Maximum principles for optimal control of FBSDE with jumps

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RICAM
Linz, October 20th 2008

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Outline

- Motivation: risk minimizing portfolio problem
- Maximum principles for optimal control of FBSDE driven by Lévy processes
  - a sufficient maximum principle
  - an equivalence principle
  - a Malliavin calculus approach
- Application to risk minimizing portfolios

Financial market set up

Filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$.
- A *risk free asset*, with unit price $S_0(t) = 1$ for all $t \in [0, T]$
- A *risky asset*, with unit price $S(t)$

$$dS(t) = S(t^-)[\mu(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}_0} \gamma(t,z)\tilde{N}(dt,dz)];\ S(0) > 0$$

- $B(t)$: $\mathcal{F}_t$-Brownian motion
- $\tilde{N}(dt,dz) = N(dt,dz) - \nu(dz)dt$: *compensation* of the *jump measure* $N(\cdot,\cdot)$ of a Lévy process $\eta(\cdot)$, ν being the *Lévy measure* of $\eta(\cdot)$.
- $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$
- $\mu(t), \sigma(t)$ and $\gamma(t,z)$: $\mathcal{F}_t$-predictable processes s.t. $\gamma(t,z) \geq -1 + \epsilon$ and

$$\int_0^T \left\{|\mu(t)| + \sigma^2(t) + \int_{\mathbb{R}_0} \gamma^2(t,z)\nu(dz)\right\}dt < \infty \quad \text{a.s.}$$

Risk minimizing portfolio problem

The wealth process A_u corresponding to a portfolio u is given by

$$\begin{cases} dA(t) = A(t^-)u(t)\left[\mu(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}_0} \gamma(t,z)\tilde{N}(dt,dz)\right] \\ A(0) = a > 0. \end{cases} \tag{1}$$

Pb: find $u^* \in \mathcal{A}_\mathcal{E}$ which minimizes the risk of the terminal wealth, i.e.

$$\inf_{u \in \mathcal{A}_\mathcal{E}} \rho(A_u(T)) = \rho(A_{u^*}(T))$$

where ρ is a *convex risk measure*, i.e. a map satisfying convexity, monotonicity and translation properties.

A representation of convex risk measures

A convex risk measure ρ can be represented as:

$$\rho(F) = \sup_{Q \in \mathcal{P}} \{E_Q[-F] - \zeta(Q)\} \tag{2}$$

for some family $\mathcal{P}$ of probability measures absolutely continuous wrt P and some convex "penalty" function $\zeta : \mathcal{P} \to \mathbb{R}$.
For example, the *entropic risk measure* is defined by:

$$\rho(F) := \sup_{Q \ll P} \{E_Q[-F] - H(Q, P)\}$$

where H is *the relative entropy*

$$H(Q, P) = E\left[\frac{dQ}{dP} \ln\left(\frac{dQ}{dP}\right)\right]$$

With the representation (2), the problem of *minimizing the risk of the terminal wealth* leads to a *stochastic differential game*.
(Mataramvura-Øksendal)

Representation of risk measures by BSDE

Definition: Define the *risk* $\rho_g(F)$ (associated to a convex function g) of a financial position F as

$$\rho_g(F) := \mathcal{E}_g[-F] := X_g^{-F}(0) \in \mathbb{R} \tag{3}$$

where $X_g^{-F}(0)$ is the value at $t = 0$ of the solution $X(t)$ of the BSDE:

$$\begin{cases} dX(t) = -g(X(t))dt + Y(t)dB(t) + \int_{\mathbb{R}_0} K(t,z)\tilde{N}(dt,dz) \\ X(T) = -F. \end{cases}$$

Remark: When $g(x) = \frac{1}{2}x^2$, then ρ_g coincides with the *entropic risk measure*.

Now, the risk minimizing portfolio problem

$$\inf_{u \in \mathcal{A}} \rho_g(A_u(T))$$

is equivalent to

$$\inf_{u \in \mathcal{A}} X_g^{-A_u(T)}(0) \tag{4}$$

where $X_g^{-A_u(T)}(t)$ is given by the BSDE

$$\begin{cases} dX(t) = -g(X(t))dt + Y(t)dB(t) + \int_{\mathbb{R}_0} K(t,z)\tilde{N}(dt,dz) \\ X(T) = -A_u(T). \end{cases}$$

and $A(t)$ is given by a SDE.
This is an example of a *stochastic control problem of a system of FBSDEs driven by Lévy processes*.

Forward system in the unknown process $A(t)$

$$\begin{cases} dA(t) = b(t, A(t), u(t))dt + \sigma(t, A(t), u(t))dB(t) \\ \qquad + \int_{\mathbb{R}} \gamma(t, A(t), u(t), z)\tilde{N}(dt, dz); \quad t \in [0, T] \\ A(0) = a \in \mathbb{R} \end{cases} \tag{5}$$

Backward system in the unknown processes $X(t), Y(t), K(t, z)$

$$\begin{cases} dX(t) = -g(t, A(t), X(t), Y(t), u(t))dt + Y(t)dB(t) \\ \qquad + \int_{\mathbb{R}} K(t, z)\tilde{N}(dt, dz); \quad t \in [0, T] \\ X(T) = cA(T), \quad c \in \mathbb{R} \setminus \{0\} \end{cases} \tag{6}$$

Admissible controls

Consider a subfiltration $\mathcal{E}_t \subseteq \mathcal{F}_t$ representing the information available to the controller at time t, e.g.

$$\mathcal{E}_t = \mathcal{F}_{(t-\delta)^+} \qquad (\delta > 0 \text{ constant})$$

i.e. the controller gets a *delayed* information flow

• Let $\mathcal{A}_{\mathcal{E}}$ denote the family of admissible controls, contained in the set of $\mathcal{E}_t$-predictable controls $u(\cdot)$ such that the system (5)–(6) has a unique strong solution.

• U: given convex set s.t. $u(t) \in U, \;\; \forall t \in [0, T]$

Optimal control problem

Performance functional:

$$J(u) = E\Big[\int_0^T f(t, A(t), X(t), Y(t), K(t,\cdot), u(t))dt \qquad (7)$$
$$+ h_1(X(0)) + h_2(A(T))\Big]; \qquad u \in \mathcal{A}_{\mathcal{E}}$$

where f, h_1, h_2 are given functions s.t.

$$E\Big[\int_0^T |f(t, A(t), X(t), Y(t), K(t,\cdot), u(t))|dt + |h_1(X(0))| + |h_2(A(T))|\Big] < \infty.$$

Find $\Phi_{\mathcal{E}} \in \mathbb{R}$ and $u^* \in \mathcal{A}_{\mathcal{E}}$ such that

$$\Phi_{\mathcal{E}} = \sup_{u \in \mathcal{A}_{\mathcal{E}}} J(u) = J(u^*) \qquad (8)$$

Hamiltonian

The Hamiltonian is defined by

$$
\begin{aligned}
& H(t, a, x, y, k, u, \lambda, p, q, r) \\
& \quad = f(t, a, x, y, k, u) + g(t, a, x, y, u)\lambda + b(t, a, u)p \\
& \quad + \sigma(t, a, u)q + \int_{\mathbb{R}_0} \gamma(t, a, u, z)r(z)\nu(dz)
\end{aligned} \tag{9}
$$

We assume that H is Frechet differentiable (C^1) in the variables a, x, y, k.

Pair of FBSDEs in the adjoint processes

Forward system in the unknown process $\lambda(t)$

$$\begin{cases} d\lambda(t) = & \dfrac{\partial H}{\partial x}(t, A(t), X(t), Y(t), K(t,\cdot), u(t), \lambda(t), p(t), q(t), r(t,\cdot))dt \\ & +\dfrac{\partial H}{\partial y}()dB(t) + \displaystyle\int_{\mathbb{R}_0} \nabla_k H() \tilde{N}(dt, dz) \\ \lambda(0) = & h_1'(X(0)) \quad (= \dfrac{dh_1}{dx}(X(0))) \end{cases} \tag{10}$$

Backward system in the unknown processes $p(t), q(t), r(t,\cdot)$

$$\begin{cases} dp(t) = & -\dfrac{\partial H}{\partial a}()dt + q(t)dB(t) + \displaystyle\int_{\mathbb{R}} r(t,z)\tilde{N}(dt, dz); \quad t \in [0, T] \\ p(T) = & c\lambda(T) + h_2'(A(T)) \end{cases} \tag{11}$$

Sufficient conditional maximum principle

Theorem 1: Let $\hat{u} \in \mathcal{A}_{\mathcal{E}}$ with corresponding solutions $\hat{A}, \hat{X}, \hat{Y}, \hat{K}, \hat{\lambda}, \hat{p}, \hat{q}, \hat{r}$. Suppose that

- The functions $x \to h_i(x)$, $i = 1, 2$ and

$$(a, x, y, k, u) \to H(t, a, x, y, k, u, \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot))$$

are *concave*, for all $t \in [0, T]$

- $\hat{u}(t) \in$ $\mathrm{argmax}_{v \in U} E[H(t, \hat{A}(t), \hat{X}(t), \hat{Y}(t), \hat{K}(t, \cdot), v, \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot)) \mid \mathcal{E}_t]$

Then (under some growth conditions) $\hat{u}(t)$ is an optimal control i.e.

$$J(\hat{u}) = \sup_{u \in \mathcal{A}_{\mathcal{E}}} J(u).$$

Proof

Choose $u \in \mathcal{A}$ with corresponding solutions $A, X, Y, K, \lambda, p, q, r$.
We write

$$\hat{H}(t) = H(t, \hat{A}(t), \hat{X}(t), \hat{K}(t, \cdot), \hat{u}(t), \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot))$$

$$H(t) = H(t, A(t), X(t), Y(t), K(t, \cdot), u(t), \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t)(t, \cdot))$$

and similarly with $\hat{f}(t), f(t), \ldots$ etc.

$$J(\hat{u}) - J(u) = I_1 + I_2,$$

where

$$I_1 = E\Big[\int_0^T \{\hat{f}(t) - f(t)\}dt\Big]$$

and

$$I_2 = E[h_1(\hat{X}(0)) - h_1(X(0)) + h_2(\hat{A}(T)) - h_2(A(T))].$$

Using definition of $H = f + g\lambda + bp + \sigma q + \int_{\mathbb{R}_0} \gamma r \nu(dz)$ we have

$$\begin{aligned} I_1 = E\Big[\int_0^T \{ & \hat{H}(t) - H(t) - (\hat{g}(t) - g(t))\hat{\lambda}(t) \\ & - (\hat{b}(t) - b(t))\hat{p}(t) - (\hat{\sigma}(t) - \sigma(t))\hat{q}(t) \\ & - \int_{\mathbb{R}_0} (\hat{\gamma}(t,z) - \gamma(t,z))\hat{r}(t,z)\nu(dz)\}dt\Big]. \end{aligned}$$

Since h_1 and h_2 are concave, we have

$$h_1(\hat{X}(0)) - h_1(X(0)) \geq (\hat{X}(0) - X(0))h_1'(\hat{X}(0)) = (\hat{X}(0) - X(0))\hat{\lambda}(0).$$

$$h_2(\hat{A}(T)) - h_2(A(T)) \geq (\hat{A}(T) - A(T))h_2'(\hat{A}(T)).$$

By the Itô formula and (6) and (11) we get

$$E[(\hat{X}(0 - X(0))\hat{\lambda}(0)] = E[(\hat{X}(T) - X(T))\hat{\lambda}(T)] \; (1)$$

$$- E\Big[\int_0^T (\hat{X}(t) - X(t))d\hat{\lambda}(t) + \int_0^T \hat{\lambda}(t)d(\hat{X}(t) - X(t)) \; (2)$$

$$+ \int_0^T \frac{\partial \hat{H}}{\partial y}(t)(\hat{Y}(t) - Y(t))dt + \int_0^T \int_{\mathbb{R}_0} \nabla_k \hat{H}(t,z)(\hat{K}(t,z) - K(t,z))\nu(dz)dt\Big]$$

$$(1) = E[(\hat{A}(T) - A(T))(\hat{p}(T) - h_2'(\hat{A}(T)))]$$

$$= E\Big[\int_0^T (\hat{A}(t) - A(t))d\hat{p}(t) + \int_0^T \hat{p}(t)d(\hat{A}(t) - A(t)) \; (3)$$

$$+ \int_0^T (\hat{\sigma}(t) - \sigma(t))\hat{q}(t)dt + \int_0^T \int_{\mathbb{R}_0} (\hat{\gamma}(t,z) - \gamma(t,z))\hat{r}(t,z)\nu(dz)dt$$

$$- E[(\hat{A}(T) - A(T))h_2'(\hat{A}(T))].$$

$$(3) = E\Big[\int_0^T \Big\{(\hat{b}(t) - b(t))\hat{p}(t) - (\hat{A}(t) - A(t))\frac{\partial \hat{H}}{\partial a}(t)\Big\}]$$

$$(2) = - E\Big[\int_0^T \Big\{(\hat{X}(t) - X(t))\frac{\partial \hat{H}}{\partial x}(t) - \hat{\lambda}(t)(\hat{g}(t) - g(t))\Big\} dt$$

we get

$$
\begin{aligned}
J(\hat{u}) - J(u) \\
&\geq E\Big[\int_0^T \Big\{\hat{H}(t) - H(t) - (\hat{A}(t) - A(t))\frac{\partial \hat{H}}{\partial a}(t) - (\hat{X}(t) - X(t))\frac{\partial \hat{H}}{\partial x}(t) \\
&- (\hat{Y}(t) - Y(t))\frac{\partial \hat{H}}{\partial y}(t) - \int_{\mathbb{R}_0} \nabla_k \hat{H}(t,z)(\hat{K}(t,z) - K(t,z))\nu(dz)\Big\} dt\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= E\Big[\int_0^T E\Big[\Big\{\hat{H}(t) - H(t) - (\hat{A}(t) - A(t))\frac{\partial \hat{H}}{\partial a}(t) - (\hat{X}(t) - X(t))\frac{\partial \hat{H}}{\partial x}(t) \\
&- (\hat{Y}(t) - Y(t))\frac{\partial \hat{H}}{\partial y}(t) - \int_{\mathbb{R}_0} \nabla_k \hat{H}(t,z)(\hat{K}(t,z) - K(t,z))\nu(dz)\Big\} \mid \mathcal{E}_t\Big] dt\Big].
\end{aligned}
$$

Since the function

$$(a,x,y,k,u) \to H(t,a,x,y,k,u,\hat{\lambda}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))$$

is concave, we have

$$\begin{aligned}\hat{H}(t) - H(t) \geq {} & \frac{\partial \hat{H}}{\partial a}(t)(\hat{A}(t) - A(t) + \frac{\partial \hat{H}}{\partial x}(t)(\hat{X}(t) - X(t)) \\ & + \frac{\partial \hat{H}}{\partial y}(t)(\hat{Y}(t) - Y(t)) + \int_{\mathbb{R}_0} \nabla_k \hat{H}(t,z)(\hat{K}(t,z) - K(t,z))\nu(dz) \\ & + \frac{\partial \hat{H}}{\partial u}(t)(\hat{u}(t) - u(t)). \qquad (12)\end{aligned}$$

Since $\hat{u}(t) \in$ $\operatorname{argmax} E[H(t,\hat{A}(t),\hat{X}(t),\hat{Y}(t),\hat{K}(t,\cdot),u,\hat{\lambda}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)) \mid \mathcal{E}_t]$ we deduce that

$$\begin{aligned}\frac{d}{du}E[H(t,\hat{A}(t),\hat{X}(t),\hat{Y}(t),\hat{K}(t,\cdot),u, \\ \hat{\lambda}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)) \mid \mathcal{E}_t]_{u=\hat{u}(t)}(\hat{u}(t) - u(t)) \geq 0\end{aligned}$$

i.e.

$$E\Big[\frac{\partial \hat{H}}{\partial u}(t)(\hat{u}(t) - u(t)) \mid \mathcal{E}_t\Big] \geq 0.$$

We conclude that

$$J(\hat{u}) - J(u) \geq 0.$$

Since this holds for all $u \in \mathcal{A}_{\mathcal{E}}$, $\hat{u}$ is optimal.

A partial information equivalence principle for FBSDE's

Drawback with the previous result: the concavity assumption. Now we remove this assumption and assume the following instead:

(A1) $\forall s \in [0, T)$ and all bounded $\mathcal{E}_s$-measurable RV $\theta(\omega)$ the control

$$\beta_s(t) = \theta(\omega)\chi_{(s,T]}(t) \; ; \; t \in [0, T]$$

is in $\mathcal{A}_{\mathcal{E}}$.

(A2) $\forall u, \beta \in \mathcal{A}_{\mathcal{E}}$ where β is bounded, $\exists \delta > 0$ s.t. the control

$$u(t) + y\beta(t) \; ; \; t \in [0, T]$$

belongs to $\mathcal{A}_{\mathcal{E}}$ $\forall y \in (-\delta, \delta)$.

Partial information equivalence principle

Theorem 2: suppose $u \in \mathcal{A}_{\mathcal{E}}$ with corresponding solutions $A, X, Y, K, \lambda, p, q, r$. Then the following are equivalent:

(i) $\frac{d}{dy} J(u + y\beta) \mid_{y=0} = 0$ for all $\beta \in \mathcal{A}_{\mathcal{E}}$

(ii)

$$E\left[\frac{\partial}{\partial u} H(t, A(t), X(t), Y(t), K(t,\cdot), u, \lambda(t), p(t), q(t), r(t,\cdot)) \mid \mathcal{E}_t\right] = 0$$

Sketch of Proof

Define

$$\alpha(t) = \frac{d}{dy} A_{u+y\beta}(t) \mid_{y=0}; \quad \xi(t) = \frac{d}{dy} X_{u+y\beta}(t) \mid_{y=0}$$

$$\eta(t) = \frac{d}{dy} Y_{u+y\beta}(t) \mid_{y=0}; \quad \zeta(t) = \frac{d}{dy} K_{u+y\beta}(t,z) \mid_{y=0}$$

Note that

$$\alpha(0) = 0$$

$$\alpha(T) = \frac{1}{c} \frac{d}{dy} X_{u+y\beta}(T) \mid_{y=0} = \frac{1}{c} \xi(T)$$

Assume that (i) holds. Then

$$0 = \frac{d}{dy} J(u + y\beta) \mid_{y=0}$$

$$= E\left[\int_0^T \left\{ \frac{\partial f}{\partial a}(t)\alpha(t) + \frac{\partial f}{\partial x}\xi(t) + \frac{\partial f}{\partial y}(t)\eta(t) + \nabla_k f(t,z)\zeta(t,z) + \frac{\partial f}{\partial u}(t)\beta(t)\right.\right.$$

$$+ h_1'(X(0))\xi(0) + h_2'(A(T))\alpha(T)]$$

Using the Itô formula and after some computations, we get

$$E\left[\int_0^T \frac{\partial H}{\partial u}(t)\beta(t)dt\right] = 0 \ ; \ \beta \in \mathcal{A}_{\mathcal{E}} \text{ bounded .}$$

In particular, this holds for all $\beta \in \mathcal{A}_{\mathcal{E}}$ of the form

$$\beta(t) = \beta_s(t,\omega) = \theta(\omega)\chi_{[s,T]}(t) \ ; \ t \in [0,T] \tag{13}$$

for a fixed $s \in [0,T)$ where $\theta(\omega)$ is a bounded $\mathcal{E}_s$-measurable RV.

This gives

$$E\left[\int_s^T \frac{\partial H}{\partial u}(t)\theta dt\right] = 0.$$

Differentiating with respect to s we arrive at

$$E\left[\frac{\partial H}{\partial u}(s)\theta\right] = 0.$$

Since this holds for all bounded $\mathcal{E}_s$-measurable random variables θ, we conclude that

$$E\left[\frac{\partial H}{\partial u}(s) \mid \mathcal{E}_s\right] = 0.$$

This proves that (i) $\Rightarrow$ (ii).
Conversely, since every bounded $\beta \in \mathcal{A}_{\mathcal{E}}$ can be approximated by linear combinations of controls β_s of the form (13), we can prove that (ii) $\Rightarrow$ (i) by reversing the above argument.

A Malliavin calculus approach

- Replace the adjoint processes p, q, r given by BSDEs by $\tilde{p}$, $\tilde{q}$, $\tilde{r}$ given directly in terms of the parameters and state of the system.
- Moreover, this approach allows non-Markovian systems.

$$\begin{cases} dA(t) = b(t, A(t), u(t), \omega)dt + \sigma(t, A(t), u(t), \omega)dB(t) \\ \qquad + \int_{\mathbb{R}} \gamma(t, A(t), u(t), z, \omega)\tilde{N}(dt, dz); \quad t \in [0, T] \\ A(0) = a \in \mathbb{R} \end{cases}$$

$$\begin{cases} dX(t) = -g(t, A(t), X(t), Y(t), u(t), \omega)dt + Y(t)dB(t) \\ \qquad + \int_{\mathbb{R}} K(t, z)\tilde{N}(dt, dz); \quad t \in [0, T] \\ X(T) = cA(T), \quad c \in \mathbb{R} \setminus \{0\} \end{cases}$$

$$J(u) = E\Big[\int_0^T f(t, A(t), X(t), Y(t), K(t, \cdot), u(t), \omega)dt + h_1(X(0)) + h_2(A(T), \omega)\Big];$$

$b(t, a, u, \omega)$ is $\mathcal{F}_t$-measurable for each constant a, u and similarly

$$\tilde{H}(t,a,x,y,k,u,\lambda,\omega) = f + \lambda g + b\tilde{p}(t) + \sigma\tilde{q}(t) + \int_{\mathbb{R}_0} \gamma\tilde{r}(t,z)\nu(dz),$$

where

$$\begin{aligned}\tilde{p}(t) &= K(t) + \int_t^T \frac{\partial H_0}{\partial a}(s)G(t,s)ds\\ \tilde{q}(t) &= D_t\tilde{p}(t)\\ \tilde{r}(t,z) &= D_{t,z}\tilde{p}(t)\end{aligned}$$

D_tF : *Malliavin derivative wrt* $B(\cdot)$ *(at* t*)* of a rv F.
$D_{t,z}F$: *Malliavin derivative wrt* $\tilde{N}(\cdot,\cdot)$ *(at* t,z*)* of F.

$$\tilde{p}(t) = K(t) + \int_t^T \frac{\partial H_0}{\partial a}(s)G(t,s)ds$$

with

$$K(t) = h_2'(A(T)) + c\tilde{\lambda}(T) + \int_t^T \frac{\partial f}{\partial a}(s)ds$$

$$H_0(s,a,x,u) = \tilde{\lambda}(s)g + K(s)b + D_sK(s)\sigma + \int_{\mathbb{R}_0} D_{s,z}K(s)\gamma\nu(dz)$$

$$G(t,s) = \exp\left(\int_t^s \{\frac{\partial b}{\partial a}(r) - \frac{1}{2}\left(\frac{\partial \sigma}{\partial a}(r)\right)^2\}dr + \int_t^s \frac{\partial \sigma}{\partial a}(r)dB(r)\right.$$

$$\left. + \int_t^s \int_{\mathbb{R}_0} \ln(1 + \frac{\partial \gamma}{\partial a}(r,z))\tilde{N}(dr,dz) + \int_0^t \int_{\mathbb{R}_0} [\ln(1 + \frac{\partial \gamma}{\partial a}) - \frac{\partial \gamma}{\partial a}]\nu(dz)dr \; ; s > t\right.$$

$\tilde{\lambda}(t)$ given by the same (forward) equation as for λ but H replaced by $\tilde{H}$:

$$d\tilde{\lambda}(t) = \frac{\partial \tilde{H}}{\partial x}(t, A(t), X(t), Y(t), K(t,\cdot), u(t), \tilde{\lambda}(t))dt$$

$$+ \frac{\partial \tilde{H}}{\partial y}()dB(t) + \int_{\mathbb{R}_0} \nabla_k \tilde{H}()\tilde{N}(dr,dz) \; ; \; t \in [0,T]$$

Main result

Theorem 3: Let $u \in \mathcal{A}_\varepsilon$ with corresponding solutions $A(t), X(t), Y(t), K(t,z)$ and $\tilde{\lambda}(t)$. Assume that $h_2'(A(T))$, $\lambda(T)$, $\frac{\partial f}{\partial a}(t)$ and $\frac{\partial H_0}{\partial a}(s)G(t,s)$ are Malliavin differentiable to all $s > t$ and satisfy the conditions for the use of duality formulae when necessary. Then the following are equivalent:

1. $\frac{d}{dy}J(u + y\beta) \mid_{y=0} = 0$ for all bounded $\beta \in \mathcal{A}_\mathcal{E}$
2. $E\left[\frac{d}{du}\tilde{H}(t, A(t), X(t), Y(t), K(t), u, \tilde{\lambda}(t))_{u=u(t)} \mid \mathcal{E}_t\right] = 0$ for a.a. $(t,\omega) \in [O,T] \times \Omega$.

Duality formulae for Malliavin derivatives

$$E\left[F\int_0^T \varphi(s)dB(s)\right] = E\left[\int_0^T \varphi(s)D_sFds\right]$$

$$E\left[F\int_0^T\int_{\mathbb{R}_0} \psi(s,z)\tilde{N}(ds,dz)\right] = E\left[\int_0^T\int_{\mathbb{R}_0} \psi(s,z)D_{s,z}F\nu(dz)ds\right],$$

valid for all Malliavin differentiable F and $\mathcal{F}_t$-predictable processes φ and ψ such that the integrals on the right converge absolutely.

Application to risk minimizing portfolios

- Wealth process $A(t) = A_u(t)$:

$$\begin{cases} dA(t) = u(t)\left[\alpha(t)dt + \beta(t)dB(t) + \int_{\mathbb{R}_0} \theta(t,z)\tilde{N}(dt,dz)\right] \\ A(0) = a > 0 \end{cases} \tag{15}$$

where α, β and θ are given predictable processes.
$u(t) = \pi(t)A(t^-)$: *amount* invested in the risky asset at time t.

- Corresponding BSDE for (X, Y, K) :

$$\begin{cases} dX(t) = -g(t, X(t), \omega)dt + Y(t)dB(t) + \int_{\mathbb{R}_0} K(t,z)\tilde{N}(dt,dz) \\ X(T) = -A_u(T) \end{cases} \tag{16}$$

g is a function s.t. $(t, \omega) \to g(t, x, \omega)$ is $\mathcal{F}_t$-predictable $\forall x$.

- Performance functional:

$$J(u) = X_u(0)$$

Modified Hamiltonian:

$$\tilde{H}(t,a,x,y,k,u,\lambda,\omega)$$
$$= \lambda g(t,x) + u\alpha(t)\tilde{p}(t) + u\beta(t)\tilde{q}(t) + \int_{\mathbb{R}_0} u\theta(t,z)\tilde{r}(t,z)\nu(dz),$$

where

$$\tilde{p}(t) = K(t) + \int_t^T \frac{\partial H_0}{\partial a}(s)G(t,s)ds = -\tilde{\lambda}(T)$$
$$\tilde{q}(t) = -D_t\tilde{\lambda}(T)$$
$$\tilde{r}(t,z) = -D_{t,z}\tilde{\lambda}(T)$$

and $\tilde{\lambda}(t)$ is given by:

$$d\tilde{\lambda}(t) = \tilde{\lambda}(t)g'(t,\hat{X}(t))dt; \quad \tilde{\lambda}(0) = 1,$$

Hence

$$\tilde{\lambda}(t) = \exp\left(\int_0^t g'(s,\hat{X}(s))ds\right) \; ; \; 0 \leq t \leq T.$$

Condition for an optimal control $\hat{u}(t)$:

$$E\left[\alpha(t)\tilde{\lambda}(T) + \beta(t)D_t\tilde{\lambda}(T) + \int_{\mathbb{R}_0} \theta(t,z)D_{t,z}\tilde{\lambda}(T)\nu(dz) \mid \mathcal{E}_t\right] = 0,$$

We can solve this *Malliavin-differential eq* in the unknown rv $\tilde{\lambda}(T)$:

$$\begin{aligned}\tilde{\lambda}(T) = E[\tilde{\lambda}(T)]\exp\Bigg(&\int_0^T \sigma(s)dB(s) - \frac{1}{2}\int_0^T \sigma^2(s)ds \\ &+ \int_0^T \int_{\mathbb{R}_0} \ln(1+\gamma(s,z))\tilde{N}(ds,dz) \\ &+ \int_0^T \int_{\mathbb{R}_0} \{\ln(1+\gamma(s,z)) - \gamma(s,z)\}\nu(dz)ds\Bigg)\end{aligned}$$

for some $\mathcal{F}_t$-predictable processes $\sigma(t)$ and $\gamma(t,z)$ such that

$$\alpha(t) + \beta(t)\sigma(t) + \int_{\mathbb{R}_0} \theta(t,z)\gamma(t,z)\nu(dz) = 0 \text{ for a.a. } t,\omega. \tag{17}$$

Condition (17) says that the measure Q defined by $dQ(\omega) = \frac{\tilde{\lambda}(T)}{E[\tilde{\lambda}(T)]}dP(\omega)$ on $\mathcal{F}_T$ is an ELMM for the process $A(t)$.

A special case

Suppose $g(t,x,\omega) = -c_0(t) + c(t)x$; $\mathcal{E}_t = \mathcal{F}_t$. Then the solution X_u of

$$dX(t) = -g(t,X(t),\omega)dt + Y(t)dB(t) + \int_{\mathbb{R}_0} K(t,z)\tilde{N}(dt,dz);\ X(T) = -A_u(T)$$

satisfies

$$X_u(0) = -E\left[\lambda_u(T)A_u(T) + \int_0^T \lambda_u(t)c_0(t)dt\right] \quad \text{with}$$

$$\lambda_u(t) = \exp\left(\int_0^t g'(s,X_u(s))ds\right) = \exp\left(\int_0^t c(s)ds\right).$$

The minimal risk is

$$\begin{aligned} X_{\hat{u}}(0) &= -E[\lambda_{\hat{u}}(T)]E_Q[A_{\hat{u}}(T)] - E\left[\int_0^T \lambda_{\hat{u}}(t)c_0(t)dt\right] \\ &= -aE[\lambda_{\hat{u}}(T)] - E\left[\int_0^T \lambda_{\hat{u}}(t)c_0(t)dr\right] \\ &= -aE\left[\exp\left(\int_0^T c(s)ds\right)\right] - \int_0^T E\left[c_0(t)\exp\left(\int_0^t c(s)ds\right)\right]dt. \end{aligned}$$