Discrete hedging in models with jumps

Peter Tankov

CMAP, Ecole Polytechnique

Partly joint work with E. Voltchkova (Université Toulouse 1)

Workshop on Optimization and Optimal Control, Linz, October 20–24, 2008

Hedging in incomplete markets

- Incomplete market: exact replication impossible.
- Hedging is now an approximation problem.
- Industry practice: sensitivities to risk factors

$$\text{Delta} = \frac{\partial C(t, S_t)}{\partial S}: \quad \text{infinitesimal moves, hedge with stock}$$

$$\text{Gamma} = \frac{\partial^2 C(t, S_t)}{\partial S^2}: \quad \text{bigger moves; hedge with liquid options}$$

- Quadratic hedging: control the residual error

$$\min_{\phi} E\left(c + \int_0^T \phi_t dS_t - Y\right)^2$$

All these strategies require a continuously rebalanced portfolio.

Discrete hedging

- Continuous rebalancing is unfeasible: in practice, the strategy ϕ_t is replaced with a discrete strategy, leading to the hedging error of the "second type": error of approximating the continuous portfolio with a discrete one.
- The simplest choice is $\phi^n_t := \phi_{h[t/h]}$, $h = T/n$.
- This discretization error has only been studied in the case of continuous processes.
- Two main approaches: weak convergence (CLT for hedging error) and L^2 convergence

Discrete hedging: the complete market case

- Bertsimas, Kogan and Lo '98 introduced an *asymptotic approach* allowing to study discrete hedging in continuous time.

Suppose

$$\frac{dS_t}{S_t} = \mu(t, S_t)dt + \sigma(t, S_t)dW_t$$

and we want to hedge a European option with payoff $h(S_T)$ using delta-hedging $\phi_t = \frac{\partial C}{\partial S}$.

CLT for hedging error

The discrete hedging error is defined by

$$\varepsilon_T^n = h(S_T) - \int_0^T \phi_t^n dS_t$$

Then $\varepsilon_T^n \to 0$ but the renormalized error $\sqrt{n}\varepsilon_T^n$ converges to

$$\sqrt{\frac{T}{2}} \int_0^T \frac{\partial^2 C}{\partial S^2} S_t^2 \sigma_t^2 dW_t^*,$$

where W^* is a Brownian motion independent of W.

- Hedging error decays as $\sqrt{h}$.
- It is orthogonal to the stock price.
- The amplitude is determined by the gamma $\frac{\partial^2 C}{\partial S^2}$

Approximating hedging portfolios

Hayashi and Mykland '05 interpreted the discrete hedging error as the error of approximating the "ideal" hedging portfolio $\int_0^T \phi_t dS_t$ with a feasible hedging portfolio $\int_0^T \phi_t^n dS_t$

- This makes sense in incomplete markets

Suppose ϕ and S are Itô process:
$d\phi_t = \tilde{\mu}_t dt + \tilde{\sigma}_t dW_t$ and $dS_t = \mu_t dt + \sigma_t dW_t$. Then

$$\sqrt{n}\varepsilon_t^n \Rightarrow \sqrt{\frac{T}{2}} \int_0^t \tilde{\sigma}_s \sigma_s dW_s^*, \qquad \left(\tilde{\sigma}_t = \frac{\partial^2 C}{\partial S^2} S_t \sigma_t\right)$$

$$\text{where} \quad \varepsilon_t^n := \int_0^t (\phi_t - \phi_t^n) dS_t.$$

- Weak convergence of processes in the Skorokhod topology on the space $\mathbb{D}$ of càdlàg functions

L^2 hedging error for continuous processes

- Result by Zhang (1999): for call/put options, the L^2 hedging error converges to the expected square of the weak limit.

$$\lim_{n\to\infty} nE[(\varepsilon^n_T)^2] = \frac{T}{2}E\left[\int_0^T \left(\frac{\partial^2 C}{\partial S^2}\right)^2 S_t^4\sigma_s^4 ds\right].$$

- The constant may be improved by an intelligent choice of rebalancing dates (Brodén and Wiktorsson '08) but the convergence rate cannot be improved.
- See also related results by Gobet and Temam (01) and Geiss (02), (06), (07).

Discretization error in presence of jumps

Our idea: study the discretization error

$$\varepsilon_t^n := \int_0^t (\phi_{t-} - \phi_{t-}^n) dS_t$$

in presence of jumps in the underlying and the hedging strategy.

- Approximation error of the Lévy-driven Euler scheme: Jacod and Protter (98), Jacod (04)
- Related results in the approximation of quadratic variation by realized volatility

$$X_T^2 = X_0^2 + 2 \int_0^T X_{t-} dX_t + [X, X]_T$$

- Limit theorems for the approximation error of quadratic variation: Jacod (08).

Model setup: Lévy-Itô processes

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{|z|\leq 1} \gamma_s(z)\tilde{J}(ds \times dz) + \int_0^t \int_{|z|>1} \gamma_s(z) J(ds \times dz).$$

- J: Poisson random measure with intensity $dt \times \nu$
- μ and σ are càdlàg $(\mathcal{F}_t)$-adapted
- γ: $\Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ is such that $(\omega, z) \mapsto \gamma_t(z)$ is $\mathcal{F}_t \times \mathcal{B}(\mathbb{R})$-measurable $\forall t$ and $t \to \gamma_t(z)$ is càglàd $\forall \omega, z$;

$$\gamma_t(z)^2 \leq A_t \rho(z), \qquad \int_{|z|\leq 1} \rho(z)\nu(dz) < \infty$$

with ρ positive deterministic and A càglàd $(\mathcal{F}_t)$-adapted.

Model setup

- The stock price S is a Lévy-Itô process with coefficients μ, σ, γ;
- The continuous-time strategy ϕ is a Lévy-Itô process with coefficients $\tilde{\mu}, \tilde{\sigma}, \tilde{\gamma}$.
- The agent uses the discrete strategy $\phi_t^n := \phi_{h[t/h]}$ instead of the continuous strategy ϕ_t.

Weak convergence: the normalizing sequence

The normalizing factor need not be equal to $\sqrt{n}$.
Suppose ϕ and S move only by finite-intensity jumps. If there is only one jump between t_i and t_{i+1},

$$\int_{t_i}^{t_{i+1}} \phi_{t-} dS_t = \int_{t_i}^{t_{i+1}} \phi^n_{t-} dS_t$$

Therefore $P[\varepsilon^n_t \neq 0] = O(1/n)$ and

$$n^\alpha \varepsilon^n_t \to 0$$

in probability $\forall \alpha$.
More generally, if S and ϕ are Lévy-Itô processes without diffusion parts,

$$\sqrt{n}\varepsilon^n_t \to 0$$

in probability uniformly on t.

Weak convergence

The discretization error satisfies

$$\sqrt{n}\varepsilon_t^n \to \sqrt{\frac{T}{2}} \int_0^t \sigma_s \tilde{\sigma}_s dW_s^* + \sqrt{T} \sum_{i:T_i \le t} \Delta\phi_{T_i} \sqrt{\zeta_i} \xi_i \sigma_{T_i} + \sqrt{T} \sum_{i:T_i \le t} \Delta S_{T_i} \sqrt{1-\zeta_i} \xi_i' \tilde{\sigma}_{T_i-}.$$

W^* is a standard BM independent from W and J,
$(\xi_k)_{k\ge 1}$ and $(\xi_k')_{k\ge 1}$ are two sequences of independent $N(0,1)$,
$(\zeta_k)_{k\ge 1}$ is sequence of independent $U([0,1])$
$(T_i)_{i\ge 1}$ are the jump times of J enumerated in any order.

Remarks on convergence

- The hedging error $\sqrt{n}\varepsilon_t^n$ converges weakly in finite-dimensional laws but not in Skorohod topology.
- The discretized error process $\sqrt{n}\varepsilon_{h[t/h]}^n$ converges in Skorohod topology to the same limit.

Idea of the proof

Main tool: if (X^n) and (Y^n) are two sequences of processes such that

$$\sup_t |X^n_t - Y^n_t| \to 0 \quad \text{in probability}$$

and $X^n \to X$ weakly then $Y^n \to X$ weakly.

Idea of the proof

Step 1 Remove the big jumps

Step 2 Remove the small jumps

Step 3 Now we can write

$$S_t = S_0 + S_t^d + S_t^c + S_t^j$$

$$S_t^d = \int_0^t \left(\mu_s + \int \gamma_s(z)\nu(dz) \right) ds$$

$$S_t^c = \int_0^t \sigma_s dW_s$$

$$S_t^j = \int_0^t \int \gamma_s(z) J(ds \times dz)$$

and $\phi_t = \phi_0 + \phi_t^d + \phi_t^c + \phi_t^j$.

Idea of the proof

The leading terms in the hedging error are

$$\sqrt{n}\int(\phi_t^c-\phi_t^{c,n})dS_t^c \to \sqrt{\frac{T}{2}}\int_0^t \sigma_s\tilde{\sigma}_s dW_s^*$$

$$\sqrt{n}\int(\phi_t^j-\phi_t^{j,n})dS_t^c = \sum_i \sqrt{n}\Delta\phi_{T_i}\int_{T_i}^{r(T_i)}\sigma_s dW_s$$

$$\to \sqrt{T}\sum_{i:T_i\leq t}\Delta\phi_{T_i}\sqrt{\zeta_i}\xi_i\sigma_{T_i}$$

$$\sqrt{n}\int(\phi_t^c-\phi_t^{c,n})dS_t^j = \sum_i \sqrt{n}\Delta S_{T_i}\int_{l(T_i)}^{T_i}\tilde{\sigma}_s dW_s$$

$$\to \sqrt{T}\sum_{i:T_i\leq t}\Delta S_{T_i}\sqrt{1-\zeta_i}\xi_i'\tilde{\sigma}_{T_i-}.$$

Application: delta-hedging in a Lévy market

$$S_t = S_0 e^{X_t}, \quad X_t = bt + \sigma W_t + \int zJ(ds \times dz)$$

$$C(t,S) = E^Q[H(Se^{X_{T-t}})], \quad \phi_t = \frac{\partial C}{\partial S}(t, S_t)$$

Suppose

- The Lévy measure is finite and has a regular density (e.g. Merton model).
- The payoff function H is piecewise smooth with a finite number of discontinuities.

Application: delta-hedging in a Lévy market

Apply the Itô formula to get the decomposition for ϕ:

$$d\phi_t = d\frac{\partial C(t,S_t)}{\partial S} = \left\{\frac{\partial^2 C}{\partial t \partial S} + (b+\sigma^2/2)\frac{\partial^2 C}{\partial S^2}S_t + \frac{\sigma^2}{2}\frac{\partial^3 C}{\partial S^3}S_t^2\right\}dt$$
$$+\sigma\frac{\partial^2 C}{\partial S^2}S_t dW_t + \int_{\mathbb{R}}\left(\frac{\partial C}{\partial S}(t,S_{t-}e^z) - \frac{\partial C}{\partial S}(t,S_{t-})\right)J(dt\times dz)$$

Under the hypotheses on H and ν it can be shown that the coefficients do not explode in T: almost all trajectories end in a point where H is smooth.

Application: delta-hedging in a Lévy market

The main result then implies $\sqrt{n}\varepsilon^n_t \to Z_t$ with

$$Z_t = \sqrt{\frac{T}{2}}\int_0^t \sigma^2 S_s^2 \frac{\partial^2 C}{\partial S^2} dW^*_s + \sqrt{T}\sum \Delta \frac{\partial C}{\partial S}\sqrt{\zeta_i}\xi_i \sigma S_s + \sqrt{T}\sum \Delta S_s \sqrt{1-\zeta_i}\xi'_i \sigma S_{s-}\frac{\partial^2 C}{\partial S^2}(s, S_{s-})$$

Application: risk of a hedged option position

If $E[Z_t^2] < \infty$, we can estimate the risk of a hedged option position using

$$P[|\varepsilon_t^n| \geq \delta] \approx P[|Z_t/\sqrt{n}| \geq \delta] \leq \frac{1}{\delta\sqrt{n}} E[Z_t^2]^{1/2}$$

with (small jump size approximation)

$$E[Z_t^2] \approx \frac{T}{2}\int_0^t E\left[S_s^4\left(\frac{\partial^2 C}{\partial S^2}\right)^2\right](\sigma^4 + \sigma^2 \int (e^z - 1)^2 (e^{2z} + 1)\nu(dx)).$$

Convergence of L^2 error

We have proved the weak convergence

$$\sqrt{n}\varepsilon_T^n \to Z_T := \sqrt{\frac{T}{2}} \int_0^T \sigma_s \tilde{\sigma}_s dW_s^* + \sqrt{T} \sum_i \Delta\phi_{T_i} \sqrt{\zeta_i} \xi_i \sigma_{T_i} + \sqrt{T} \sum_i \Delta S_{T_i} \sqrt{1-\zeta_i} \xi_i' \tilde{\sigma}_{T_i -},$$

but for some applications it is more convenient to have

$$E[(\sqrt{n}\varepsilon_T^n)^2] \to E[Z_T^2].$$

Surprising result: Even in the most simple cases, the L^2 error does not converge to the expected square of the weak limit if there are jumps *both* in S and in ϕ.

L^2 convergence: example

Suppose

$$\phi_t = S_t = N_t,$$

with N_t a Poisson process with intensity λ. Then

$$P\left[\int_0^T (N_{t-} - N_t^n)dN_t \neq 0\right] = O\left(\frac{1}{n}\right)$$

but

$$\lim_{n\to\infty} E\left[\left(\sqrt{n}\int_0^T (N_{t-} - N_t^n)dN_t\right)^2\right] = \frac{\lambda^2 T^2}{2}.$$

Convergence of L^2 error: general case

Let

$$A_t = \frac{d\langle S\rangle_t}{dt} = \sigma_t^2 + \int_{\mathbb{R}} \gamma_t^2(z)\nu(dz)$$

and

$$\tilde{A}_t = \frac{d\langle \phi\rangle_t}{dt} = \tilde{\sigma}_t^2 + \int_{\mathbb{R}} \tilde{\gamma}_t^2(z)\nu(dz)$$

and suppose

- (A_t) and (μ_t) are themselves Lévy-Itô processes.
- Integrabiliity assumptions on A, μ and ϕ.

Then

$$\lim_n E[(\sqrt{n}\varepsilon_T^n)^2] = E\left[\frac{T}{2}\int_0^T A_t\tilde{A}_t dt + \frac{T}{2}\sum_{t\leq T}\Delta\phi_t^2\Delta A_t\right].$$

Comparaison of L^2 and weak convergence

Denote

$$A_t = \sigma_t^2 + \int_{\mathbb{R}} \gamma_t^2(z)\nu(dz) := A_t^\sigma + A_t^\gamma$$

and similarly $\tilde{A}_t := \tilde{A}_t^\sigma + \tilde{A}_t^\gamma$. Then

$$E[Z^2] = \frac{T}{2} E\left[\int_0^T (A_t^\sigma \tilde{A}_t^\sigma + A_t^\sigma \tilde{A}_t^\gamma + A_t^\gamma \tilde{A}_t^\sigma) ds + \sum_{t \le T} \Delta\phi_t^2 \Delta A_t^\sigma\right]$$

whereas

$$\lim_n E[(\sqrt{n}\varepsilon_T^n)^2] = \frac{T}{2} E\left[\int_0^T (A_t^\sigma \tilde{A}_t^\sigma + A_t^\sigma \tilde{A}_t^\gamma + A_t^\gamma \tilde{A}_t^\sigma + A_t^\gamma \tilde{A}_t^\gamma) ds \right.$$

$$\left. + \sum_{t \le T} \Delta\phi_t^2 (\Delta A_t^\sigma + \Delta A_t^\gamma)\right].$$

The rebalancing strategy

- Suppose that ϕ and S are piecewise constant and that the portfolio is rebalanced after each jump of ϕ. Then the hedging error is zero with a finite number of rebalancing dates.
- In the general case, we suppose that the rebalancing is done at deterministic dates $T_i = \frac{i}{n}T$ and at random dates $\tilde{T}_i$ given by the jump times of the Poisson process $N_t^\varepsilon := J([0,t] \times (-\infty, -\varepsilon) \cup (\varepsilon, \infty))$.
- When $n \to \infty$ and $\varepsilon \to 0$, the convergence rate can be defined in terms of the expected number of rebalancings $n + T\lambda_\varepsilon$, with $\lambda_\varepsilon = \nu(\mathbb{R} \setminus [-\varepsilon, \varepsilon])$.

The limit theorem

Denote $A_t^* = A_t + \mu_t^2$ and let

$$\phi_t^\varepsilon := \int_0^t \tilde{\mu}_s^\varepsilon ds + \int_0^t \tilde{\sigma}_s dW_s + \int_0^t \int_{|z|\leq\varepsilon} \tilde{\gamma}_{s-}(z)\tilde{J}(ds\times dz),$$

$$\tilde{\mu}_s^\varepsilon = \tilde{\mu}_s - \int_{|z|>\varepsilon} \gamma_s(z)\nu(dz), \qquad \tilde{A}_t^\varepsilon = \tilde{\sigma}_s^2 + \int_{|z|\leq\varepsilon} \tilde{\gamma}_s^2(z)\nu(dz).$$

Then there exists $C < \infty$ such that for ε sufficiently small, $\forall n$,

$$E\left[\left(\int_0^T (\phi_{t-} - \phi_{l(t)})dS_t\right)^2\right] \leq \frac{C}{n} E[\sup_t A_t^* \sup_t \tilde{A}_t^\varepsilon] + \frac{C}{n^2} E[\sup_t A_t^* \sup_t (\tilde{\mu}_t^\varepsilon)^2],$$

where $l(t)$ is the closest rebalancing date to the left of t.

Example: exponential Lévy model

Let S follow a pure-jump exponential Lévy model

$$\frac{dS_t}{S_{t-}} = dX_t$$

$$X_t = bt + \int_0^t \int_{\mathbb{R}} z\tilde{J}(ds \times dz),$$

with the Lévy measure ν satisfying

$$\int_{|z|>1} |z|^p \nu(dz) < \infty, \quad p \geq 1.$$

Further, let $\phi_t = \phi(t, S_t)$ with $\phi(\cdot, \cdot)$ smooth on $[0, T] \times \mathbb{R}_+$ such that there exist $p \geq 0$ and $C < \infty$ with

$$\left|\frac{\partial \phi(t, S)}{\partial t}\right| + \left|\frac{\partial \phi(t, S)}{\partial S}\right| + \left|\frac{\partial^2 \phi(t, S)}{\partial S^2}\right| \leq C(1 + |S|^p).$$

Example: exponential Lévy model

In this example, $A^*_t = S_t^2(b^2 + \int_{\mathbb{R}} z^2\nu(dz))$, and it is easy to get

$$\tilde{A}^\varepsilon_t \leq C(1+|S_t|^p)\int_{|z|\leq\varepsilon} z^2\nu(dz),$$

$$|\tilde{\mu}^\varepsilon_t| \leq C(1+|S_t|^p)(1+\int_{|z|>\varepsilon}|z|\nu(dz)),$$

therefore

$$E\left[\left(\int_0^T (\phi_{t-}-\phi_{l(t)})dS_t\right)^2\right]$$

$$\leq \frac{C}{n}\int_{|z|\leq\varepsilon} z^2\nu(dz) + \frac{C}{n^2}\left(1+\int_{|z|>\varepsilon}|z|\nu(dz)\right)^2.$$

Example: exponential Lévy model

Suppose $\nu(\mathbb{R}) = \infty$, and let ε be chosen such that $T\lambda_\varepsilon = n$. Since

$$\lim_{\varepsilon \to 0} \int_{|z| \leq \varepsilon} z^2 \nu(dz) = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{1}{\lambda_\varepsilon} \left(1 + \int_{|z| > \varepsilon} |z| \nu(dz) \right)^2 = 0,$$

we have that

$$E\left[\left(\int_0^T (\phi_{t-} - \phi_{l(t)}) dS_t \right)^2 \right] = o((n + T\lambda_\varepsilon)^{-1}).$$

Moreover, if $\nu(dx) \sim \frac{dx}{|x|^{1+\alpha}}$ near zero then

$$E\left[\left(\int_0^T (\phi_{t-} - \phi_{l(t)}) dS_t \right)^2 \right] = O((n + T\lambda_\varepsilon)^{-2(1 \wedge 1/\alpha)}).$$

- Implementing this strategy does not require the knowledge of ν.

Concluding remarks

- For bounded functionals (e.g., for estimating the Value at Risk of a hedged position), the discretization error is dominated by the diffusion component.
- For unbounded functionals, the contribution of jumps is equally important.
- In pure jump models, the rate of L^2 convergence can be improved by jump-adapted rebalancing strategies.