On CAT Options and Bonds

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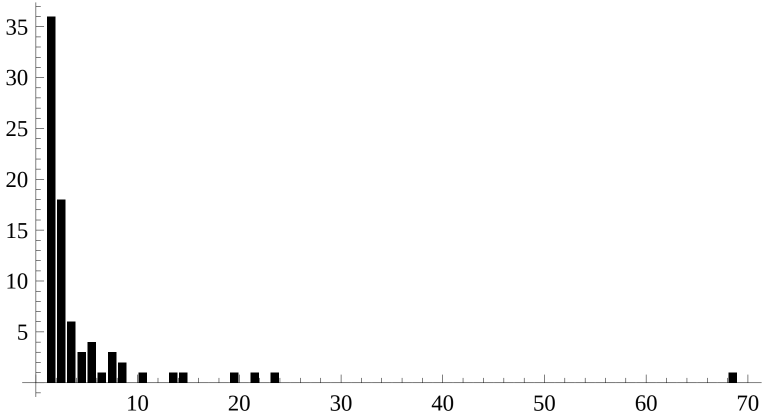
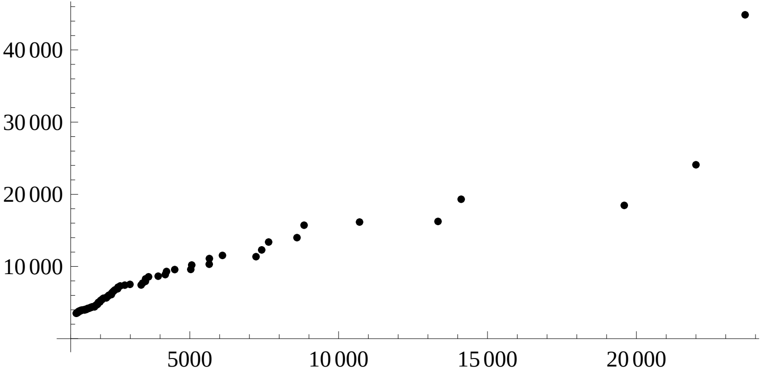
Catastrophic Claims: In Mil. USD

Event	Date	Claim
Hurricane Katrina	25.08.05	68 515
Hurricane Andrew	23.08.92	23 654
Terrorist attacks in US	11.09.01	21 999
Northridge Earthquake	17.01.94	19 593
Hurricane Ivan	02.09.04	14 115
Hurricane Wilma	19.10.05	13 339
Hurricane Rita	20.09.05	10 704
Hurricane Charley	11.08.04	8 840
Typhoon Mireille	27.09.91	8 599
Hurricane Hugo	15.09.89	7 650
Storm Daria	25.01.90	7 413
Storm Lothar	25.12.99	7 223

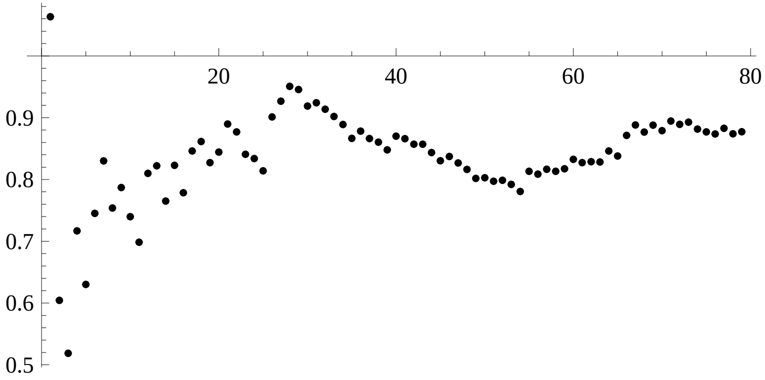
Catastrophic Claims (80 Events)



Empirical Mean Residual Life



The Hill Plot



The Index of Regular Variation

Estimation of the index of regular variation $\alpha = 1.14027$.

Variance does not exist!

Mean value of the estimated distribution: 8 441 Mil

Empirical mean 4 653 Mil. (-45%)

New Record

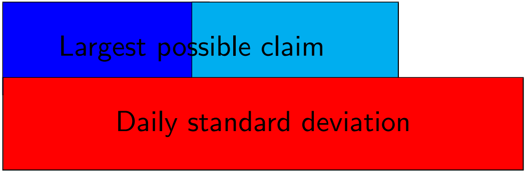
Mean value of a new record: 556 968. (8.1 times Hurricane Katrina, 23.5 times Hurricane Andrew)

Reinsurance? Risk is too large for the insurance industry.

Financial market could bear risk without problems.

Need for new financial products that take over the rôle of reinsurance.

Largest Possible Claim — Financial Market



The Use of CAT Products

Insurers use the CAT product as a substitute for reinsurance.

Insurance claims are supposed to be nearly independent of the financial market. Kobe earthquake? 9/11?

Counterparty risk is reduced because the credit risk is spread amongst investors.

Investors can use CAT products for diversification.

The ISO Index

The underlying index for the CAT future is the ISO (Insurance Service Office, a statistical agent) index.

About 100 American insurance companies report property loss data to the ISO. ISO then selects for each of the used indices a pool of at least ten of these companies on the basis of size, diversity of business, and quality of reported data. The ISO index is then

$$I_t = \frac{\text{reported incurred losses}}{\text{earned premiums}} .$$

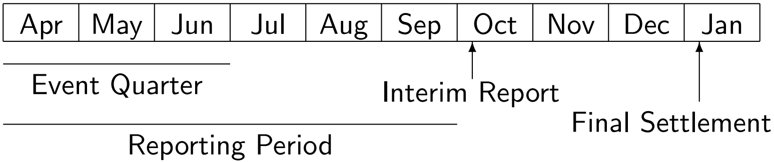
The pool is known at the beginning of the trading period.

The CAT Futures

$$F_{T_1} = \min\left\{\frac{I_{T_1}}{\Pi}, 2\right\} \times 25\,000\$$$

Π: Premium volume

I: ISO index



Hurricane Andrew: $I = 1.7\Pi$.

The CAT Futures

No success because poorly designed.
Reasons:

- Index only announced twice
- Information asymmetry
- Lack of realistic models
- Moral hazard problem
- Index may not match losses (slow reporting)
- The insurer cannot define a layer for which the protection holds.

The PCS Indices

A PCS index is an estimate of (insurance) losses in 100 Mio$ occuring from catastrophes in a certain region in a certain period. There are 9 indices:

- National
- Eastern
- Northeastern
- Southeastern
- Midwestern
- Western
- California
- Florida
- Texas

The PCS Option

A PCS option is a spread traded on a PCS index

$$\begin{aligned} F_{T_2} &= \min\{\max\{L_{T_2} - A, 0\}, K - A\} \\ &= \max\{L_{T_2} - A, 0\} - \max\{L_{T_2} - K, 0\}\,. \end{aligned}$$

L_{T_2} is PCS's estimate at time T_2 of the losses from catastrophes occurring in $(0, T_1]$ in a certain region.
The occurrence period $(0, T_1]$ is 3,6 or 12 months.
The development period $(T_1, T_2]$ is 6 or 12 months.

The PCS Option

The value of a basis point is $200.

When a catastrophe occurs, PCS makes a first estimate and then continues to reestimate the claim.

The option expires after a development period of at least six months following the occurrence period.

The index is announced daily which simplifies trading.

Moreover, there is no information asymmetry.

How Does the PCS Option Work

An insurer chooses first the layer. Then he estimates the market share and its loss experience compared to the whole market. From that the strike values and the number of spreads is calculated.

In this way one gets the desired reinsurance if the estimates coincide with the incurred liabilities.

How Does the PCS Option Work

Example

An insurer wants to hedge catastrophes.

- The layer should be 6 Mio in excess of 4 Mio, i.e. with strike values 4 Mio and 10 Mio.
- He estimates the market share to 0.2%.
- He estimates his exposure to 80% of the industry in average.

How Does the PCS Option Work

Example (cont)

Lower strike:

$$4 \times \frac{1}{0.002} \times \frac{1}{0.8} = 2500 = 25\text{pt} \ .$$

Upper strike:

$$10 \times \frac{1}{0.002} \times \frac{1}{0.8} = 6250 = 62.5\text{pt} \ .$$

Strikes are only available at 5 pt intervals, thus 25/65 are chosen as strike prices (10.4 Mio upper strike value).

How Does the PCS Option Work

Example (cont)

The number of spreads needed is

$$\frac{6\,000\,000}{200(65-25)} = 750\,.$$

The Act-of-God Bond

A bond with coupons, usually at a high rate.

The coupons and/or principal are at risk,
i.e. if a well-specified event occurs the coupon(s)/principal will not be paid (back).

Possible variant, principal will be paid back with a delay.

For the insurer, the coupons (and the principal) serve as a sort of reinsurance payment.

The Act-of-God Bond: An Example

Riskless interest rate:	2%
Coupon rate:	4%
Probability of the event:	5%

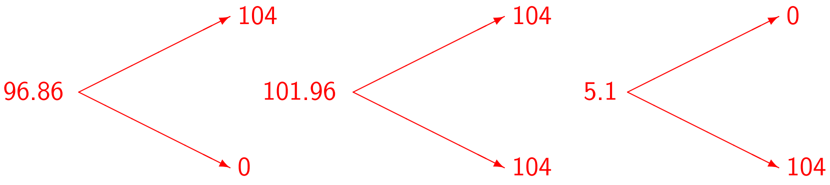
Price of the Act-of-God bond:

$$\frac{1}{1.02}(0.95 \times 104 + 0.05 \times 0) = 96.86\,.$$

Price of a riskless bond with coupon rate 4%:

$$\frac{1}{1.02}104 = 101.96\,.$$

The Act-of-God Bond: An Example



The Act-of-God Bond: An Example

Suppose the principal is (in case of the event) paid back in ten years, but the coupon is not valid. The price becomes then

$$0.95\frac{1}{1.02}104 + 0.05\frac{1}{(1.02)^{10}}100 = 100.96\,.$$

That is the insurer gets a reinsurance of 4 and an interest-free loan of 100 in the case of the event. The price of this agreement is 1.

Examples of Act-of-God Bonds

- USAA hurricane bonds issued in 1997 secured losses due to a Class-3 or stronger hurricane on the Gulf or East coast. If losses exceed 1 billion $ the coupon rate starts to be reduced, at 1.5 billion $ the coupons are completely lost.
- Winterthur Hailstorm bonds issued in 1997. Coupon lost if a (hail) storm damaged more than 6000 cars in the portfolio of Winterthur. There was also a conversion option.
- Swiss Re California Earthquake bonds: Based on PCS index.
- Swiss Re Tokyo region Earthquake bonds: Triggering event is a certain strength on the Richter scale.

Assumptions

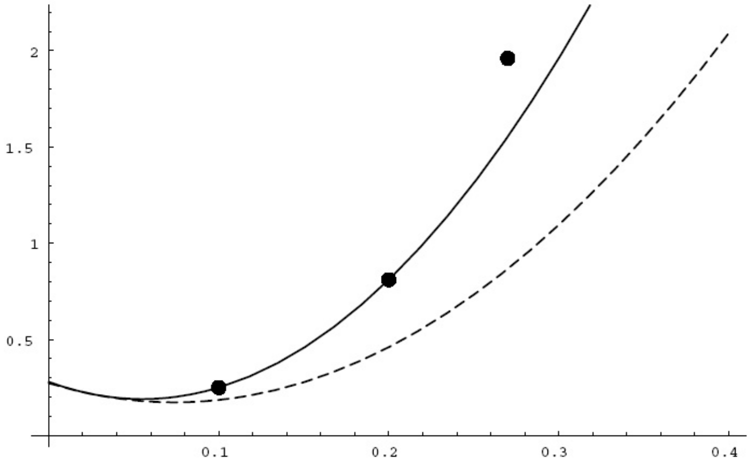
We assume that insurance market and financial market are independent. Kobe earthquake? 9/11?

There is no credit risk.

Markets are liquid and efficient.

Problem: Products are risky and therefore low rated. Many investment funds or pension schemes are not allowed to invest in products lower than A rated.

Improvement of the Efficient Frontier



Cummins and Geman (1993)

First model described in the literature.

$$I_t = \int_0^t S_s \, \mathrm{d}s$$

where

$$\mathrm{d}S_t = \mu S_t \, \mathrm{d}t + \sigma S_t \, \mathrm{d}W_t + k \, \mathrm{d}N_t$$

$\{W_t\}$: is a standard Brownian motion.
$\{N_t\}$ is a Poisson process.
Application of techniques from pricing Asian options.
Model is far from reality, but was used in practise.

Aase (1994)

Model based on actuarial modelling.
Catastrophes occurring according to a Poisson process,
losses iid gamma distributed,

$$I_t = \sum_{i=1}^{N_{t \wedge T_1}} Z_i$$

Exponential utility approach.

Reporting lags: $\{Z_i\}$ should be stochastically decreasing.

Gamma assumption, heavy-tailed distribution?

γ small approximates heavy tails. OK because index is capped.
Aase chooses for γ a natural number.

Embrechts and Meister (1995)

Catastrophes (reported claims) doubly stochastic Poisson process iid claim sizes.

$$I_t = \sum_{i=1}^{N_{t \wedge T_1}} Z_i$$

Exponential utility approach.

I_{T_2} determines change of measure:
Pricing exclusively by investors?
I_{T_2} aggregate loss seen by representative agent?

Christensen and S. (2000)

Reporting lags explicitly taken into account

$$I_t = \sum_{i=1}^{N_{t\wedge T_1}} \sum_{j=1}^{M_i} Y_{ij} \mathbb{1}_{E_{ij}+\tau_i \le t}$$

$\{N_t\}$ number of catastrophes,
$\{M_i\}$ iid, number of individual claims,
$\{Y_{ij}\}$ iid, claim size
$\{E_{ij}\}$ iid, reporting lag.
$\{\tau_i\}$ time of i-th catastrophe.

Exponential utility function is chosen (based on I_∞).
Heavy tails can be approximated.

Cummins and Geman (1993) / Aase (1994)

The same models as for the CAT-futures can be used.

Re-estimation?

Schradin and Timpel (1996)

$$P_t = P_0 \exp\{X_t\}$$

$t \in (0, T_1]$: $\{X_t\}$ increasing compound Poisson process,
$t \in (T_1, T_2]$: $\{X_t\}$ is a Brownian motion.

Index behaves differently in the two periods.

Motivation: in HARA utility framework pricing by Esscher measure.

Exponential Lévy process?

Christensen (1999)

Similar model,
$t \in (T_1, T_2]$: $\{X_t\}$ compound Poisson process with normally distributed increments.

Motivated by re-estimation procedure.

Main problems of model by Schradin and Timpel not solved.

The Model

We model the PCS index as

$$P_t = \sum_{i=1}^{N_{T_1 \wedge t}} P_t^i \,.$$

$\{N_t\}$ is an inhomogeneous Poisson process counting the numbers of catastrophes.
The index of the i-th catastrophe, occurring at time τ_i is

$$P^i_{\tau_i+t} = Y_i \exp\Big\{ \int_0^t \sigma(s)\,\mathrm{d}W_s^i - \tfrac{1}{2}\int_0^t \sigma(s)^2\,\mathrm{d}s \Big\} \,.$$

Y_i are the first estimates, $\{W_t^i\}$ are independent Brownian motions.

The Model

The first estimates $\{Y_i\}$ are iid.

$\int_0^\infty \sigma(s)^2 \, \mathrm{d}s < \infty$.

Note that $\{P^i_{\tau_i+t}\}$ is a martingale, i.e. estimates are unbiased.

The final estimate P_∞ can be seen as the accumulated claims from the catastrophes.

Biagini, Bregman and Meyer-Brandis

A similar model is considered by Biagini et al. They model the index as

$$\sum_{k=1}^{N_{t \wedge T_1}} Y_k A^k_{t-T_k} \;,$$

where $\{A^k_t\}$ are independent martingales.

Cumins and Geman

For the index

$$P_t = \int_0^t S_s \, \mathrm{d}s$$

the pricing problem is the same as pricing Asian Options. Thus

$$\pi_t = \mathbb{E}_{\mathbb{Q}}\Bigl[\mathrm{e}^{-r(T_2-t)}\Bigl(\int_0^{T_2} S_s \, \mathrm{d}s - A\Bigr)_+ \Bigm| \mathcal{F}_t\Bigr] \;.$$

Cummins and Geman use the equivalent martingale measure.

Problem: neither S_t nor P_t are traded indices!

Cumins and Geman

Yor has, using results of Kingman on Bessel processes, calculated the Laplace transform

$$\int_0^\infty \mathbb{E}_{\mathbb{Q}}\Big[\mathrm{e}^{-rT_2}\Big(\int_0^{T_2} S_s \,\mathrm{d}s - A\Big)_+\Big]\mathrm{e}^{-\beta A}\,\mathrm{d}A\,.$$

Inversion gives the price at time zero.
At time t, the price can be obtained by using the strike price

$$\Big(A - \int_0^t S_s \,\mathrm{d}s\Big) \Big/ S_t\,.$$

If the strike price becomes negative, the pricing problem is simple.

Aase

In an exponential utility approach the index

$$P_t = \sum_{i=1}^{N_{t \wedge T_1}} Z_i$$

behaves under the pricing measure in the same way but with changed parameters. Thus the price is

$$\pi_t = \mathbb{E}_{\mathbb{Q}}\Bigl[\mathrm{e}^{-r(T_2-t)}\Bigl(\sum_{i=1}^{N_{T_1}} Z_i - A\Bigr)_+ \Bigm| \mathcal{F}_t\Bigr]\,.$$

Schradin and Timpel; Christensen

They use the Esscher transform for pricing. Let us first consider the interval $[0, T_1]$. Under the physical measure

$$\mathbb{E}_{\mathbb{P}}[P_0 \exp\{X_t\} e^{-rt}] = \exp\{(\beta - r)t\} \; .$$

It is therefore natural to consider the index

$$P_t e^{-\beta t} \; .$$

Let

$$M(z) = \mathbb{E}_{\mathbb{P}}[\exp\{z X_1\}]$$

denote the moment generating function.

Schradin and Timpel; Christensen

Changing the measure with $L_t = \exp\{hX_t\}/M(h)^t$ the process remains an exponential Lévy process but with moment generating function

$$M(z;h) = \frac{M(z+h)}{M(h)} \;.$$

In order that $P_t e^{-(\beta+r)t}$ is a martingale we choose h^* such that

$$M(1;h^*) = \frac{M(1+h^*)}{M(h^*)} = e^{\beta+r} \;.$$

Schradin and Timpel; Christensen

The pricing measure is now determined through

$$\mathbb{Q}[A] = \mathbb{E}_{\mathbb{P}}[\exp\{hX_t\}/M(h)^t \mathbb{1}_A]$$

on $\mathcal{F}_t$ for $t \le T_1$.

For $(T_1, T_2]$ the change of measure is constructed similarly.

Gerber and Shiu have shown that this corresponds to a power utility function for the investor.

Individual Indices: Radon-Nikodym Derivative

We want the market to see the same model. Therefore we use the Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}} = \exp\Big\{\Big(\sum_{k=1}^{N_t}\beta(Y_i) + \int_0^\infty \gamma(s)\,\mathrm{d}W_s^i\Big) - \int_0^t \lambda(s)\mathbb{E}[\Gamma\exp\{\beta(Y)\} - 1]\,\mathrm{d}s\Big\}\,.$$

Individual Indices: Radon-Nikodym Derivative

$\beta(y)$ is a function, such that $\mathbb{E}[\exp\{\beta(Y_i)\}] < \infty$.

$\gamma(s) \geq 0$ such that $\Gamma = \exp\{\frac{1}{2}\int_0^\infty \gamma(t)^2 \, \mathrm{d}t\} < \infty$.

There is a side condition

$$\Pi = \mathbb{E}^*[L_\infty] = \mathbb{E}^*\Bigl[\sum_{i=1}^{N_{T_1}} L_\infty^i\Bigr],$$

where Π is the aggregate premium for catastrophic losses in the occurrence period.

Individual Indices: Distribution under $\mathbb{P}^*$

Under the measure $\mathbb{P}^*$ the claim number process $\{N_t\}$ is a (inhomogeneous) Poisson process with rate

$$\tilde{\lambda}(t) = \Gamma \mathbb{E}[\exp\{\beta(Y)\}]\lambda(t)\,.$$

The first estimates Y_i have distribution function

$$\mathrm{d}\tilde{F}_Y(y) = \frac{\mathrm{e}^{\beta(y)}\,\mathrm{d}F_Y(y)}{\mathbb{E}[\exp\{\beta(Y)\}]}\,,$$

Individual Indices: Distribution under $\mathbb{P}^*$

and the process $\{W_t^i\}$ becomes an Itô process satisfying

$$W_t^i = \widetilde{W}_t^i + \int_0^t \gamma(s)\,\mathrm{d}t\,,$$

where $\{\widetilde{W}^i\}$ are independent standard Brownian motions under $\mathbb{P}^*$, independent of $\{Y_i\}$ and $\{N_t\}$.

Individual Indices: Distribution under $\mathbb{P}^*$

We get

$$\int_0^t \sigma(s)\,\mathrm{d}W_s^i = \int_0^t \sigma(s)\gamma(s)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}\tilde{W}_s^i\,.$$

The market adds a drift to the re-estimates.

Individual Indices: Pricing the PCS Option

- Monte-Carlo simulations
  - Variance reduction methods
  - Importance sampling
- Actuarial approximations
  - Normal approximation
  - Lognormal approximation
  - Translated Gamma approximation
  - Edgeworth approximations
  - Saddle point approximations

Biagini et al. — Pricing

Biagini et al. also use an exponential martingale for changing the measure.

They calculate the Fourier transform of $((x - K)^+ - k)$. They insert into the pricing formula the inversion formula of the Fourier transform, interchange measure and get a formula for the price of the option.

CAT Bond: Independent Triggering Event

Let $\{r_t\}$ denote the interest rate, i.e. the zero coupon prices are calculated as

$$B(0,T) = \mathbb{E}_{\mathbb{Q}}\Bigl[\exp\Bigl\{-\int_0^T r_s \,\mathrm{d}s\Bigr\}\Bigr] \;.$$

Suppose the triggering event A is independent of $\{r_t\}$ with $\mathbb{Q}[A] = 1 - q$.

We denote the return of the bond by v_c, i.e. the value at time T is $1 + v_c$.

CAT Bond: Independent Triggering Event

Then the price of the act-of-God bond becomes

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\Big[\exp\Big\{-\int_0^T r_t \,\mathrm{d}t\Big\}(1+v_c)(1-\mathbb{1}_A)\Big] \\
&= (1+v_c)\mathbb{E}_{\mathbb{Q}}\Big[\exp\Big\{-\int_0^T r_t \,\mathrm{d}t\Big\}\Big]q \\
&= (1+v_c)B(0,T)q
\end{aligned}$$

Note that only q has to be determined by the market because the zero coupon bond exists.

CAT Bond: Dependent Interest Rate and Triggering Event

Denote by $\mathbb{P}_T$ the risk adjusted forward measure, i.e. the measure obtained by using $1/B(0,T)$ as numeraire. That is,
$\mathrm{d}\mathbb{P}_T/\mathrm{d}\mathbb{Q} = \exp\{-\int_0^T r_s \,\mathrm{d}s\}/B(0,T)$.
Changing the measure we find

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\Big[\exp\Big\{-\int_0^T r_t \,\mathrm{d}t\Big\}(1+v_c)(1-1\!\!1_A)\Big] \\ = (1+v_c)B(0,T)\mathbb{E}_T[1-1\!\!1_A] = (1+v_c)B(0,T)q_T\,.\end{aligned}$$

Here $q_T = 1-\mathbb{P}_T[A]$.

Note: The formula looks simple but the problem is to calculate q_T, which is as hard as under $\mathbb{Q}$.

Setup

Time Horizons $0 < T_1 < T_2$.
Initial capital x.
Aggregate insurance loss X (not $\mathcal{F}_{T_2}$-measurable)
Price π_t of the option at time t.
Number of options κ_t in the portfolio (previsible).
Index P_t at time t.
Expiry date T_2.
Value of the option at expiry date $f(P_{T_2})$.
Utility function $u(x)$ (at time T_2).
Suppose all quantities are discounted.

The Utility Maximising Problem

The gain from trading in the option is

$$-\kappa_0\pi_0 + \int_0^{T_2} \kappa_s \,\mathrm{d}\pi_s + \kappa_{T_2} f(P_{T_2}) \;.$$

The insurer (representative agent) wants to maximise

$$\mathbb{E}\Bigl[u\Bigl(x - \kappa_0\pi_0 + \int_0^{T_2} \kappa_s \,\mathrm{d}\pi_s + \kappa_{T_2} f(P_{T_2}) - \mathbb{E}_{\mathbb{Q}}[X \mid \mathcal{F}_{T_2}]\Bigr)\Bigr] \;.$$

The Hedging Strategy

Standard methods from pricing in incomplete markets show that there is a unique trading strategy $\{\kappa_t\}$ which leads to a maximisation of the expected utitility.

If $u(x)$ is the utility function of a representative agent, then we find the indifference price

$$\pi_t = \frac{\mathbb{E}[f(P_{T_2})u'(x - \mathbb{E}_{\mathbb{Q}}[X \mid \mathcal{F}_{T_2}]) \mid \mathcal{F}_t]}{\mathbb{E}[u'(x - \mathbb{E}_{\mathbb{Q}}[X \mid \mathcal{F}_{T_2}]) \mid \mathcal{F}_t]} \ .$$

The Problem in Discrete Time

It seems simpler to consider the problem in discrete time. Let $0 = t_0 < t_1 < \cdots < t_k = T_1 < t_{k+1} < \cdots < t_n = T_2$.
π_i, κ_i, P_i, $\mathcal{F}_i$ for π_{t_i}, κ_{t_i}, P_{t_i}, $\mathcal{F}_{t_i}$.

The value at time t_i becomes

$$\begin{aligned} V_i(x) &= \sup_{\boldsymbol{\kappa}} \mathbb{E}\Bigl[u\Bigl(x - \kappa_i\pi_i - \sum_{\ell=i+1}^{n-1} (\kappa_\ell - \kappa_{\ell-1})\pi_\ell \\ &\qquad + \kappa_{n-1} f(P_n) - \mathbb{E}_{\mathbb{Q}}[X \mid \mathcal{F}_n]\Bigr) \Bigm| \mathcal{F}_i\Bigr] . \end{aligned}$$

We want to maximise $V_0(x)$.

Time t_{n-1}

We have to calculate

$$V_{n-1}(x) = \sup_k \mathbb{E}[u(x - k\pi_{n-1} + kf(P_n) - \mathbb{E}_{\mathbb{Q}}[X \mid \mathcal{F}_n]) \mid \mathcal{F}_{n-1}] \,.$$

First derivative is with respect to k

$$\mathbb{E}[(f(P_n) - \pi_{n-1})u'(x - k\pi_{n-1} + kf(P_n) - \mathbb{E}_{\mathbb{Q}}[X \mid \mathcal{F}_n]) \mid \mathcal{F}_{n-1}] \,.$$

Second derivative

$$\mathbb{E}[(f(P_n) - \pi_{n-1})^2 u''(x - k\pi_{n-1} + kf(P_n) - \mathbb{E}_{\mathbb{Q}}[X \mid \mathcal{F}_n]) \mid \mathcal{F}_{n-1}] < 0$$

Time t_{n-1}

Suppose $\lim_{x\to\infty} u'(x) = 0$ and $\lim_{x\to-\infty} u'(x) = \infty$. Then the first derivative tends to ∞ as $k \to -\infty$ and to $-\infty$ as $k \to \infty$. Thus there is a unique k where the sup is taken.

At time t_{n-1}, the wealth determines the optimal κ_{n-1}.

Time $t_i < t_{n-1}$

Define

$$V_i(x) = \sup_k \mathbb{E}[V_{i+1}(x + k(\pi_{i+1} - \pi_i)) \mid \mathcal{F}_i] \; .$$

The first derivative is

$$\mathbb{E}[(\pi_{i+1} - \pi_i) V'_{i+1}(x + k(\pi_{i+1} - \pi_i)) \mid \mathcal{F}_i] \; .$$

The second derivative is

$$\mathbb{E}[(\pi_{i+1} - \pi_i))^2 V''_{i+1}(x + k(\pi_{i+1} - \pi_i)) \mid \mathcal{F}_i] < 0 \; .$$

Also strictly concave in k.

Time $t_i < t_{n-1}$

Recursively, the first derivative tends to ∞ as $k \to -\infty$ and to $-\infty$ as $k \to \infty$. Thus there is a unique k where the sup is taken.

At time t_i, the wealth determines the optimal κ_i. In particular, there is a unique optimal strategy maximising the expected utility.