Dual Formulation of Second order Target Problems

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Special Semester in Financial Mathematics
Stochastic Control
Linz, October 19-24, 2008

Outline

Supply Function Models

Price of an order depends on volumes $\mathbb{S}_t(\omega, \nu)$. $\mathbb{S}$ may be estimated from orders book :

Quantity	10	35	20	100
Price	110	112	117	125

Note that the price by share is non-decreasing. But there is no influence of a large trade on the next moment orders book...
(Çetin-Jarrow-Protter '06, Rogers-Singh '05)
This includes Proportional Transaction Costs models

$$\mathbb{S}_t(\nu) = (1+\lambda)\mathbb{S}_t(0)\mathbb{1}_{\mathbb{R}_+}(\nu) + (1-\mu)\mathbb{S}_t(0)\mathbb{1}_{\mathbb{R}_-}(\nu)$$

The discrete-time model (Çetin, Jarrow and Protter 2004, 2006)

Risky asset price is defined by

- the marginal price $S_t,\ t \geq 0$
- the supply curve $\nu \longmapsto \mathbb{S}(.,\nu)$:

$$\mathbb{S}(S_t,\nu) \qquad \text{price per share of } \nu \text{ risky assets}$$

with $\mathbb{S}(s,0) = s$

Z_t^0 : holdings in cash, Z_t : holdings in risky asset

$$Z_{t+dt}^0 - Z_t^0 + (Z_{t+dt} - Z_t)\,\mathbb{S}\left(S_t, Z_{t+dt} - Z_t\right) = 0$$

$$\Longrightarrow \quad Z_T^0 = Z_0^0 - \sum (Z_{t+dt} - Z_t)\,\mathbb{S}\left(S_t, Z_{t+dt} - Z_t\right)$$
$$= Z_0^0 + \sum Z_t\left(S_{t+dt} - S_t\right) + \ldots$$

Continuous-time formulation of Model

Set $Y_t \;:=\; Z^0_t + Z_t S_t$, then :

$$
\begin{aligned}
Y_T \;&=\; Y_0 + \sum Z_t \left(S_{t+dt} - S_t\right) \\
&\quad - \sum \left(Z_{t+dt} - Z_t\right) \left[\mathbb{S}\left(S_t, Z_{t+dt} - Z_t\right) - \mathbb{S}\left(S_t, Z_{t+dt} - Z_t\right)\right]
\end{aligned}
$$

Assume $\nu \longmapsto \mathbb{S}(.,\nu)$ is smooth (unlike proportional transaction costs models), then :

$$
\begin{aligned}
Y_T \;=\; Y_0 \;+\; \int_0^T Z_t dS_t \quad &- \int_0^T \frac{\partial \mathbb{S}}{\partial \nu}\left(S_t, 0\right) d\langle Z^c\rangle_t \\
&- \sum_{t\leq T} \Delta Z_t \left[\mathbb{S}\left(S_t, \Delta Z_t\right) - S_t\right]
\end{aligned}
$$

- $d\langle Z^c\rangle_t = \Gamma_t^2 \; d\langle Z^c\rangle_t$: the so-called Gamma...

The Hedging Problem

Option / contingent claim : $g(S_T)$, where

$$g \quad : \quad \mathbb{R}_+ \longrightarrow \mathbb{R} \quad \text{has linear growth}$$

Super-hedging problem

$$V \quad := \quad \inf\left\{y \ : \ Y_T^{y,Z} \geq g(S_T)\ \mathbb{P}-\text{a.s. for some "admissible" } Z\right\}$$

• For this formulation to be consistent with the financial problem, we assume there is no liquidity cost at maturity T

• Here, admissibility is the crucial issue

• Non-Markov case : with new results, should be possible...

The Çetin-Jarrow-Protter Negative Result

Without further restrictions on trading strategies, **the problem reduces to Black-Scholes**! Reason for this result is the following result of Bank-Baum 04

Lemma *For predictable $W-$integ. càdlàg process ϕ, and $\varepsilon > 0$*

$$\sup_{0\le t\le 1}\left|\int_0^t \phi_r dW_r - \int_0^t \phi_r^\varepsilon dW_r\right| \;\le\; \varepsilon$$

for some a.c. predictable process $\phi_t^\varepsilon = \phi_0^\varepsilon + \int_0^t \alpha_r dr$

$\Longrightarrow$ If the "admissibility" set allows for arbitrary a.c. portfolio $Z_t = Z_0 + \int_0^t \alpha_u du$, then $V = V^{\mathrm{BS}}$ (with $\Gamma = 0$!)

A Convenient Set of Admissible Strategies

We show that **liquidity cost does affect** V, **perfect replication is possible, and hedging strategy can be described (formally)**

Definition $Z \in \mathcal{A}$ if it is of the form

$$Z_t = \sum_{n=0}^{N-1} z_n \mathbb{1}_{\{t<\tau_{n+1}\}} + \int_0^t \alpha_u du + \int_0^t \Gamma_u dS_u$$

- (τ_n) is an $\nearrow$ seq. of stop. times, z_n are $\mathcal{F}_{\tau_n}-$measurable, $\|N\|_\infty < \infty$
- Z and Γ are $\mathbb{L}^\infty-$bounded up to some polynomial of S
- $\Gamma_t = \Gamma_0 + \int_0^t a_u du + \int_0^t \xi_u dW_u$, $0 \le t \le T$, and

$$\|\alpha\|_{B,b} + \|a\|_{B,b} + \|\xi\|_{B,2} < \infty, \qquad \|\phi\|_{B,b} := \left\| \sup_{0\le t\le T} \frac{|\phi_r|}{1+S_t^B} \right\|_{\mathbb{L}^b}$$

PDE characterization

Let

$$\ell(s) \quad := \quad \left[4\frac{\partial \mathbb{S}}{\partial \nu}(s,0)\right]^{-1}$$

Theorem Let $-C \leq g(.) \leq C(1+\ .)$ for some $C > 0$. Then $V(t,s)$ is the unique continuous viscosity solution of the dynamic programming equation

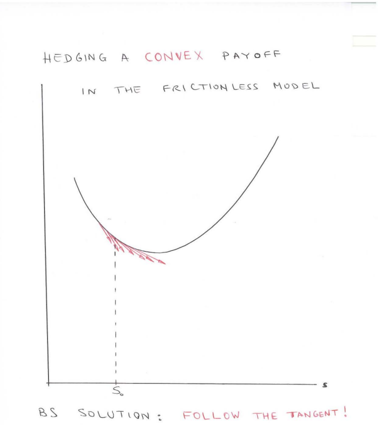
$$-V_t(t,s) + \frac{1}{4}s^2\sigma(t,s)^2\ell(s)\left[1 - \left(\frac{V_{ss}(t,s)}{\ell(s)} + 1\right)^{+^2}\right] \quad = \quad 0$$

with $V(T,s) = g(s)$ and $-C \leq V(t,s) \leq C(1+s)$ for every (t,s).

- Notice that there is no boundary layer $\Longrightarrow$ perfect hedge

Hedging a Convex Payoff in the Frictionless BS Model

For a convex payoff : only possibility to super-hedge is the Black-Scholes perfect replication strategy

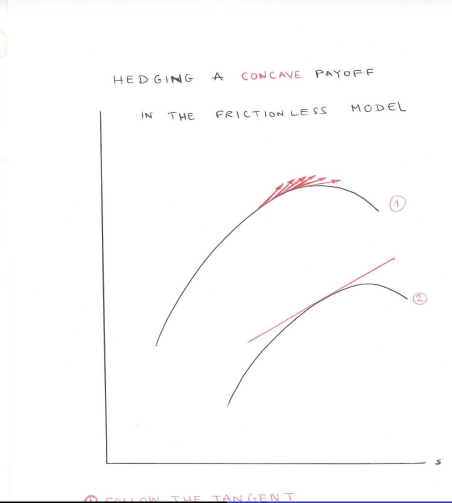


Hedging a Concave Payoff in the Frictionless BS Model

For a concave payoff : two possibilities to super-hedge

- Black-Scholes perfect replication $\Longrightarrow \Gamma \neq 0$ so pay liquidity cost
- Buy-and-hold $\Longrightarrow \Gamma = 0$ no liquidity cost, but hedge might be too expensive

Hedging a Concave Payoff in the Frictionless BS Model



Formal Description of a Hedging Strategy

- $v_{ss} < -\ell(s)$: Then the PDE satisfied by V reduces to

$$-V_t(t,s) + \frac{1}{4}s^2\sigma(t,s)^2\ell(s) \quad = \quad 0 \quad \text{(degenerate !)}$$

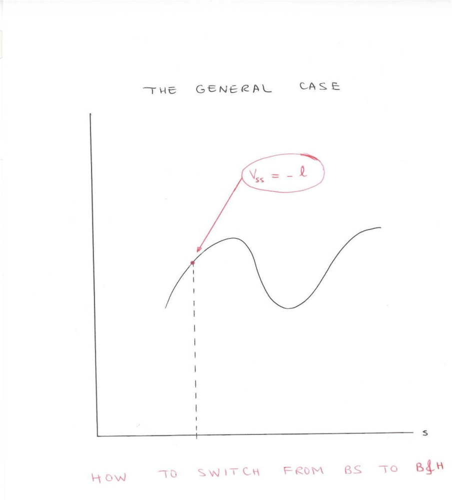
buy-and-hold strategy is more interesting because liquidity cost is too expensive

$v_{ss} \geq -\ell(s)$: Then the PDE satisfied by V reduces to

$$-V_t(t,s) - \frac{1}{2}s^2\sigma(t,s)^2 V_{ss} - \frac{s^2\sigma(t,s)^2}{4\ell(s)}V_{ss}^2 \quad = \quad 0$$

perfect replication

The Technical Difficulty



A New Formulation : Intuition

Recall the state dynamics in Stratonovitch form :

$$dY_t = Z_t \circ dS_t - \left(\frac{1}{2}\Gamma_t + \mathbb{S}_\nu(S_t, 0)\right) \sigma_t^2 S_t^2 dt$$

and the corresponding "natural" PDE :

$$\frac{\partial V}{\partial t} = -\left(\frac{1}{2} V_{ss} + \mathbb{S}_\nu(s, 0)\right) \sigma^2 s^2$$

Main observation : We would obtain the same PDE if the volatility of S is modified :

$$\begin{aligned} dY_t &= Z_t \circ dS'_t - \left(\frac{1}{2}\Gamma_t + \mathbb{S}_\nu(S'_t, 0)\right) \sigma_t^2 {S'_t}^2 dt \\ dS'_t &= \sigma'_t S'_t dW_t \end{aligned}$$

A New Formulation : Relax Controls and Change Volatility (Intuition from L. Denis and C. Martini)

Consider the super-hedging problem :

$$\hat{V} \quad := \quad \inf\left\{y : \; Y_T \geq g(S_T) \; \hat{\mathbb{P}} - \text{a.s. for some } Z \in \mathcal{S}\hat{\mathcal{M}}^2\right\}$$

where

$$dY_t = Z_t \circ dS_t - \left(\frac{1}{2}\Gamma_t + \mathbb{S}_\nu(S_t, 0)\Gamma_t^2\right) \sigma_t^2 {S_t}^2 dt$$

Compare with

$$V \quad := \quad \inf\left\{y : \; Y_T \geq g(S_T) \; \mathbb{P} - \text{a.s. for some } Z \in \mathcal{A}\right\}$$

Then, $\hat{V} = V$

Outline

A Dense Subset of Scales

$C([0,T])^d$ Canonical space, B canonical process, $\mathbb{P}$ Wiener measure

$$\mathcal{M} \quad := \quad \{\mathbb{P}' : \text{ Prob. meas. s.t. } \mathbb{P}'[B_0 = 0] = 1 \text{ and } B \in \mathcal{M}^2(\mathbb{P}')\}$$

$U \subset \mathbb{S}_d^+$ given, U_0 dense subset of U, $\mathbf{T}_0$ dense subset of $[0,1]$

$\mathcal{U}_0$, simple functions : $n \in \mathbb{N}$, $0 = t_0 < \ldots < t_n = 1$, $\varepsilon > 0$,

$$a(t) = \sum_{i=1}^{n} \alpha_i \mathbb{1}[t_{i-1}, t_i), \quad \alpha_i \in U_0 \cap [\varepsilon I_d, \varepsilon^{-1} I_d], \quad t_i \in \mathbf{T}_0$$

$\mathcal{U}_0 = \{a_i, i \geq 1\}$ countable with $\sum_{i\geq 1} 2^{-i} \int_0^1 |a_i(t)|dt < \infty$

$\bar{\mathcal{U}}_0$, simple processes : above a_i's $\in \mathbb{L}^2(\mathbb{P}, \mathcal{F}_{t_i})$

$\bar{\mathcal{U}} = \mathbb{H}^2(\mathbb{P}, \mathbb{F}, U)$

For $a \in \bar{\mathcal{U}}_0$: $\mathcal{M} \ni P^a$ = distribution of the process $\int_0^t \sqrt{a(t)}dB_t$

- since $\sum_{i\geq 1} 2^{-i} \int_0^1 |a^i(t)|dt < \infty$, the reference measure

$$\hat{\mathbb{P}} = \hat{\mathbb{P}}^{U_0,\mathbf{T}_0} \;:=\; \sum_{i\geq 1} 2^{-i}\mathbb{P}^{a_i} \;\in\; \mathcal{M}$$

- For every $i \geq 1$, $\mathbb{P}^{a_i} \lll \hat{\mathbb{P}}$

- For every $a \in \bar{\mathcal{U}}_0$, $\mathbb{P}^a \lll \hat{\mathbb{P}}$

- But for arbitrary $a \in \bar{\mathcal{U}}$, $\mathbb{P}^a \not\lll \hat{\mathbb{P}}$

Our result will however not depend on the choice of $(U_0, \mathbf{T}_0)$

Properties of $\hat{\mathbb{P}}$

- $\hat{\mathbb{P}} \in \mathcal{M}$, and $d\langle B\rangle_t = a_i(t)dt\ \mathbb{P}^{a_i}-$a.s.

Aggregartion Let $X^i \in \mathbb{H}^0(\mathbb{P}^{a_i})$ be a family of processes such that

$$X^i_s = X^j_s,\ s \le t \quad \text{whenever} \quad a_i = a_j \text{ on } [0,t]$$

Then there is a unique process $X \in \mathbb{H}^0(\hat{\mathbb{P}})$ such that

$$X = X^i \qquad dt \times d\mathbb{P}^{a_i} - \text{a.s.}$$

$\Longrightarrow d\langle B\rangle_t = \hat{a}_t dt\ \hat{\mathbb{P}}-$a.s.

Problems with Null Sets

• Our analysis requires to define objects $\hat{\mathbb{P}}$-a.s. and then to look at their decompositions under $\mathbb{P}^{a_i}$ for every $i \geq 1$
• Standard stochastic analysis results are stated under the assumption that the filtration satisfies the usual conditions...
• Let $\mathbb{F}^{\hat{\mathbb{P}}}$ be the filtration completed by $\hat{\mathbb{P}}-$null sets, then $\mathbb{F}^{\hat{\mathbb{P}}}$ is not complete for $\mathbb{P}^{a_i}$!

$\mathbb{F}^+-$**adapted modification** For any $Z \in \mathbb{H}^0(\mathbb{P}', \mathbb{F}^{\mathbb{P}'})$, there is a unique $\tilde{Z} \in \mathbb{H}^0(\mathbb{P}', \mathbb{F}^+)$ such that Z and $\tilde{Z}$ are $\mathbb{P}'-$modifications (i.e. $Z = \tilde{Z}\ dt \times d\mathbb{P}'-$a.s.)

$\Longrightarrow$ Always consider $\mathbb{F}^+-$adapted versions

Outline

Operators and Spaces

- $H_t(y, z, \gamma)$

$H : \Omega \times [0,1] \times \mathbb{R} \times \mathbb{S}_d \longrightarrow \mathbb{R}$, $\mathcal{P} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{S}_d) -$ meas.

- $F_t(y, z, a)$: conjugate wrt γ

$$F_t(y,z,a) \quad := \quad \sup_{\gamma \in \mathbb{S}_d} \left\{ \frac{1}{2} \mathrm{Tr}[a\gamma] - H_t(y,z,\gamma) \right\}$$

Assumption H uniformly Lipschitz in (y,z), has linear growth wrt γ, $H(0,0,0) \in \mathbb{H}^2(\mathbb{P}^a)$ for every $a \in \mathbb{H}^2(\mathbb{S}_d^+)$, and there is a subset $U \subset \mathbb{S}_d^+$ such that $U \subset \mathrm{dom}(F_t)$, $t \leq 1$

Consider the reference measure $\hat{\mathbb{P}} := \hat{\mathbb{P}}^{U_0, \mathbf{T}_0}$, and let

$$\hat{\mathbb{H}}^2 := \cap_{i \geq 1} \mathbb{H}^2(\mathbb{P}^{a_i}, \mathbb{F}^+), \qquad \hat{\mathcal{SM}}^2 := \cap_{i \geq 1} \mathcal{SM}^2\left(\mathbb{P}^{a_i}, \mathbb{F}^+\right)$$

Formulation of Second Order Target Problems

- For $Z \in \mathcal{S}\hat{\mathcal{M}}^2$, define the controlled state :

$$dY_t = -H_t(Y_t, Z_t, \Gamma_t)dt + Z_t \circ dB_t, \quad \hat{\mathbb{P}} - \text{a.s.}$$

where the process Γ is defined by

$$d\langle Z, B\rangle_t = \Gamma_t d\langle B\rangle_t \quad \hat{\mathbb{P}} - \text{a.s.}$$

- The target problem is :

$$\hat{V} := \inf\left\{y : \; Y_1 \geq \xi \; \hat{\mathbb{P}} - \text{a.s. for some } Z \in \mathcal{S}\hat{\mathcal{M}}^2\right\}$$

First Relaxation of Second Order Target Problems

Relax the connection between Z and Γ

- For $Z, G \in \hat{\mathbb{H}}^2$, define the controlled state :

$$\begin{aligned} dY_t^0 &= -H_t(Y_t^0, Z_t, G_t)dt + Z_t \circ dB_t, \quad \hat{\mathbb{P}} - \text{a.s.} \\ &= \frac{1}{2}\mathrm{Tr}[G_t d\langle B\rangle_t] - H_t(Y_t, Z_t, G_t)dt + Z_t \cdot dB_t, \quad \hat{\mathbb{P}} - \text{a.s.} \\ &= \left(\frac{1}{2}\mathrm{Tr}[G_t \hat{a}_t] - H_t(Y_t, Z_t, G_t)\right) dt + Z_t \cdot dB_t, \quad \hat{\mathbb{P}} - \text{a.s.} \end{aligned}$$

where the process $d\langle B\rangle_t = \hat{a}_t dt$ $\hat{\mathbb{P}}-$a.s.

- The relaxed problem is :

$$\hat{V}^0 \quad := \quad \inf\left\{y : \; Y_1^0 \geq \xi \; \hat{\mathbb{P}} - \text{a.s. for some } Z, G \in \hat{\mathbb{H}}^2\right\}$$

First Relaxation of Second Order Target Problems

Clearly : $\hat{V} \geq \hat{V}^0$

Proposition $\hat{V} = \hat{V}^0$

Based on extension of Bank and Baum to the nonlinear case : Let (Y, Z) be such that

$$Y_t = y + \int_0^t h_s(Y_s, Z_s, \hat{a}_s)ds + \int_0^t Z_s dB_s, \quad \hat{\mathbb{P}} - \text{a.s.}$$

Then, $\forall\, \varepsilon > 0$, there is a continuous FV process Z^ε such that

$$Y_t^\varepsilon = y + \int_0^t h_s(Y_s^\varepsilon, Z_s^\varepsilon, \hat{a}_s)ds + \int_0^t Z_s^\varepsilon dB_s, \quad \hat{\mathbb{P}} - \text{a.s.}$$

$$\text{and} \quad \sup_{0 \le t \le 1} |Y_t - Y_t^\varepsilon| \le \varepsilon \qquad \hat{\mathbb{P}} - \text{a.s.}$$

Further Relaxation of Second Order Target Problems

Second relaxation : forget Γ !

- Recall the (partial) convex conjugate of H :

$$F_t(y,z,a) \;:=\; \sup_{\gamma\in\mathcal{S}^d}\left\{\frac{1}{2}\mathrm{Tr}[a\gamma]-H_t(y,z,\gamma)\right\},\quad a\in\mathcal{S}^d_+$$

- For $Z\in\mathcal{H}^2(\hat{\mathbb{P}})$, define the controlled state :

$$dY^1_t \;=\; F_t(Y^1_t,Z_t,\hat{a}_t)dt+Z_t\cdot dB_t,\quad \hat{\mathbb{P}}-\text{a.s.}$$

where the process $d\langle B\rangle_t=\hat{a}_t dt$ $\hat{\mathbb{P}}-$a.s.

- The relaxed problem is :

$$\hat{V}^1 \;:=\; \inf\left\{y:\; Y^1_1\geq\xi\ \hat{\mathbb{P}}-\text{a.s. for some } Z\in\mathcal{H}^2(\hat{\mathbb{P}})\right\}$$

Further Relaxation of Second Order Target Problems

Immediately follows that : $\hat{V}^0 \geq \hat{V}^1$

Proposition $\hat{V}^0 = \hat{V}^1$

Since

$$F_t(y,z,a) \quad := \quad \sup_{\gamma \in \mathcal{S}^d} \left\{ \frac{1}{2}\mathrm{Tr}[a\gamma] - H_t(y,z,\gamma) \right\}, \quad a \in \mathcal{S}^d_+$$

one can get as close as required by choosing an almost optimal process G_t...

The Duality Result

Theorem Under some conditions, we have

$$\hat{V} \;=\; \hat{V}^0 \;=\; \hat{V}^1 \;=\; \sup_{a \in \mathcal{U}_0} Y_0^a$$

where (Y^a, Z^a) is the unique solution of the BSDE

$$dY_t^a \;=\; F_t(Y_t^a, Z_t^a, a_t)dt + Z_t^a \cdot dB_t, \quad Y_T^a \;=\; \xi, \quad \mathbb{P}^a - \text{a.s.}$$

Assume further that

$$\xi = g(B_.), \quad \text{and} \quad F_t(y,z,a) = \phi(t, B_., y, z, a)$$

for some uniformly continuous maps g and $\phi(., y, z, .)$, then

$$\hat{V} \;=\; \hat{V}^0 \;=\; \hat{V}^1 \;=\; \sup_{a \in \bar{\mathcal{U}}} Y_0^a$$

Sketch of proof of the duality result

Introduce :

$$\hat{Y}^i_t \;:=\; \mathbb{P}^{a_i}-\mathrm{ess}-\sup\left\{Y^a_t : a \in \bar{\mathcal{U}}_0, a = a_i \text{ on } [0,t]\right\}$$

• Partial dynamic programming : $\hat{Y}^i$ is a strong $F(.,a_i)-$supermartingale, i.e. $\hat{Y}^{i\tau_2}_{\tau_1} \le \hat{Y}^i_{\tau_1}$, $\mathbb{P}^{a_i}-$a.s. where

$$\hat{Y}^{i\tau_2}_t \;=\; \hat{Y}^i_{\tau_2} - \int_t^{\tau_2} F_s(\hat{Y}^{i\tau_2}_s, \hat{Z}^{i\tau_2}_t, a_i(s))ds - \int_t^{\tau_2} \hat{Z}^{i\tau_2}_s dB_s$$

• Use (an adaptation of) the nonlinear Doob-Meyer decompositon of Peng $\Longrightarrow$ for some non-decreasing process C :

$$\hat{Y}^i_t \;=\; \hat{Y}^i_0 + \int_0^t F_s(\hat{Y}^i_s, \hat{Z}^i_s, a_i(s))ds + \int_0^t \hat{Z}^i_s dB_s - C_t$$

• Aggregate the processes $\hat{Y}^i$ into a process $\hat{Y}$ under $\hat{\mathbb{P}}$ by checking the consistency condition...

Example 1 : H affine in γ

Let

$$H_t(y,z,\gamma) \quad = \quad H_t^0(y,z) + \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^{\mathrm{T}}\gamma\right]$$

Then

$$\begin{aligned} F_t(y,z,a) \quad &:= \quad \sup_{\gamma\in\mathcal{S}^d}\left(\frac{1}{2}\mathrm{Tr}[a\gamma] - H_t(y,z,\gamma)\right) \\ &= \quad \begin{cases} H_t^0(y,z) & \text{for } a = \sigma\sigma^{\mathrm{T}} \\ \infty & \text{otherwise} \end{cases} \end{aligned}$$

In this case $\hat{\mathbb{P}} = \mathbb{P}^{\sigma\sigma^{\mathrm{T}}}$

Example 2 : Uncertain volatility

Denis and Martini 1999, Peng 2007

Let $d = 1$ for simplicity, and

$$H_t(y, z, \gamma) \quad = \quad H_t^0(y, z) + \frac{1}{2}\overline{\sigma}^2\gamma^+ - \frac{1}{2}\underline{\sigma}^2\gamma^-$$

Then

$F_t(y, z, a) = H_t^0(y, z)$ for $a \in \left[\underline{\sigma}^2, \overline{\sigma}^2\right]$ and $F_t(y, z, a) = \infty$ otherwise

In this case $\hat{\mathbb{P}} = \mathbb{P}^{U_0}$ where U_0 is any dense subset of $\left[\underline{\sigma}^2, \overline{\sigma}^2\right]$

Conclusion

• Second order stochastic target problems have a suitable fomulation by allowing for model uncertainty

• From the dual formulation, we have obtained existence for the second relaxation of the target problem

• Future work : exploit this existence result to define a weak notion of second order BSDEs

• Provide a rigorous hedging strategy in the context of the Cetin-Jarrow-Protter model, and for the problem of hedging under Gamma constraints.