Optimal Decisions in Finite State Markov Chains:

Applications to Personal Finance

and Credit Risk Management

Mogens Steffensen

Linz

October 24, 2008

Picture Of A Break, Festival, 1892

G. Laska (19th C. Czech), Oil On Canvas

Private Collection, Munich, Germany

Based on (all articles joint with Holger Kraft, Goethe-University, Frankfurt)

- Optimal Consumption and Insurance: A Continuous-Time Markov Chain Approach *ASTIN Bulletin* 2008

- The Policyholder's Static and Dynamic Decision Making of Life Insurance and Pension Payments. *Blätter der DGVFM* 2008.

- Bankruptcy, Counterparty Risk, and Contagion. *Review of Finance* 2007

- Asset Allocation with Contagion and Explicit Bankruptcy Procedures. *Journal of Mathematical Economics* 2008

- Intro
- Life Insurance Modelling
- Credit Risk Modelling
- Life Insurance Decision Making
- Optimal Credit Risky Portfolios
- Outtro

Life insurance Modelling

$$\boxed{0^{\text{alive}}} \overset{\mu}{\longrightarrow} \boxed{1^{\text{dead}}}$$

Figure 1: Survival model

$$
\begin{aligned}
N(t) &= \#\{s \in (0,t], Z(s-) \neq 0, Z(s) = 1\} \\
M(t) &= N(t) - \int_0^t I(s)\,\mu(s, Y(s))\,ds \\
dB(t) &= \text{'death sum'}\,dN(t) \\
&\quad + I(t)\,\text{'pension sum'}\,d\varepsilon(t,n) \\
&\quad - I(t)\,\text{'premium rate'}\,dt
\end{aligned}
$$

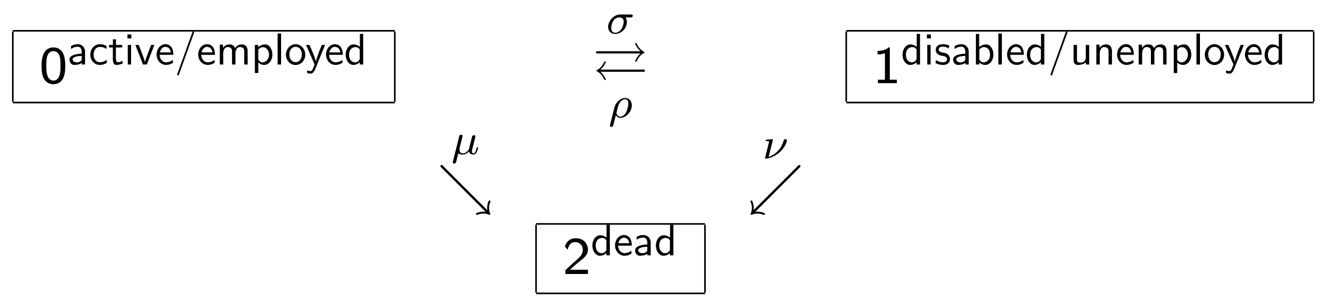


Figure 2: Disability/Unemployment model

$$
\begin{aligned}
N^k(t) &= \#\left\{s \in (0,t], Z(s-) \neq k, Z(s) = k\right\} \\
M^k(t) &= N^k(t) - \int_0^t \mu^k(s, Z(s), Y(s))\, ds \\
dB(t) &= \sum_{k: k \neq Z(t-)} b^{Z(t-)k} dN^k(t) + \left(\Delta B^{Z(t)}(t)\, d\varepsilon(t,n) + b^{Z(t)}(t)\, dt\right)
\end{aligned}
$$

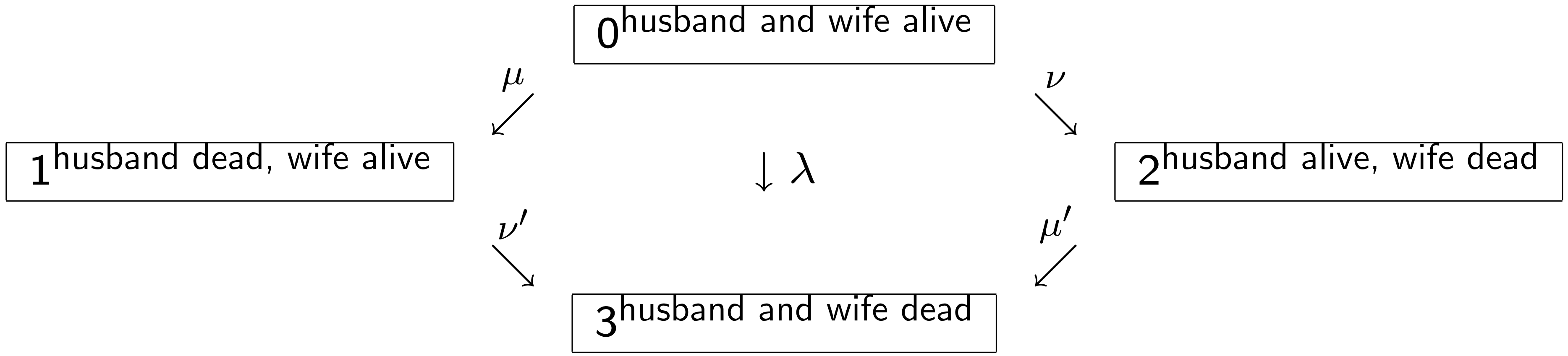


Figure 3: Dependent lives

'Weak' dependence induced by Y

'Medium' dependence induced by e.g. $\mu' > \mu$ or $\nu' < \nu$

'Strong' dependence induced by $\lambda > 0$

Credit Risk Modelling

Everybody knows by now what a 'survival model' looks like so therefore I show you instead a...

...'Default model'

$$\boxed{0^{\text{solvent}}} \overset{\mu}{\longrightarrow} \boxed{1^{\text{default}}}$$

Figure 4: Default model

$$\begin{aligned} N(t) &= \#\{s \in (0,t], Z(s-) \neq 0, Z(s) = 1\} \\ M(t) &= N(t) - \int_0^t I(s)\,\mu(s, Y(s))\,ds \\ dB(t) &= \text{'recovery payment'} dN(t) \\ &\quad + I(t)\,\text{'lump sum coupon'} d\varepsilon(t,n) \\ &\quad + I(t)\,\text{'coupon rate'} dt \end{aligned}$$

Everybody knows by now what a 'disability model' looks like so therefore I show you instead a...

...'Corporate rating model':

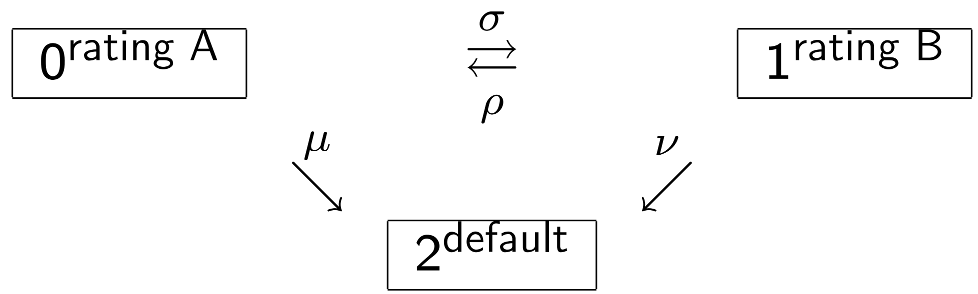


Figure 5: Rating model

$$
\begin{aligned}
N^k(t) &= \#\left\{s \in (0,t], Z(s-) \neq k, Z(s) = k\right\} \\
M^k(t) &= N^k(t) - \int_0^t \mu^k\left(s, Z(s), Y(s)\right) ds \\
dB(t) &= \sum_{k:k\neq Z(t-)} b^{Z(t-)k} dN^k(t) + \left(\Delta B^{Z(t)}(t)\, d\varepsilon(t,n) + b^{Z(t)}(t)\, dt\right)
\end{aligned}
$$

Everybody knows by now what a 'dependent lives model' looks like so therefore I show you instead a...

...'Dependent corporates model':

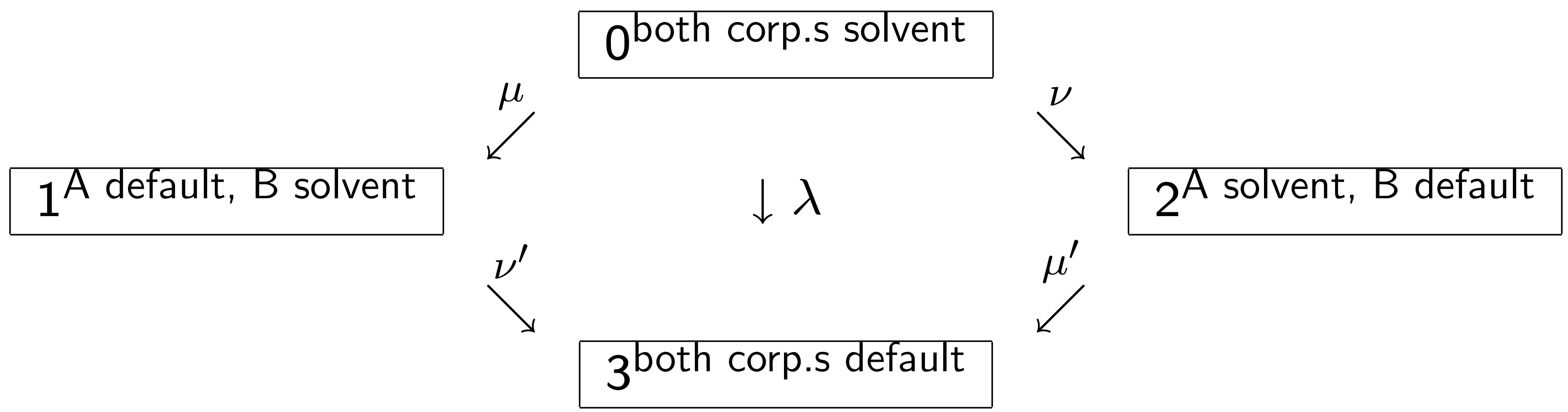


Figure 6: Dependent corporates

'Weak' dependence induced by Y

'Medium' dependence induced by e.g. $\mu' > \mu$ or $\nu' < \nu$

'Strong' dependence induced by $\lambda > 0$

Further comments to (lack of) analogies

- Evolutionary finance (life cycle, fertility, infection, adaptation etc.)
- Mortality linked derivatives (N or Y) versus credit derivatives (N or Y)
- Statistical inference versus calibration

Life Insurance Decision Making

The model,

$$
\begin{aligned}
dA(t) &= a^{Z(t)}(t)\,dt + \textstyle\sum_k a^{Z(t-)k}(t)\,dN^k(t) \\
dC(t) &= c^{Z(t)}(t)\,dt + \textstyle\sum_k c^{Z(t-)k}(t)\,dN^k(t)
\end{aligned}
$$

$$
dX(t) = rX(t)\,dt + dA(t) - dC(t) + \textstyle\sum_{k:k\neq Z(t-)} R^{Z(t-)k}(t)\,dM^{*k}(t)
$$

- the problem,

$$
\begin{aligned}
dU(t) &= \frac{1}{\gamma} w^{Z(t)}(t)^{1-\gamma} c(t)^{\gamma} dt \\
&+ \sum_k \frac{1}{\gamma} w^{Z(t-)k}(t)^{1-\gamma} c^k(t)^{\gamma} dN^k(t) \\
&+ \frac{1}{\gamma} \Delta W^{Z(t)}(t)^{1-\gamma} X(t)^{\gamma} d\varepsilon(t,n)
\end{aligned}
$$

$$
\sup_{R,C} E\left[\int_0^n dU(t)\right]
$$

- and its solution,

$$R^{jk}(t,x) + x + g^{k}(t) = \frac{f^{k}(t)}{f^{j}(t)} h^{jk}(t) \left(x + g^{j}(t)\right)$$

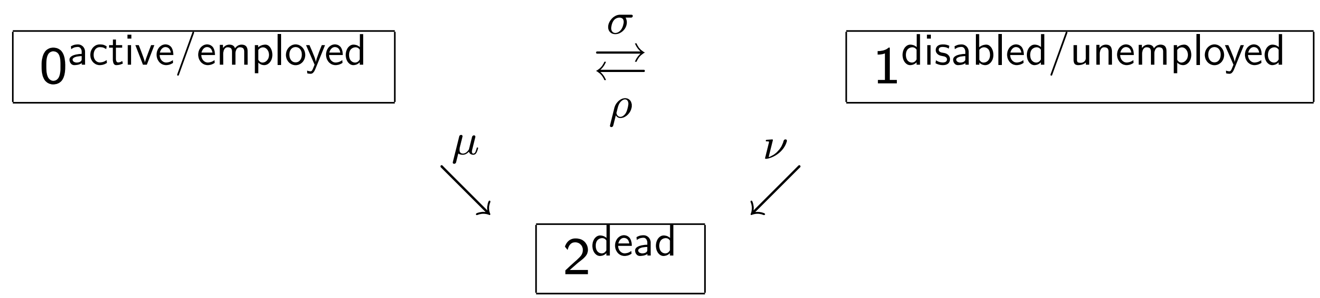
g is human wealth

f is a 'value' of future utility weights

h is a price ratio

Insure to protect your wealth up to a utility ratio and a price ratio

Example: Optimal disability annuity - for the survival model see Richard (1975) -



$$R^{01}(t,x) + x = \frac{f^1(t)}{f^0(t)} h^{01}(t) \left(x + g^0(t) \right)$$

Achieved by demanding e.g. the disability annuity rate $b^1(t)$ solving the relation

$$x + b^1(t) \int_t^n e^{-\int_t^s r+\nu^*} ds = \frac{f^1(t)}{f^0(t)} h^{01}(t) \left(x + g^0(t) \right)$$

Optimal Credit Risky Portfolios

The model (in case of no macro risk),

$$\frac{dX(t)}{X(t-)} = \left(r - \underbrace{\sum_k \left(\eta^k(t) - 1\right) \lambda^k(t) \sum_i \pi_i(t) R_i^k(t)}_{\text{excess return from event risk}} \right) dt + \sum_i \sum_k \pi_i(t-) \underbrace{R_i^k(t-)}_{\text{net gain from bond } i \text{ from event } k} dM^k(t)$$

$$\eta^k - 1 = \text{market price of risk}$$

- the problem,

$$\begin{aligned} G^{j}(t,x) &= \sup_{\pi} E_{t,x,j}\left[\int_{t}^{T}\frac{1}{\gamma}X^{\gamma}(T)\right] \\ &= \frac{1}{\gamma}x^{\gamma}f^{j}(t)^{1-\gamma}, \end{aligned}$$

- and its solution

Incomplete market: I (number of assets) non-linear equations with I unknowns

$$\sum_{k\neq j} R_v^{jk}\lambda^{jk}\eta^{jk} = \sum_{k\neq j} R_v^{jk}\lambda^{jk}\left(1+\sum_i \pi_i^j R_i^{jk}\right)^{\gamma-1}\left(\frac{f^k}{f^j}\right)^{1-\gamma}$$

Complete market: I linear equations with I unknowns

$$1+\sum_{i=1}^{I}\pi_i^j(t)\,R_i^{jk}(t) = \frac{f^k(t)}{f^j(t)}h^{jk}(t)$$

f is a 'value' of a terminal endowment 1

h is a price ratio (related to η)

Invest to protect your wealth up to a utility ratio and a price ratio

Example: Optimal investment under contagion

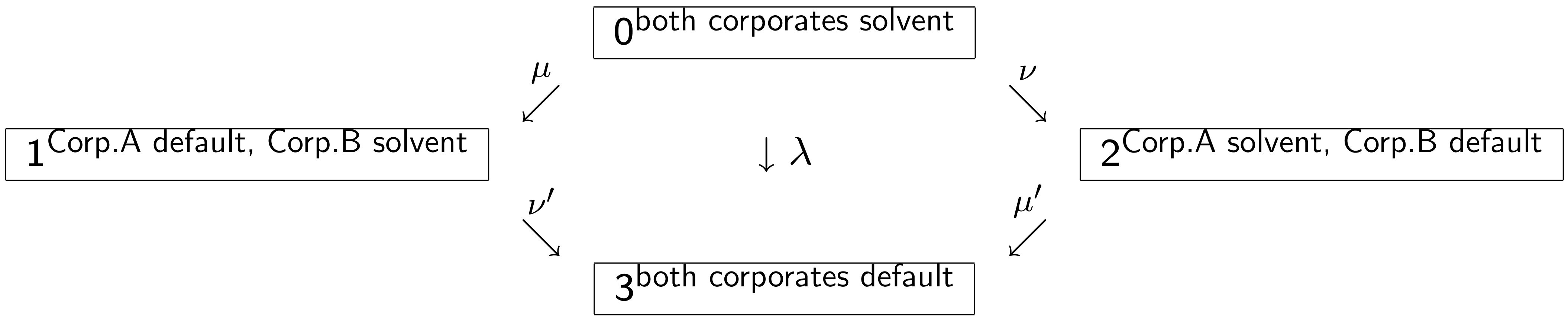


Figure 7: Dependent corporates

$$
\begin{aligned}
1 + \pi_A^0(t) R_A^{01}(t) + \pi_B^0(t) R_B^{01}(t) &= \frac{f^1(t)}{f^0(t)} h^{01}(t) \\
1 + \pi_A^0(t) R_A^{02}(t) + \pi_B^0(t) R_B^{02}(t) &= \frac{f^2(t)}{f^0(t)} h^{02}(t)
\end{aligned}
$$

FOC revisited to discuss the different market situations and decision processes

Insurance FOC without income:

$$
\begin{aligned}
R^{jk}(t,x) + x + \underbrace{g^{k}(t)}_{=0} &= \frac{f^{k}(t)}{f^{j}(t)} h^{jk}(t) \left(x + \underbrace{g^{j}(t)}_{=0} \right) \\
1 + \frac{R^{jk}(t,x)}{x} &= \frac{f^{k}(t)}{f^{j}(t)} h^{jk}(t)
\end{aligned}
$$

Credit FOC:

$$
1 + \sum_{i=1}^{I} \pi_i^{j}(t) R_i^{jk}(t) = \frac{f^{k}(t)}{f^{j}(t)} h^{jk}(t)
$$

Outtro

- Constraints (IME)
- Static problems (Blätter)
- Product design

Everybody knows what a 'Bellman equation' looks like so therefore I show you instead...

Figure 8: Richard Ernest Bellman (1920-1984)

"*I believe that the growth of vital mathematics depends crucially on continuing interaction with the real world" (1966)*