t a_u du + \int_0^t \xi_u dW_u$, $0 \leq t \leq T$, and

$$\|\alpha\|_{B,b} + \|a\|_{B,b} + \|\xi\|_{B,2} < \infty, \quad \|\phi\|_{B,b} := \sup_{0\leq t\leq T} \left\| \frac{|\phi_r|}{1 + S_t^B} \right\|_{\mathbb{L}_b}$$
PDE characterization

Let

\[ \ell(s) := \left[ 4 \frac{\partial S}{\partial \nu}(s, 0) \right]^{-1} \]

**Theorem** Let \(-C \leq g(.) \leq C(1 + .)\) for some \(C > 0\). Then \(V(t, s)\) is the unique continuous viscosity solution of the dynamic programming equation

\[ -V_t(t, s) + \frac{1}{4} s^2 \sigma(t, s)^2 \ell(s) \left[ 1 - \left( \frac{V_{ss}(t, s)}{\ell(s)} + 1 \right)^2 \right] = 0 \]

with \(V(T, s) = g(s)\) and \(-C \leq V(t, s) \leq C(1 + s)\) for every \((t, s)\).

- Notice that there is no boundary layer \(\implies\) perfect hedge
Hedging a Convex Payoff in the Frictionless BS Model

For a convex payoff: only possibility to super-hedge is the Black-Scholes perfect replication strategy.
Hedging a Concave Payoff in the Frictionless BS Model

For a concave payoff: two possibilities to super-hedge
- Black-Scholes perfect replication $\implies \Gamma \neq 0$ so pay liquidity cost
- Buy-and-hold $\implies \Gamma = 0$ no liquidity cost, but hedge might be too expensive
Hedging a Concave Payoff in the Frictionless BS Model
Formal Description of a Hedging Strategy

- $v_{ss} < -\ell(s)$: Then the PDE satisfied by $V$ reduces to

$$-V_t(t, s) + \frac{1}{4}s^2\sigma(t, s)^2\ell(s) = 0 \text{ (degenerate!)}$$

buy-and-hold strategy is more interesting because liquidity cost is too expensive

- $v_{ss} \geq -\ell(s)$: Then the PDE satisfied by $V$ reduces to

$$-V_t(t, s) - \frac{1}{2}s^2\sigma(t, s)^2V_{ss} - \frac{s^2\sigma(t, s)^2}{4\ell(s)}V_{ss}^2 = 0$$

perfect replication
The Technical Difficulty

How to switch from BS to B\&H

\( V_{ss} = -l \)

The General Case

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Dual Formulation of Second order Target Problems
A New Formulation : Intuition

Recall the state dynamics in Stratonovitch form :

\[ dY_t = Z_t \circ dS_t - \left( \frac{1}{2} \Gamma_t + S_{\nu}(S_t, 0) \right) \sigma_t^2 S_t^2 dt \]

and the corresponding "natural" PDE :

\[ \frac{\partial V}{\partial t} = - \left( \frac{1}{2} V_{ss} + S_{\nu}(s, 0) \right) \sigma^2 s^2 \]

**Main observation** : We would obtain the same PDE if the volatility of \( S \) is modified :

\[ dY_t = Z_t \circ dS'_t - \left( \frac{1}{2} \Gamma_t + S_{\nu}(S'_t, 0) \right) \sigma'_t S'_t^2 dt \]

\[ dS'_t = \sigma'_t S'_t dW_t \]
A New Formulation: Relax Controls and Change Volatility (Intuition from L. Denis and C. Martini)

Consider the super-hedging problem:

\[ \hat{V} := \inf \left\{ y : Y_T \geq g(S_T) \, \hat{\mathbb{P}} - \text{a.s. for some } Z \in \mathcal{SM}^2 \right\} \]

where

\[ dY_t = Z_t \circ dS_t - \left( \frac{1}{2} \Gamma_t + S_\nu(S_t, 0) \Gamma_t^2 \right) \sigma^2_t S_t^2 dt \]

Compare with

\[ V := \inf \left\{ y : Y_T \geq g(S_T) \, \mathbb{P} - \text{a.s. for some } Z \in \mathcal{A} \right\} \]

Then, \( \hat{V} = V \)
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A Dense Subset of Scales

$C([0, T])^d$ Canonical space, $B$ canonical process, $\mathbb{P}$ Wiener measure

$\mathcal{M} := \{ \mathbb{P}': \text{Prob. meas. s.t. } \mathbb{P}'[B_0 = 0] = 1 \text{ and } B \in \mathcal{M}^2(\mathbb{P}') \}$

$U \subset S_d^+$ given, $U_0$ dense subset of $U$, $T_0$ dense subset of $[0, 1]$

$U_0$, simple functions: $n \in \mathbb{N}$, $0 = t_0 < \ldots < t_n = 1$, $\varepsilon > 0$,

$$a(t) = \sum_{i=1}^{n} \alpha_i \mathbb{1}[t_{i-1}, t_i), \quad \alpha_i \in U_0 \cap [\varepsilon l_d, \varepsilon^{-1} l_d], \quad t_i \in T_0$$

$U_0 = \{ a_i, i \geq 1 \}$ countable with $\sum_{i \geq 1} 2^{-i} \int_{0}^{1} |a_i(t)| dt < \infty$

$\tilde{U}_0$, simple processes: above $a_i$’s $\in L^2(\mathbb{P}, \mathcal{F}_{t_i})$

$\tilde{U} = H^2(\mathbb{P}, \mathcal{F}, U)$
For $a \in \bar{U}_0 : \mathcal{M} \ni P^a = \text{distribution of the process } \int_0^t \sqrt{a(t)} dB_t$

• since $\sum_{i \geq 1} 2^{-i} \int_0^1 |a^i(t)| dt < \infty$, the reference measure

\[ \hat{P} = \hat{P}^{U_0,T_0} := \sum_{i \geq 1} 2^{-i} P^{a_i} \in \mathcal{M} \]

• For every $i \geq 1$, $P^{a_i} \ll \hat{P}$

• For every $a \in \bar{U}_0$, $P^a \ll \hat{P}$

• But for arbitrary $a \in \bar{U}$, $P^a \not\ll \hat{P}$

Our result will however not depend on the choice of $(U_0, T_0)$
Properties of $\hat{\mathbb{P}}$

- $\hat{\mathbb{P}} \in \mathcal{M}$, and $d\langle B \rangle_t = a_i(t)dt \ \mathbb{P}^a_i - \text{a.s.}$

**Aggregation** Let $X^i \in \mathbb{H}^0(\mathbb{P}^a_i)$ be a family of processes such that

$$X^i_s = X^j_s, \quad s \leq t \quad \text{whenever} \quad a_i = a_j \text{ on } [0, t]$$

Then there is a unique process $X \in \mathbb{H}^0(\hat{\mathbb{P}})$ such that

$$X = X^i \quad dt \times d\mathbb{P}^a_i - \text{a.s.}$$

$$\implies d\langle B \rangle_t = \hat{a}_t dt \hat{\mathbb{P}} - \text{a.s.}$$
Problems with Null Sets

- Our analysis requires to define objects $\hat{P}$-a.s. and then to look at their decompositions under $\mathbb{P}^{a_i}$ for every $i \geq 1$
- Standard stochastic analysis results are stated under the assumption that the filtration satisfies the usual conditions...
- Let $\mathbb{F}^{\hat{P}}$ be the filtration completed by $\hat{P}$—null sets, then $\mathbb{F}^{\hat{P}}$ is not complete for $\mathbb{P}^{a_i}$!

$\mathbb{F}^+—\text{adapted modification}$ For any $Z \in \mathbb{H}^0(\mathbb{P}', \mathbb{F}^{'})$, there is a unique $\tilde{Z} \in \mathbb{H}^0(\mathbb{P}', \mathbb{F}^+)$ such that $Z$ and $\tilde{Z}$ are $\mathbb{P}'$—modifications (i.e. $Z = \tilde{Z} \ dt \times d\mathbb{P}'$—a.s.)

$\implies$ Always consider $\mathbb{F}^+—\text{adapted versions}$
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Operators and Spaces

- \( H_t(y, z, \gamma) \)

\[
H : \Omega \times [0, 1] \times \mathbb{R} \times \mathbb{S}_d \longrightarrow \mathbb{R}, \quad \mathcal{P} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{S}_d) - \text{meas.}
\]

- \( F_t(y, z, a) : \) conjugate wrt \( \gamma \)

\[
F_t(y, z, a) := \sup_{\gamma \in \mathbb{S}_d} \left\{ \frac{1}{2} \text{Tr}[a\gamma] - H_t(y, z, \gamma) \right\}
\]

**Assumption**  \( H \) uniformly Lipschitz in \((y, z)\), has linear growth wrt \( \gamma \), \( H(0, 0, 0) \in \mathbb{H}^2(\mathbb{P}^a) \) for every \( a \in \mathbb{H}^2(\mathbb{S}_d^+) \), and there is a subset \( U \subset \mathbb{S}_d^+ \) such that \( U \subset \text{dom}(F_t), \ t \leq 1 \)

Consider the reference measure \( \hat{\mathbb{P}} := \hat{\mathbb{P}}_{U_0, T_0} \), and let

\[
\hat{\mathbb{H}}^2 := \bigcap_{i \geq 1} \mathbb{H}^2(\mathbb{P}^{a_i}, \mathbb{F}^+), \quad \hat{\mathbb{S}}\mathbb{M}^2 := \bigcap_{i \geq 1} \mathbb{S}\mathbb{M}^2(\mathbb{P}^{a_i}, \mathbb{F}^+)
\]
• For $Z \in S\hat{M}^2$, define the controlled state:

$$dY_t = -H_t(Y_t, Z_t, \Gamma_t)dt + Z_t \circ dB_t, \quad \hat{P} - \text{a.s.}$$

where the process $\Gamma$ is defined by

$$d\langle Z, B \rangle_t = \Gamma_t d\langle B \rangle_t \quad \hat{P} - \text{a.s.}$$

• The target problem is:

$$\hat{V} := \inf \left\{ y : Y_1 \geq \xi \hat{P} - \text{a.s. for some } Z \in S\hat{M}^2 \right\}$$
First Relaxation of Second Order Target Problems

Relax the connection between $Z$ and $\Gamma$

- For $Z, G \in \hat{H}^2$, define the controlled state:

\[
\begin{align*}
\frac{dY_0^t}{dt} &= -H_t(Y_0^t, Z_t, G_t)dt + Z_t \circ dB_t, \quad \hat{\mathbb{P}} - \text{a.s.} \\
&= \frac{1}{2} \text{Tr}[G_t d\langle B \rangle_t] - H_t(Y_t, Z_t, G_t)dt + Z_t \cdot dB_t, \quad \hat{\mathbb{P}} - \text{a.s.} \\
&= \left( \frac{1}{2} \text{Tr}[G_t \hat{a}_t] - H_t(Y_t, Z_t, G_t) \right) dt + Z_t \cdot dB_t, \quad \hat{\mathbb{P}} - \text{a.s.}
\end{align*}
\]

where the process $d\langle B \rangle_t = \hat{a}_t dt \hat{\mathbb{P}} - \text{a.s.}$

- The relaxed problem is:

\[
\hat{\mathcal{V}}_0 := \inf \left\{ y : Y_1^0 \geq \xi, \hat{\mathbb{P}} - \text{a.s. for some } Z, G \in \hat{H}^2 \right\}
\]
First Relaxation of Second Order Target Problems

Clearly: \( \hat{V} \geq \hat{V}^0 \)

**Proposition** \( \hat{V} = \hat{V}^0 \)

Based on extension of Bank and Baum to the nonlinear case: Let \( (Y, Z) \) be such that

\[
Y_t = y + \int_0^t h_s(Y_s, Z_s, \hat{a}_s) ds + \int_0^t Z_s dB_s, \quad \hat{P} - \text{a.s.}
\]

Then, \( \forall \, \varepsilon > 0 \), there is a continuous FV process \( Z^\varepsilon \) such that

\[
Y_t^\varepsilon = y + \int_0^t h_s(Y_s^\varepsilon, Z_s^\varepsilon, \hat{a}_s) ds + \int_0^t Z_s^\varepsilon dB_s, \quad \hat{P} - \text{a.s.}
\]

and

\[
\sup_{0 \leq t \leq 1} |Y_t - Y_t^\varepsilon| \leq \varepsilon \quad \hat{P} - \text{a.s.}
\]
Further Relaxation of Second Order Target Problems

Second relaxation: forget $\Gamma$!

- Recall the (partial) convex conjugate of $H$:

$$
F_t(y, z, a) := \sup_{\gamma \in S^d} \left\{ \frac{1}{2} \Tr[a\gamma] - H_t(y, z, \gamma) \right\}, \quad a \in S^+_

- For $Z \in \mathcal{H}^2(\hat{P})$, define the controlled state:

$$
dY^1_t = F_t(Y^1_t, Z_t, \hat{a}_t)dt + Z_t \cdot dB_t, \quad \hat{P} - \text{a.s.}
$$

where the process $d\langle B \rangle_t = \hat{a}_t dt \hat{P} - \text{a.s.}$

- The relaxed problem is:

$$
\hat{V}^1 := \inf \left\{ y : Y^1_t \geq \xi \hat{P} - \text{a.s. for some } Z \in \mathcal{H}^2(\hat{P}) \right\}
$$
Immediately follows that: $\hat{\mathcal{V}}^0 \geq \hat{\mathcal{V}}^1$

**Proposition** $\hat{\mathcal{V}}^0 = \hat{\mathcal{V}}^1$

Since

$$F_t(y, z, a) := \sup_{\gamma \in S^d} \left\{ \frac{1}{2} \text{Tr}[a \gamma] - H_t(y, z, \gamma) \right\}, \quad a \in S^d_+$$

one can get as close as required by choosing an almost optimal process $G_t$...
The Duality Result

**Theorem** Under some conditions, we have

\[ \hat{V} = \hat{V}^0 = \hat{V}^1 = \sup_{a \in \bar{U}} Y_0^a \]

where \((Y^a, Z^a)\) is the unique solution of the BSDE

\[ dY_t^a = F_t(Y_t^a, Z_t^a, a_t)dt + Z_t^a \cdot dB_t, \quad Y_T^a = \xi, \quad \mathbb{P}^a - \text{a.s.} \]

Assume further that

\[ \xi = g(B.), \quad \text{and} \quad F_t(y, z, a) = \phi(t, B., y, z, a) \]

for some uniformly continuous maps \(g\) and \(\phi(., y, z, .)\), then

\[ \hat{V} = \hat{V}^0 = \hat{V}^1 = \sup_{a \in \bar{U}} Y_0^a \]
Sketch of proof of the duality result

Introduce:

\[ \hat{Y}_t^i := \mathbb{P}^{a_i} - \text{ess-sup} \{ Y_t^a : a \in \bar{U}_0, a = a_i \text{ on } [0, t] \} \]

- Partial dynamic programming: \( \hat{Y}^i \) is a strong
  \( F(., a_i) \)-supermartingale, i.e. \( \hat{Y}_{\tau_1}^{i_{\tau_2}} \leq \hat{Y}_{\tau_1}^i \), \( \mathbb{P}^{a_i} \)-a.s. where

\[ \hat{Y}_{\tau_2}^{i_{\tau_2}} = \hat{Y}_{\tau_2}^i - \int_{\tau_2}^{\tau_2} F_s(\hat{Y}_{\tau_2}^{i_{\tau_2}}, \hat{Z}_{\tau_2}^{i_{\tau_2}}, a_i(s))ds - \int_{\tau_2}^{\tau_2} \hat{Z}_{\tau_2}^{i_{\tau_2}} dB_s \]

- Use (an adaptation of) the nonlinear Doob-Meyer decomposition
  of Peng \( \iff \) for some non-decreasing process \( C \):

\[ \hat{Y}_t^i = \hat{Y}_0^i + \int_{0}^{t} F_s(\hat{Y}_s^i, \hat{Z}_s^i, a_i(s))ds + \int_{0}^{t} \hat{Z}_s^i dB_s - C_t \]

- Aggregate the processes \( \hat{Y}^i \) into a process \( \hat{Y} \) under \( \hat{P} \) by checking
  the consistency condition...
Example 1: $H$ affine in $\gamma$

Let

$$H_t(y, z, \gamma) = H_t^0(y, z) + \frac{1}{2} \text{Tr} \left[ \sigma \sigma^T \gamma \right]$$

Then

$$F_t(y, z, a) := \sup_{\gamma \in S^d} \left( \frac{1}{2} \text{Tr}[a\gamma] - H_t(y, z, \gamma) \right)$$

$$= \begin{cases} 
H_t^0(y, z) & \text{for } a = \sigma\sigma^T \\
\infty & \text{otherwise}
\end{cases}$$

In this case $\hat{\mathbb{P}} = \mathbb{P}\sigma\sigma^T$
Example 2: Uncertain volatility

Denis and Martini 1999, Peng 2007

Let \( d = 1 \) for simplicity, and

\[
H_t(y, z, \gamma) = H_t^0(y, z) + \frac{1}{2} \sigma^2 \gamma^+ - \frac{1}{2} \sigma^2 \gamma^- \]

Then

\[
F_t(y, z, a) = H_t^0(y, z) \quad \text{for} \quad a \in [\sigma^2, \bar{\sigma}^2] \quad \text{and} \quad F_t(y, z, a) = \infty \quad \text{otherwise}
\]

In this case \( \hat{\mathbb{P}} = \mathbb{P}^{U_0} \) where \( U_0 \) is any dense subset of \([\sigma^2, \bar{\sigma}^2]\)
Conclusion

- Second order stochastic target problems have a suitable formulation by allowing for model uncertainty.

- From the dual formulation, we have obtained existence for the second relaxation of the target problem.

- Future work: exploit this existence result to define a weak notion of second order BSDEs.

- Provide a rigorous hedging strategy in the context of the Cetin-Jarrow-Protter model, and for the problem of hedging under Gamma constraints.