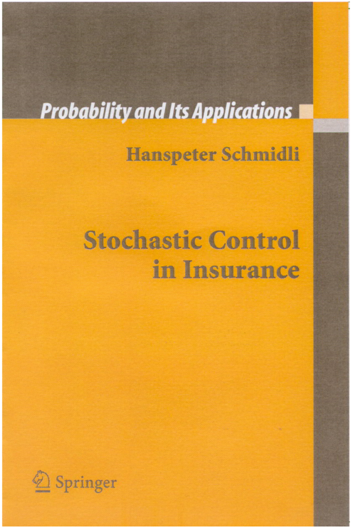
Controlled Risk Processes and Large Claims

Hanspeter Schmidli

University of Cologne

Optimization and Optimal Control
Linz, 21st of October


Probability and Its Applications
Hanspeter Schmidli
Stochastic Control
in Insurance
Springer

The Classical Risk Model

Lundberg (1903) intruduced the model

$$S_t = \sum_{i=1}^{N_t} Y_i \;,$$

$$X_t^0 = x + ct - S_t \;.$$

- x: Initial capital.
- c: Rate of the linear income, premium rate.
- $\{N_t\}$: Poisson process with rate λ.
- $\{Y_i\}$: iid sequence, distribution function $G(y)$, $G(0) = 0$.
- $\{N_t\}$ and $\{Y_i\}$ are independent.

Proportional Reinsurance

The insurer can buy proportional reinsurance, i.e. the insurer pays bY, the reinsurer pays $(1 - b)Y$ of a claim of size Y.

There is a premium at rate $c - c(b)$ the insurer has to pay. We assume

- $c(b)$ continuous.
- $c(b)$ increasing.
- $c(1) = c$.
- $c(0) < 0$.

Proportional Reinsurance

The insurer can at any time choose the retention level $b_t \in [0,1]$. Then the surplus process becomes

$$X_t^b = x + \int_0^t c(b_s)\,\mathrm{d}s - \sum_{i=1}^{N_t} b_{T_i-} Y_i\ .$$

Investment

The insurer can invest the surplus into a risky asset (Black-Scholes model)

$$Z_t = \exp\{(m - \tfrac{1}{2}\sigma^2)t + \sigma W_t\}\,.$$

$\{W_t\}$ and $\{S_t\}$ are independent.
Choosing a strategy $\{A_t\}$ the surplus process becomes

$$\mathrm{d}X_t^A = (c + A_t m)\,\mathrm{d}t + \sigma A_t\,\mathrm{d}W_t - \mathrm{d}S_t\;, \qquad X_0^A = x$$

Proportional Reinsurance and Investment

If the insurer can buy reinsurance and invest the surplus process fulfils

$$\mathrm{d}X_t^{A,b} = (c(b_t) + A_t m)\,\mathrm{d}t + \sigma A_t\,\mathrm{d}W_t - b_{t-}\mathrm{d}S_t\;, \qquad X_0^{A,b} = x$$

Heavy-Tailed Claims

We say a distribution function F is *heavy-tailed ($F \in \mathcal{H}$)* if

$$M_F(r) := \int_0^\infty \mathrm{e}^{rx} \, \mathrm{d}F(x) = \infty$$

for all $r > 0$.

We say a distribution function F is *long-tailed ($F \in \mathcal{L}$)* if

$$\lim_{x \to \infty} \frac{1 - F(x+y)}{1 - F(x)} = 1$$

for all $y \in \mathbb{R}$.

Subexponential Distributions

A distribution function $F(x)$ with $F(0) = 0$ is called *subexponential* $(F \in \mathcal{S})$ if

$$\lim_{x\to\infty} \frac{1 - F^{*n}(x)}{1 - F(x)} = n$$

for some (and therefore all) $n \geq 2$.
The definition can be interpreted as

$$\mathbb{P}\Big[\sum_{i=1}^{n} X_i > x\Big] \sim \mathbb{P}[\max\{X_1, \dots, X_n\} > x] \;.$$

The Class $\mathcal{S}^*$

A distribution function $F(x)$ is in $\mathcal{S}^*$ if it has finite mean μ_F and

$$\lim_{x\to\infty} \int_0^x \frac{(1-F(x-y))(1-F(y))}{1-F(x)} \,\mathrm{d}y = 2\mu_F \;.$$

Regularly Varying Tail

A distribution function $F(x)$ has a *regularly varying tail* with index $-\alpha$ ($F \in \mathcal{R}_{-\alpha}$) if

$$\lim_{x\to\infty} \frac{1-F(tx)}{1-F(x)} = t^{-\alpha} .$$

$$\mathcal{R}_{-\alpha} \subset \mathcal{S}^* \subset \mathcal{S} \subset \mathcal{L} \subset \mathcal{H} .$$

Moreover, the log-normal and the heavy-tailed Weibull distributions belong to $\mathcal{S}^*$. Thus $\mathcal{S}^*$ contains all heavy-tailed distribution functions of interest.

The Optimisation Problem

$$\tau^{Ab} := \inf\{t \geq 0 : X_t^{Ab} < 0\}\text{: time of ruin}$$
$$\psi^{Ab}(x) := \mathbb{P}[\tau^{Ab} < \infty]\text{: ruin probability}$$

Goal: Minimisation of the ruin probability

$$\psi(x) = \inf_{A,b} \psi^{Ab}(x)\ .$$

The problem is connected to the Hamilton-Jacobi-Bellman (HJB) equation

$$\inf_{b \in [0,1]} c(b)\psi'(x) + \lambda \mathbb{E}[\psi(x - bY) - \psi(x)] = 0\ .$$

Verification Theorem

Theorem (Hipp and Plum, S.)

Suppose there is an increasing (and twice continuously differentiable) function $f(x)$ such that $1 - f(x)/f(\infty)$ solves HJB. Then $f(x)$ is bounded and $f(x) = f(\infty)(1 - \psi(x))$. Moreover,

$$(A(X_t), b(X_t))$$

is an optimal strategy, where $A(x), b(x)$ are the arguments where the minimum in the HJB is taken.

Sketch of Proof

Proof.

Case with investment:
The process

$$f(X^A_{t\wedge\tau}) - \int_0^{t\wedge\tau} \Big\{ \frac{\sigma^2}{2} A_s^2 f''(X^A_s) + (c + mA_s) f'(X^A_s) + \lambda \mathbb{E}[f(X^A_s - Y) - f(X^A_s)] \Big\} \,\mathrm{d}s$$

is a local martingale. Using the HJB it follows that $f(X^A_{t\wedge\tau})$ is a supermartingale.

Sketch of Proof

Proof (continued).

Thus $f(x) \geq \mathbb{E}[f(X^A_{t\wedge\tau})]$. Choosing a strategy for which ruin is not certain shows that $f(x)$ is bounded. Thus

$$f(x) \geq f(\infty)\mathbb{P}[\tau = \infty] = f(\infty)(1 - \psi^A(x)) \; .$$

Choosing the optimal strategy gives that $f(X^*_{\tau\wedge t})$ is a bounded martingale. Thus

$$f(x) = \mathbb{E}[f(X^*_\tau)] = f(\infty)(1 - \psi^*(x)) \; .$$

□

Existence of a Solution

Theorem (S.)

Suppose $G(x)$ is continuous. Then $\psi(x)$ solves the HJB.

Theorem (Hipp and Plum, S.)

Suppose $G(x)$ is absolutely continuous with a bounded density. Then $\psi(x)$ is twice continuously differentiable and solves the HJB.

Proof.

Contraction arguments. □

Claim Sizes with a Regularly Varying Tail

Theorem

Suppose that the tail of the claim size distribution is regularly varying with index α*. Then*

$$\lim_{x\to\infty} \frac{\psi(x)}{\int_x^\infty (1-G(z))\,\mathrm{d}z} = \inf_{b\in(0,1]} \frac{\lambda b^\alpha}{(c(b)-\lambda\mu b)^+}\;.$$

If there is a unique value b^* *for which the infimum is taken we also have convergence of the strategy* $\lim_{x\to\infty} b(x) = b^*$.

Proof.

Choose $b_t = b$ constant such that $c(b) > \lambda\mu b$.

$$\lim_{x\to\infty} \frac{\psi^b(x)}{\int_x^\infty (1 - G(z))\,\mathrm{d}z} = \lim_{x\to\infty} \frac{\psi^b(x) b^\alpha}{\int_x^\infty (1 - G(z/b))\,\mathrm{d}z} = \frac{\lambda b^\alpha}{c(b) - \lambda\mu b}$$

Thus

$$\limsup_{x\to\infty} \frac{\psi(x)}{\int_x^\infty (1 - G(z))\,\mathrm{d}z} \le \inf_{b\in(0,1]} \frac{\lambda b^\alpha}{(c(b) - \lambda\mu b)^+}\;.$$

Proof (continued).

Let $g(x) = -\psi'(x)/(1 - G(x))$. Then for $b = b(x)$

$$\lambda\Big[\int_0^x g(z)\frac{(1-G(z))(1-G((x-z)/b))}{1-G(x)}\,\mathrm{d}z + \delta(0)\frac{1-G(x/b)}{1-G(x)}\Big] - c(b)g(x) = 0\,.$$

$g_0 = \liminf_{x\to\infty} g(x) > 0$.

Take a sequence $\{x_n\}$ such that $g(x_n) \to g_0$.

Take a subsequence such that $b(x_n) \to b_0$.

Proof (continued).

$$\int_0^{x_n/2} g(z)(1-G(z))\frac{1-G((x_n-z)/b(x_n))}{1-G(x_n)}\,\mathrm{d}z \le -C\int_0^{x_n/2}\psi'(z)\,\mathrm{d}z\;.$$

$$\int_0^{x_n/2} g(z)(1-G(z))\frac{1-G((x_n-z)/b(x_n))}{1-G(x_n)}\,\mathrm{d}z \to b_0^\alpha\psi(0)\;.$$

Proof (continued).

The second part of the integral is

$$\int_0^{x_n/2} g(x_n - z)\frac{1 - G(x_n - z)}{1 - G(x_n)}[1 - G(z/b(x_n))]\,\mathrm{d}z\;.$$

$\geq g_0 - \varepsilon$ $\qquad \geq 1$

lim inf bounded from below by $g_0 b_0 \mu$.

$$\lambda b_0^\alpha + \lambda\mu b_0 g_0 - c(b_0)g_0 \leq 0\;.$$

This shows that $c(b_0) - \lambda\mu b_0 > 0$

Proof (continued).

Thus

$$\liminf_{x\to\infty} \frac{-\psi'(x)}{1-G(x)} \ge \frac{\lambda b_0^\alpha}{c(b_0)-\lambda\mu b_0}\;.$$

Integration yields

$$\liminf_{x\to\infty} \frac{\psi(x)}{\int_x^\infty (1-G(z))\,\mathrm{d}z} \ge \frac{\lambda b_0^\alpha}{c(b_0)-\lambda\mu b_0}\;.$$

Proof (continued).

From

$$\lambda\Big[\int_0^x g(z)\frac{(1-G(z))(1-G((x-z)/b_0))}{1-G(x)}\,\mathrm{d}z + \delta(0)\frac{1-G(x/b_0)}{1-G(x)}\Big] - c(b_0)g(x) \quad\geq\quad 0\,.$$

we conclude that $g(x)$ is bounded.

Let $g_1 = \limsup_{x\to\infty} g(x)$ and choose a sequence $\{x_n\}$ such that $g(x_n) \to g_1$.

In the limit we get $\lambda b_0^\alpha + \lambda\mu b_0 g_1 - c(b_0)g_1 \geq 0$.

Proof (continued).

$$g_1 \le \frac{\lambda b_0^\alpha}{c(b_0) - \lambda\mu b_0} \; .$$

Thus $g(x) \to g_0$ converges.

Choose a sequence $\{x_n\}$ such that $b(x_n) \to b_1$.

$$\lambda b_1^\alpha + \lambda\mu b_1 g_0 - c(b_1) g_0 = 0 \; .$$

$$g_0 = \frac{\lambda b_1^\alpha}{c(b_1) - \lambda\mu b_1} \; .$$

If $b^* = b_0$ is unique then $b_1 = b^*$. □

Claim Sizes with a Rapidly Varying Tail

Proposition

Suppose that

$$\lim_{x\to\infty} \int_0^x \frac{(1-G(z))(1-G(x-z))}{1-G(x)} \, \mathrm{d}x = 2\mu$$

and that the distribution tail $1 - G(x)$ *is of rapid variation. Let* $b_0 = \inf\{b : c(b) > \lambda\mu b\}$. *Then for any* $b > b_0$

$$\lim_{x\to\infty} \frac{\psi(x)}{\int_x^\infty (1 - G(z/b)) \, \mathrm{d}z} = 0 \, .$$

For the strategy we obtain that $\limsup_{x\to\infty} b(x) = b_0$.

Comparision Rapid and Regular Variation

Remark

Regularly varying tails are more dangerous. But one chooses more reinsurance for rapidly varying tails. Strange?
Reinsurance makes tail considerably smaller for rapid variation whereas the premium is more important in the regular varying case.

Proof.

Let $b > b_0$.

Choose $b_0 < b_1 < b$ such that $c(b_1) > \lambda\mu b_1$.

Note that

$$\lim_{x\to\infty} \frac{\int_x^\infty 1 - G(z/b_1)\,\mathrm{d}z}{\int_x^\infty 1 - G(z/b)\,\mathrm{d}z} = 0\;.$$

Thus we can assume that $c(b) > \lambda\mu b$.

Proof (continued).

Let $g(x) = -\psi'(x)/(1 - G(x/b))$.

$$\lambda\Big[\int_0^x g(z)\frac{(1-G(z/b))(1-G((x-z)/b_1))}{1-G(x/b)}\,\mathrm{d}z + \delta(0)\frac{1-G(x/b_1)}{1-G(x/b)}\Big] - c(b_1)g(x) \quad \geq \quad 0\,.$$

We start showing that $g(x)$ is bounded.

Proof (continued).

Otherwise there is a sequence $\{x_n\}$ such that $g(x_n) = \sup_{0 \le x \le x_n} g(x)$.

$$\begin{aligned}
&\int_0^{x_n/2} \frac{g(z)}{g(x_n)} \frac{(1-G(z/b))(1-G((x_n-z)/b_1))}{1-G(x_n/b)} \,\mathrm{d}z \\
&\quad\le \varepsilon \int_0^{x_n/2} \frac{(1-G(z/b))(1-G((x_n-z)/b))}{1-G(x_n/b)} \,\mathrm{d}z \\
&\quad= \varepsilon b \int_0^{x_n/(2b)} \frac{(1-G(z))(1-G(x_n/b-z))}{1-G(x_n/b)} \,\mathrm{d}z\;.
\end{aligned}$$

$$\int_0^{x_n/2} \frac{g(z)}{g(x_n)} \frac{(1-G(z/b))(1-G((x_n-z)/b_1))}{1-G(x_n/b)} \,\mathrm{d}z \to 0\;.$$

Proof (continued).

The other part of the integral is

$$\int_0^{x_n/2} \frac{g(x_n - z)}{g(x_n)} \frac{(1 - G((x_n - z)/b))(1 - G(z/b_1))}{1 - G(x_n/b)} \,\mathrm{d}z$$

$$\leq \int_0^{x_n/2} \frac{(1 - G((x_n - z)/b))(1 - G(z/b))}{1 - G(x_n/b)} \,\mathrm{d}z$$

$$\to b\mu\ .$$

We find $\lambda\mu b - c(b) \geq 0$.

Thus $g(x)$ must be bounded.

Proof (continued).

Let $g_0 = \limsup_{x\to\infty} g(x)$.
In the limit we get $(\lambda\mu b - c(b))g_0 \ge 0$.

Thus $g_0 = 0$. Integration over (x, ∞) gives the result.

Suppose $b(x_n) \to b_1 > b_0$. $b_2 = (2b_0 + b_1)/3$, $b_3 = (b_0 + 2b_1)/3$.
$g(x) = -\psi'(x)/(1 - G(x/b_2)) \to 0$. But

$$\lim_{n\to\infty} \delta(0)\frac{1 - G(x_n/b(x_n))}{1 - G(x_n/b_2)} \ge \lim_{n\to\infty} \delta(0)\frac{1 - G(x_n/b_3)}{1 - G(x_n/b_2)} = \infty\ .$$

Thus $\limsup b(x) \le b_0$.
$\limsup b(x) < b_0$ would imply $\psi(x) = 1$. □

The HJB Equation

We suppose now that $\psi(x)$ solves HJB.
Taking the infimum, i.e. inserting

$$A(x) = -\frac{m\psi'(x)}{\sigma^2\psi''(x)}\;.$$

yields

$$-\frac{m^2\psi'(x)^2}{2\sigma^2\psi''(x)} + c\psi'(x) + \lambda\Bigl[\int_0^x \psi(x-y)\,\mathrm{d}G(y) + 1 - G(x) - \psi(x)\Bigr] = 0$$

The HJB Equation

Integration by parts yields

$$-\frac{m^2\psi'(x)^2}{2\sigma^2\psi''(x)} + c\psi'(x) + \lambda\delta(0)(1 - G(x)) - \lambda\int_0^x \psi'(x-y)(1-G(y))\,\mathrm{d}y = 0$$

$\delta(0) = 1 - \psi(0) \in (0, 1)$.

Regularly Varying Tail

Let

$$\kappa = \frac{2\lambda\sigma^2}{m^2}$$

Theorem (Gaier and Grandits)

Suppose $G \in \mathcal{R}_{-\alpha}$ with $\alpha > 1$. Then

$$\lim_{x\to\infty} \frac{\psi(x)}{1 - G(x)} = \kappa\,\frac{\alpha + 1}{\alpha}\;.$$

Complicated proof.

Subexponential Claim Sizes

One could now expect that

$$\psi(x) \sim C(1 - G(x))$$

for some $C > 0$ and all subexponential claims.

This is 'almost' true.

Asymptotics: Strong Conditions

If $G(x)$ is absolutely continuous we define the hazard rate

$$\ell(x) = \frac{G'(x)}{1 - G(x)}$$

Suppose $G \in \mathcal{S}^*$ and $\lim_{x\to\infty} \ell(x) = 0$.

Let $g(x) = -\psi'(x)/(1 - G(x))$. (stupid choice?)

Asymptotics: Strong Conditions

HJB divided by $1 - G(x)$

$$-\frac{m^2}{2\sigma^2}\frac{g(x)}{\ell(x) - \frac{g'(x)}{g(x)}} - cg(x) + \lambda\delta(0) + \lambda\int_0^x g(x-y)\frac{(1-G(x-y))(1-G(y))}{1-G(x)}\,\mathrm{d}y = 0$$

It follows that $\lim_{x\to\infty} g(x) = 0$.

Asymptotics: Strong Conditions

Integral

$$\int_0^{x/2} g(x-y)\frac{(1-G(x-y))(1-G(y))}{1-G(x)}\,\mathrm{d}y \to 0$$

$$\int_{x/2}^{x-x_0} g(x-y)\frac{(1-G(x-y))(1-G(y))}{1-G(x)}\,\mathrm{d}y \qquad \text{small}$$

$$\int_{x-x_0}^{x} g(x-y)\frac{(1-G(x-y))(1-G(y))}{1-G(x)}\,\mathrm{d}y$$

$$\to \int_0^{x_0} g(y)(1-G(y))\,\mathrm{d}y = \psi(0)-\psi(x_0)$$

Asymptotics: Strong Conditions

$$\int_0^x g(x-y)\frac{(1-G(x-y))(1-G(y))}{1-G(x)}\,\mathrm{d}y \to \psi(0)$$

$$\lim_{x\to\infty}\frac{\psi'(x)^2}{\psi''(x)(1-G(x))} = \frac{2\sigma^2\lambda}{m^2} = \kappa$$

Asymptotics: $\mathcal{S}^*$

Integration yields

$$\psi(x) \sim \kappa \int_x^\infty \frac{1}{\int_0^y \frac{1}{1-G(z)}\,\mathrm{d}z}\,\mathrm{d}y$$

By tail equivalence the result holds for all $G \in \mathcal{S}^*$.

Some sort of smothed version of the tail.

Asymptotics: Regularly Varying Tail

$$1 - G(x) \in \mathcal{R}_{-\alpha}$$
$$\int_0^y \frac{1}{1 - G(z)}\,\mathrm{d}z \in \mathcal{R}_{\alpha+1}$$
$$\int_x^\infty \frac{1}{\int_0^y \frac{1}{1 - G(z)}\,\mathrm{d}z}\,\mathrm{d}y \in \mathcal{R}_{-\alpha}$$

$$\lim_{x\to\infty} \frac{\psi(x)}{1 - G(x)} = \kappa\,\frac{\alpha+1}{\alpha}$$

Asymptotics: $\mathrm{MDA}(\exp\{-\mathrm{e}^{-x}\})$

Suppose that $G(y) \in \mathrm{MDA}(\exp\{-\mathrm{e}^{-x}\})$. Then $G(x)$ has the representation

$$1 - G(x) = c(x) \exp\Bigl\{-\int_0^x a(z)\,\mathrm{d}z\Bigr\}\,,$$

$c(x) \to 1$, $a(x) > 0$ absolutely continuous such that the density of $1/a(x)$ tends to zero. Because $G(y) \in \mathcal{S}^*$, $a(x)$ tends to zero.

Asymptotics: $\mathrm{MDA}(\exp\{-\mathrm{e}^{-x}\})$

Tail equivalent to

$$1 - \tilde{G}(x) = \exp\Bigl\{ - \int_0^x a(z) \,\mathrm{d}z \Bigr\} ,$$

$$\lim_{x\to\infty} -\frac{\tilde{G}''(x)(1-\tilde{G}(x))}{\tilde{G}'(x)^2} = 1 .$$

From L'Hospital's rule we conclude that

$$\lim_{x\to\infty} \frac{\psi(x)}{1-G(x)} = \kappa .$$

$\psi(x) \sim C(1 - G(x))$?

Suppose $G(y) \in \mathcal{S}$ and that $\psi(x) \sim C(1 - G(x))$ for some $C > 0$. Then

$$\limsup_{x\to\infty} \frac{-\psi'(x)}{1 - G(x)} \le \lim_{x\to\infty} \frac{\psi(x-1) - \psi(x)}{1 - G(x)} = 0 \;.$$

Analogously as before

$$\int_0^x \frac{\psi(x-y)}{1 - G(x)} \,\mathrm{d}G(y) \to C \;.$$

Thus

$$\lim_{x\to\infty} \frac{\psi'(x)^2}{\psi''(x)(1 - G(x))} = \frac{2\sigma^2\lambda}{m^2} = \kappa \;.$$

$\psi(x) \sim C(1 - G(x))$?

We conclude that either $1 - G(x) \in \mathcal{R}_{-\alpha}$ or $G(y) \in \mathrm{MDA}(\exp\{-\mathrm{e}^{-x}\})$. Thus $C = \kappa(\alpha + 1)/\alpha$ or $C = \kappa$.

Asymptotics of $A(x)$

Integration of

$$\lim_{x\to\infty} \frac{\psi'(x)^2}{\psi''(x)(1-G(x))} = \frac{2\sigma^2\lambda}{m^2} = \kappa$$

yields

$$A(x) = -\frac{m}{\sigma^2}\frac{\psi'(x)}{\psi''(x)} \sim \frac{m}{\sigma^2}\int_0^x \frac{1-G(x)}{1-G(z)}\,\mathrm{d}z$$

In particular, $A(x) \to \infty$.

Asymptotics of $A(x)$

If $G \in \mathcal{R}_{-\alpha}$, $\alpha \geq 1$, $\mu_G < \infty$ then

$$\lim_{x \to \infty} \frac{A(x)}{x} = \frac{m}{(\alpha + 1)\sigma^2} \; .$$

The strategy $A(x) = mx/((\alpha + 1)\sigma^2)$ yields an asymptotically optimal ruin probability.
Proof is analogously.

Asymptotics of $A(x)$

If $G(y) \in \mathcal{S}^* \cap \mathrm{MDA}(\exp\{-\mathrm{e}^{-x}\})$ then

$$A(x) \sim \frac{m}{\sigma^2} \int_0^x \exp\Bigl\{-\int_y^x a(z)\,\mathrm{d}z\Bigr\}\,\mathrm{d}y \sim \frac{m}{\sigma^2 a(x)}\;.$$

In particular, $A(x)/x \to 0$.

Also here, the strategy $A_t = m/(\sigma^2 a(X_t))$ yields asymptotically the optimal ruin probability.

Investment and Reinsurance

It is possible to reinsure the whole portfolio and then to speculate on the market.
Thus the ruin probability is basically the ruin probability of a Brownian motion with drift.
In particular, $\psi(x)$ is decreasing exponentially fast.

We assume that no exponential moments exist, i.e. $\mathbb{E}[e^{rY}] = \infty$ for all $r > 0$.

The Adjustment Coefficient

Let $b^* = 0$. For a constant strategy $A > -c(0)/m$ the adjustment coefficient is

$$R(A,0) = \frac{2(mA + c(0))}{\sigma^2 A^2}\,.$$

$R(A,0)$ becomes maximal for $A^* = -2c(0)/m$, thus $R = R(A^*,0) = -m^2/(2\sigma^2 c(0))$.

We find $\psi(x) < \psi^{A^*0}(x) = \mathrm{e}^{-Rx}$.

The HJB Equation

Let $f(x) = \psi(x)\mathrm{e}^{Rx}$ and $g(x) = -\psi'(x)\mathrm{e}^{Rx} = Rf(x) - f'(x)$. The HJB equation can be written as

$$\begin{aligned} -\frac{m^2}{2\sigma^2}\frac{g(x)^2}{Rg(x)-g'(x)} - c(b(x))g(x) \\ + \lambda \int_0^x g(x-y)[1-G(y/b(x))]\mathrm{e}^{Ry}\,\mathrm{d}y \\ + \lambda\delta(0)[1-G(x/b(x))]\mathrm{e}^{Rx} = 0\;. \end{aligned}$$

Asymptotics

Replacing $b(x)$ by 0

$$-\frac{m^2}{2\sigma^2}\frac{g(x)^2}{Rg(x)-g'(x)} - c(0)g(x) \ge 0\;.$$

This is equivalent to

$$0 \le -\frac{m^2}{2\sigma^2}\frac{Rg(x)}{Rg(x)-g'(x)} - Rc(0) = -\frac{m^2}{2\sigma^2}\frac{g'(x)}{Rg(x)-g'(x)}\;.$$

We see that $g'(x) \le 0$, that is $g(x)$ is decreasing. But then also $f'(x) \le 0$, that is $f(x)$ is decreasing.

Asymptotics

We have proved

Proposition

The functions $\psi(x)e^{Rx}$ *and* $-\psi'(x)e^{Rx}$ *are decreasing. In particular, there is a constant* $\zeta \in [0, \psi(0))$ *such that*

$$\lim_{x\to\infty} \psi(x)e^{Rx} = \zeta\;.$$

Asymptotics of $b(x)$

Suppose $b(x) \geq b > 0$.

$$\begin{aligned}\lambda \int_0^x g(x-y)[1-G(y/b(x))]\mathrm{e}^{Ry}\,\mathrm{d}y \\ &\geq \lambda \int_0^x g(x-y)[1-G(y/b)]\mathrm{e}^{Ry}\,\mathrm{d}y \\ &\geq \lambda \int_0^x [1-G(y/b)]\mathrm{e}^{Ry}\,\mathrm{d}y\; g(x)\;.\end{aligned}$$

Asymptotics of $b(x)$

Thus

$$\begin{aligned}\lambda \int_0^x g(x-y)[1-G(y/b(x))]\mathrm{e}^{Ry}\,\mathrm{d}y \\ + \lambda\delta(0)[1-G(x/b(x))]\mathrm{e}^{Rx} - c(b(x))g(x) \\ \geq \quad g(x)\Bigl[\lambda \int_0^x [1-G(y/b)]\mathrm{e}^{Ry}\,\mathrm{d}y - c\Bigr]\,.\end{aligned}$$

For x large enough this is larger than $-c(0)g(x)$.
Thus $b(x) < b$ for x large enough.
We have proved that

$$\lim_{x\to\infty} b(x) = 0\,.$$

Asymptotics of $A(x)$

The definition of R gives

$$\Bigl(c(0) + \frac{m^2}{2\sigma^2 R}\Bigr) g(x) = 0 \;.$$

Adding this to the HJB equation

$$-\frac{m^2}{2\sigma^2 R} \frac{g'(x)g(x)}{Rg(x) - g'(x)} - (c(b(x)) - c(0))g(x) + \lambda \int_0^x g(x-y)[1 - G(y/b(x))]\mathrm{e}^{Ry} \,\mathrm{d}y + \lambda\delta(0)[1 - G(x/b(x))]\mathrm{e}^{Rx} = 0 \;.$$

The only negative part is

$$-(c(b(x)) - c(0))g(x) \;.$$

Is R the Correct Exponent?

We know that $b(x) \to 0$. Choose b_0 such that $c(b_0) < 0$. There is x_0, such that $b(x) < b_0$ for all $x \geq x_0$.

Consider the following risk process:

$$b_t = \begin{cases} b(X_t)\;, & \text{if } x < x_0, \\ 0\;, & \text{if } x \geq x_0. \end{cases}$$

$$c(x) = \begin{cases} c(b(X_t))\;, & \text{if } x < x_0, \\ c(b_0)\;, & \text{if } x \geq x_0. \end{cases}$$

The investment A_t is chosen in an optimal way, in particular, $A(x) = -2c(b_0)/m$ for $x \geq x_0$.

Then $\hat{\psi}(x) < \psi(x)$.

Is R the Correct Exponent?

If $x \ge x_0$, ruin occurs by passing the capital x_0. Thus

$$\psi(x) > \hat{\psi}(x) = \hat{\psi}(x_0) \exp\Big\{ \frac{2m^2}{2\sigma^2 c(b_0)} (x - x_0) \Big\}$$

for $x \ge x_0$.
This means

$$\lim_{x \to \infty} \frac{-\log \psi(x)}{x} \le -\frac{2m^2}{2\sigma^2 c(b_0)} \;.$$

This holds for all b_0, thus

$$\lim_{x \to \infty} \frac{-\log \psi(x)}{x} \le -\frac{2m^2}{2\sigma^2 c(0)} = R \;.$$

Positive Limit

Proposition

Suppose that there exists $K > 0$ *such that* $c(b) - c(0) \le Kb$*. Suppose, moreover, that there are constants* $\alpha > 0$ *and* $0 < \gamma < \frac{1}{2}$ *such that*

$$1 - G(y) \ge \alpha \exp\{-x^\gamma\} .$$

Then $\zeta = \lim_{x\to\infty} \psi(x) e^{Rx} > 0$*.*

Full Reinsurance

Proposition

1. *If*

$$\limsup_{b\downarrow 0} \frac{c(b)-c(0)}{b} > \lambda \mathbb{E}[Y]\,,$$

then $b(x) > 0$ *for all* x.

2. *If*

$$\limsup_{b\downarrow 0} \frac{c(b)-c(0)}{b} < \lambda \mathbb{E}[Y]\,,$$

then $b(x) = 0$ *for all* x *large enough.*

References

- Gaier, J. and Grandits, P. (2002). Ruin probabilities in the presence of regularly varying tails and optimal investment. *Insurance Math. Econom.* **30**, 211–217.
- Hipp, C. and Plum, M. (2000). Optimal investment for insurers. *Insurance Math. Econom.* **27**, 215–228.
- Hipp, C. and Schmidli, H. (2004). Asymptotics of ruin probabilities for controlled risk processes in the small claims case. *Scand. Actuarial J.*, 321–335.
- Schmidli, H. (2001). Optimal proportional reinsurance policies in a dynamic setting. *Scand. Actuarial J.*, 55–68.

References

- Schmidli, H. (2002). On minimising the ruin probability by investment and reinsurance. *Ann. Appl. Probab.* **12**, 890–907.
- Schmidli, H. (2004). Asymptotics of ruin probabilities for risk processes under optimal reinsurance policies: the large claim case. *Queueing Syst. Theory Appl.* **46**, 149–157.
- Schmidli, H. (2005). On optimal investment and subexponential claims. *Insurance Math. Econom.* **36**, 25–35.
- Schmidli, H. (2008). *Stochastic Control in Insurance.* Springer-Verlag, London.